Lecture 5 1/20/2015

Agenda:

→ Random Walks.
→ Hitting / commute / cover times
→ Effective Resistance

→ Applications

Thursday: Midterm Feb. 12^{th} In class.

SUBGRAPH: Subset of nodes and edges (subject to incidence)

(eg. spanning tree weights come with edges)

INDUCED SUBGRAPH: Subset of nodes only. Edges inferred as all edges between nodes in subset.
(eg. Ramsey graph).

→ Random Walk: Start somewhere on undirected unweighted graph, and step on edges uniformly.

unsymmetric ← h_{ij} = E[# steps starting from i to get to j for the first time.]

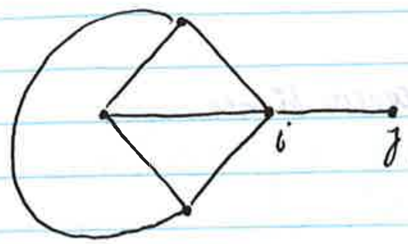
$h_{ii} = 0$.

commute time

$c_{ij} = h_{ij} + h_{ji}$ {also a random variable}.

↳ (symmetric $c_{ij} = c_{ji}$)

Example:



$h_{ji} = 1$

cover time $C_v(G)$: E [To visit all nodes, starting from v]

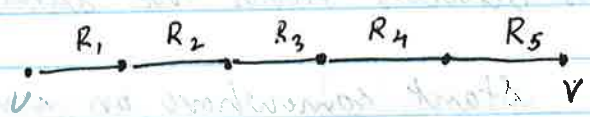
$C(G) = \max_v C_v(G)$

EFFECTIVE RESISTANCE

→ Turn graph into circuit by replacing edge with 1 ohm resistors.

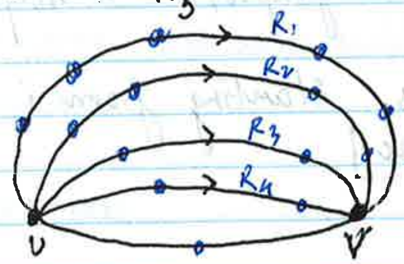
→ Two rules effective resistance

Series:



$R_{uv} = R_1 + \ldots R_5 = 1 + 1 \ldots 1 = 5$

Parallel



$$R_{uv} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \ldots \frac{1}{R_4}}$$

→ Smaller than

→ R_{max} (max of all R)

$R_{uv} \leq \frac{R_{max}}{K}$

Start upper bound

GOAL: Fundamental Theorem

☆ $\boxed{C_{ij} = 2m R_{ij}}$ (1996)

First set of equns.

~~$h_{ij} = \sum_w$~~

$h_{uv} = \sum_{w \in \Gamma(u)} (1 + h_{wv}) \cdot \frac{1}{\deg(u)}$.

$1 + E[$

$h_{vv} = 0 \quad \forall v \in 1 \ldots n$.

Connection to circuits:

Circuit 1 Inject $d(i)$ amps at all node i.
Extract $2m$ amps from target v.

Voltage

$\Rightarrow h_{uv} = \phi_{uv}$

$\phi_{vv} = 0$

Circuit 2 Extract $d(i)$ amps from all nodes i
Inject $2m$ at u.

$\Rightarrow h_{vu} = \phi_{uv}$

$\phi_{uu} = 0$

Superimposition

ϕ in superimposed circuit

$\phi^{sic}_{uv} = 0$

Voltage.

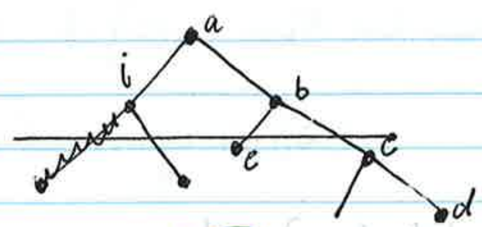
$\phi_{uv} = h_{uv} + h_{vu} \quad (= C_{uv})$

$$\underset{\text{voltage}}{C_{uv}} = \underset{\text{current}}{2m} \quad \underset{\text{effective resistance}}{R_{uv}}$$

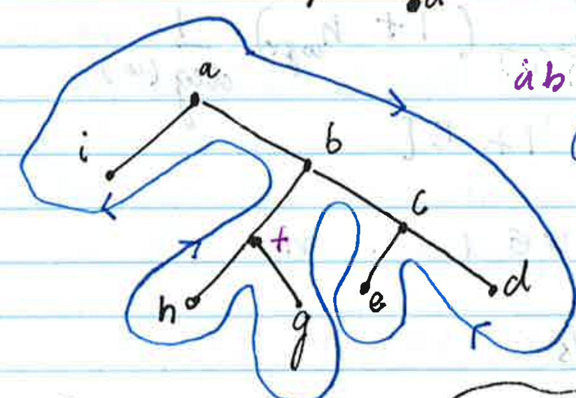
→ Commute time to covering & effective resistance.

Prove : $C(G) \leq 2m(n-1)$

Given a arbitrary graph G. Take spanning tree of G



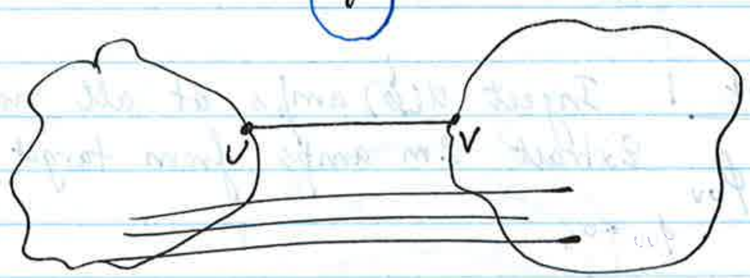
Spanning tree of G



a b c d c e c

(a,b) ... (c,d) (d,c)

(c,e) (e,c) (c,b)



$C_{uv} \leq 2m$

(because $R_{uv} \leq 1$)

$$C^{RV}(G) \leq h^{RV}_{ab} + h^{RV}_{bc} + h^{RV}_{cd} + \dots h^{RV}_{ia}$$

(pair up to get commute time)

$$= C^{RV}_{ab} + C^{RV}_{bc} + \dots + C^{RV}_{ia}$$

in expectation

$$\overset{E}{\leq} 2m + \dots 2m = 2m(n-1)$$

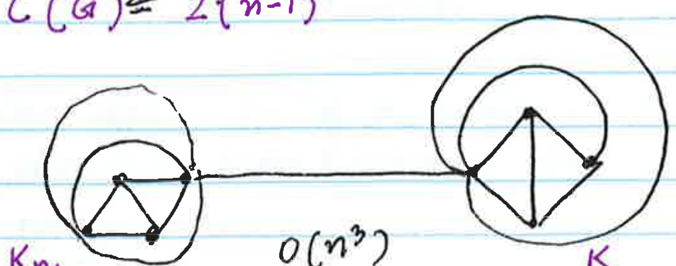
(n-1) times.

$C = 2m\,k_{ij} = 2m(n-1) = 2(n-1)^2$

It's a tree.

What is the cover time?

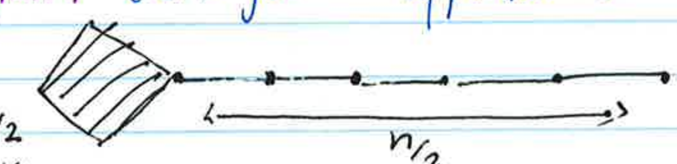
$C(G) \leq 2(n-1)^2$.



$K_{n/2}$ $\quad O(n^3)$ $\quad K$ $\quad \Theta(n^2)$. → (Double check).

Barbell Graph.

LOLLIPOP GRAPH Let's get an upperbound.



$K_{n/2}$

Klique $n/2$.

$m = \Theta(n^2)$ → tightly bound.

$m = \binom{n/2}{2} + \frac{n}{2}$

Θ =

O ≤

o <

Ω ≥

ω >

$C(G) = O(n^3)$

Cover time of a complete graph. $O(n\log(n))$)

Why?

Covering time of a complete graph

$C(K_n) = \sum_{i=1}^{n-1} \text{Geom}\left(\frac{1}{i}\right)$

Expectations of these things...

$$E = [\quad] \leq n \sum_{i=1}^{n} \frac{1}{i} = \Theta(n\log(n)).$$

? with calculus $O(n^3)$ X.