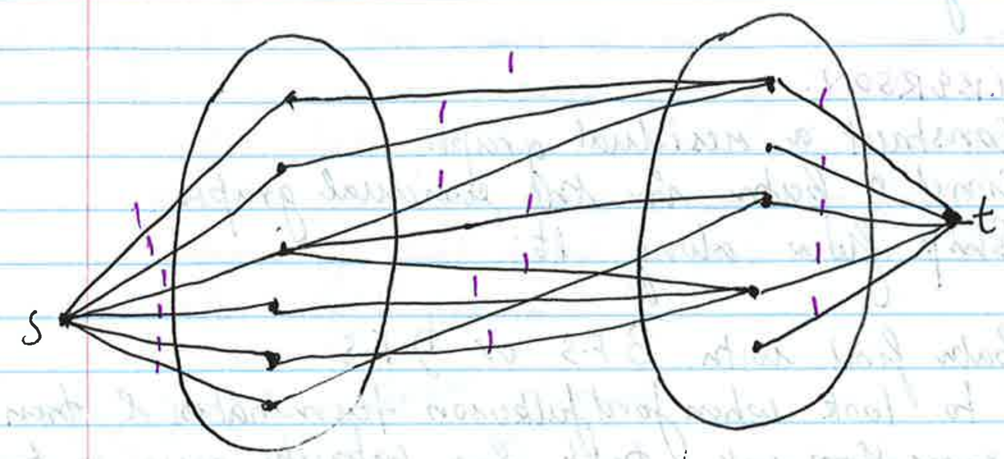
Lecture - 3
1/13/2015

- Applications of Max flow / min cut.
- The probabilistic method.
  1. Max Cut 2. Erdos Renyi Graphs.
  3. Ramsey Numbers.

MCE.

Matching: Subset of edges
S-t no node is incident on more than one edge in M.



1. use
Perfect matching:
all nodes get assigned.
→ Bipartite graphs: min cut / max flow.
→ General, Flower, Blossom & Trees.

Maximum matching is different from maximal matching.

1. Use min or max?
2. How to encode constraints?
3. where to put source and target?
S t

Remark: Ford fulkerson returns integer max-flow with integer capacities.

— max matching ≠ maximal matching
less inter.

→ Multi source/target.

PROBABILISTIC METHOD

Prove this thing exists. (Flip some coins determinis-tically in said team).

→ Prove things exist by providing a probabilistic procedure that has non-zero probability of outputting desired object.

Max-cut CUT-EXISTENCE

Prove that every unweighted undirected graph has a cut of atleast size $\frac{m}{2} = \left(\frac{|E|}{2}\right)$

DMC → Dumb Max Cut

PROCEDURE: $S = \{\ \}$

For $v \in V$

with probability $1/2$

$S \leftarrow S \cup \{v\}$

What is the expected cut size?

X: Cut size produced by DMC

$$E[X] = \sum_{i=1}^{m} E[\ \mathbb{I}\{\text{edge is cut}\}\ ]$$ ← Indicator fn.

Expectation of indicator variable = Probability

$$= \sum_{i=1}^{m} P_r[\text{edge } i \text{ is cut}] = \sum_{i=1}^{m} \frac{1}{2} = \frac{m}{2}$$

$$E[X] = \frac{m}{2} \Rightarrow P[X \geq \frac{m}{2}] > 0$$

↳ write down what expectation means?

$$E[X] = \sum_{i=1}^{m} P[X = x]\,x = \frac{m}{2}$$

u — v with prob. $1/2$ u, v are on different sides of S.

$\Rightarrow \exists$ cut with size $\geq \frac{m}{2}$

By the probabilistic method

→ ~~For sufficiently~~

→ Ramsey number thing.

Every graph on 6-nodes has K_3 or $\bar{K}_3$ as subgraph

K_3: Komplete graph on 3 nodes

$\bar{K}_3$: Komplete disconnected graph on 3 nodes (Also, called independent set)

→ for sufficiently n, there always exists a subgraph of K_r or $\bar{K}_r$

→ $R(r)$ = smallest n for which is true for all graphs.

$R(3) = 6$

$R(4) = 18$

$R(5) = 43 - 49$

$R(6) = 102 - 165$

→ Prove $\underbrace{2^{r/2} \leq R(r)}_{\text{Improv this one with the probabilistic method.}} \leq 2^{2r}$

→ We are going to generate random graphs

ERDOS - RENYI $G(n,p)$

→ ~~Fix~~ Fix n nodes

For each potential edge flip a coin with prob. p.

☆ if $p > \frac{\ln(n)}{n}$ then $G(n,p)$ is connected almost surely as $n \to \infty$

Probability of ($G(n,p)$ has a k_r in it) =

$\leq \binom{n}{r} p^{\binom{r}{2}}$ ← fixing r nodes.

$P(G(n,p)$ has a $\bar{k}_r) \leq \binom{n}{r}(1-p)^{\binom{r}{2}}$

Set $p = \frac{1}{2}$

$P_r(G(n,p)$ has either k_r or $\bar{k}_r) \leq 2\binom{n}{r}\left(\frac{1}{2}\right)^{\binom{r}{2}}$

$P_r[X_1 \cup X_2 \cup \ldots] \leq \sum_i P[X_i]$

$\leq 2\,\frac{n^r}{2^r}\left(\frac{1}{2}\right)^{\frac{r(r-1)}{2}}$

First bound:

$\binom{n}{r} \leq \frac{n^r}{r!} \leq \frac{n^r}{2^r}$

$\leq 2\,\frac{(2^{r/2})^r}{2^r}\left(\frac{1}{2}\right)^{\frac{r\cdot(r-1)}{2}}$

$= 2 \cdot \frac{2^{r^2/2}}{2^r}\, 2^{-r(r-1)/2}$

$= 2 \cdot 2^{-r/2} < 1$

$P(G(n,p)$ for $n \leq 2^{r/2}$ has k_r o $\bar{k}_r) < 1$

$\Rightarrow P(G(n,p)$ for $n \leq 2^{r/2}$ has neither k_r or $\bar{k}_r) > 0$

$\Rightarrow \exists$ some graph of size $2^{r/2}$ that violates ramsey condition.

$\Rightarrow R(r) \geq 2^{r/2}$.

New bounds:

$\underbrace{\frac{\sqrt{2}}{e}\, r\, 2^{r/2} \leq R(r)}_{\text{Probabilistic.}} \leq r^{\frac{-\log r}{\log\log r}} \cdot 2^{2r}$