Lecture-6 ~~10/1/2015~~ 1/22/2015

Cover times

$C_{uv} = 2m R_{uv}$

$C(G) \le 2m(n-1)$

Today: - Markov Inequality
- Tighter covertime bounds
- Applications.

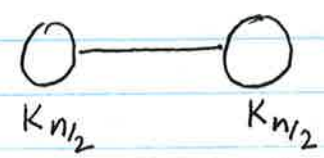
COMPLETE GRAPH

$C(G) \sim \text{Geom}\left(\frac{1}{n-1}\right) + \text{Geom}\left(\frac{2}{n-1}\right) + \dots + \text{Geom}\left(\frac{n-1}{n-1}\right)$

$\Rightarrow C(G) = \theta(n \log n)$

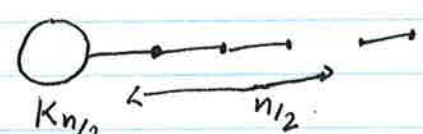
Bound gives $O(n^3)$ not tight

BARBELL GRAPH



Runtime: Bound $O(n^3)$ better: $O(n^2 \log n)$

LOLLIPOP GRAPH



Bound: $O(n^3)$ is tight! Proof?

CHAIN

Bound $O(n^2)$ is tight! proof?

MARKOV INEQUALITY

R.variable

$X \ge 0$

$\text{Prob.}[X \ge a] \le \frac{E[X]}{a}$

useful: a is large "$E(X)$" is not too big.

Proof: $a\, \mathbb{I}_{\{X \ge a\}} \le X$

Take expectation of both sides.

$E[\cdot] \le E[\cdot]$

$a\, E[\cancel{a} \cdot \mathbb{I}_{\{\dots\}}] \le E[X]$

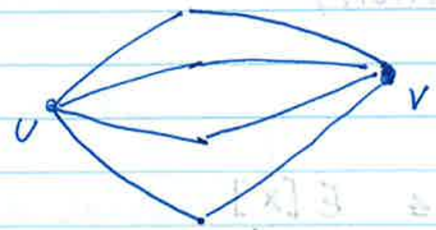
$P_r[C(G) \ge 2\, C(G)] \le$

$P_r[\overset{R.V}{C(G)} \ge 2E[C(G)]] \le \frac{E[C(G)]}{2E[C(G)]} = \frac{1}{2}$

expect to understand.

MAX. RESISTANCE: $R(G) = \max_{u,v} R_{u,v}$ {largest effective resistance}.

Theorem: $mR(G) \le C(G) \le 2e^3 m R(G) \ln(n) + n$



$R_{uv} = \dfrac{1}{\frac{1}{2} + \cdots + \frac{1}{2}} = \dfrac{2}{n-2} = \Theta\left(\dfrac{1}{n}\right)$

Proof: Fix vertex v.

~~P(not visiting v after~~

P_r(starting from any u, not visiting v after $e^3 h_{uv}$ steps)

$$P_r(X_{uv} \geq e^3 \underbrace{h_{uv}}_{E[X_{uv}]}) \leq \frac{1}{e^3}$$

This is effectively setting $a = e^3 h_{uv}$ in markov inequality.

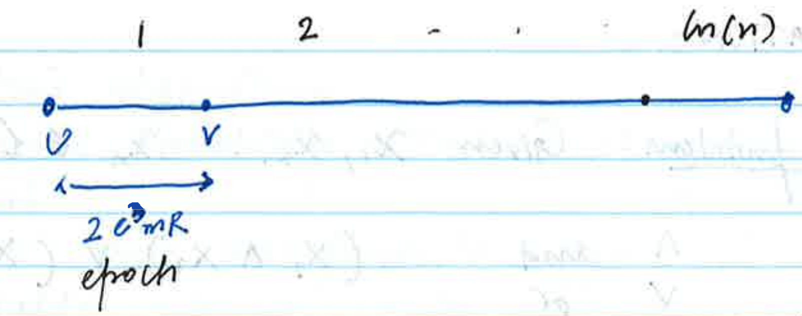
Proof strategy: $2e^3 m R(G) \ln(n)$

Look at this many steps and bound probability that there exists a node that has not been visited.

chunk epochs: $2e^3 m R(G)$

→ What do I know about h_{uv}?

$h_{uv} \leq 2mR$ ~~$\ln(n)$~~



Pr(not visiting v during any of ln(n) epochs) $\leq \left(\dfrac{1}{e^3}\right)^{\ln(n)}$

$= \dfrac{1}{n^3}$

→ Prob.[$\exists v$ not visited during ln(n) epochs] $\leq n \cdot \dfrac{1}{n^3} = \dfrac{1}{n^2}$

→ $C(G) \leq 2e^3 m R \ln(n) + \dfrac{1}{n^2} \cdot n^3$ ← absolute

$= \quad 〃 \quad + n$

→ ~~APPLICATIONS~~

Proving the lower bound:

$mR(G) \leq C(G)$

$h_{uv} \leq C(G)$

Say The largest R_{uv} graph is between u,v

$h_{uv} + h_{vu} = C_{uv} = 2mR \qquad \max(h_{uv}, h_{vu}) \geq \dfrac{\cancel{2}mR(G)}{\cancel{2}}$

APPLICATION:

2-SAT problem: Given $x_1, x_2, \ldots x_n \in \{True, False\}$

$\wedge$ and $\qquad (x_1 \wedge x_3) \vee (x_5 \wedge \neg x_1)$

$\vee$ or

$\neg$ not $\qquad$ Satisfiability

Given boolean formula, can we set $x_1 \cdots x_n$ to make formula true?

This is the SAT problem.

Problem we are going to consider:

$\boxed{\text{2-SAT}}$ doable.

Restriction allowed on

Formula must be the conjunction (and) of 2-literal ~~variable~~ disjunctions.(or)

Literal : $\neg x_1$ or x_1

2-SAT

$$\underbrace{(x_1 \vee \neg x_5)}_{\text{clause}} \wedge (x_3 \vee x_4) \wedge (\neg x_{11} \vee x_1) \wedge \cdots$$

Given 2-SAT formula, is it satisfiable?

ALGORITHM:

Assume satisfying assignment exists.

While ~~Then while~~ formula is not satisfied.

Pick a unsatisfied clause.

Flip ~~over~~ one literal uniform randomly

end

In this algorithm

with probability atleast $\frac{1}{2}$ we get a correct literal.

Let X = RV that corresponds to number of correct variables with regards to satisfying assignment.

$X \leq n$

cover time $\Theta(n^2)$