CME 305: Discrete Mathematics and Algorithms

Instructor: Reza Zadeh (rezab@stanford.edu)

HW#1 – Due at the beginning of class Thursday 01/23/14

1. **(Lovasz, Pelikan, and Vesztergombi 7.3.9)** Prove that at least one of G and $\overline{G}$ is connected. Here, $\overline{G}$ is a graph on the vertices of G such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

2. A vertex in G is *central* if its greatest distance from any other vertex is as small as possible. This distance is the *radius* of G.

   (a) Prove that for every graph G

   $$\text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G$$

   (b) Prove that a graph G of radius at most k and maximum degree at most $d \geq 3$ has fewer than $\frac{d}{d-2}(d-1)^k$ vertices.

3. An oriented incidence matrix B of a directed graph $G(V, E)$ is a matrix with $n = |V|$ rows and $m = |E|$ columns with entry B_{ve} equal to 1 if edge e enters vertex v and -1 if it leaves vertex v. For an undirected graph, we will use an arbitrary orientation of the edges. Let $M = BB^T$. Note, that M (or Laplacian) is independent of the orientation of the edges. Prove that $rank(M) = n - w$ where w is the number of connected components of G.

4. A simple graph $G(V, E)$ is called Hamiltonian if it contains a cycle which visits all nodes exactly once. Prove that if every vertex has degree at least $|V|/2$, then G is Hamiltonian.

5. Let $G = (V, E)$ be a graph and $w : E \to R^+$ be an assignment of nonnegative weights to its edges. For $u, v \in V$ let $f(u, v)$ denote the weight of a minimum $u - v$ cut in G.

   (a) Let $u, v, w \in V$, and suppose $f(u, v) \leq f(u, w) \leq f(v, w)$. Show that $f(u, v) = f(u, w)$, i.e., the two smaller numbers are equal.

   (b) Show that among the $\binom{n}{2}$ values $f(u, v)$, for all pairs $u, v \in V$, there are at most $n - 1$ distinct values.

6. Let V be a finite set. A function $f \colon 2^V \to R$ is submodular iff for any $A, B \subseteq V$, we have

   $$f(A \cap B) + f(A \cup B) \leq f(A) + f(B)$$

   Now consider a graph with nodes V. For any set of vertices $S \subseteq V$ let $f(S)$ denote the number of edges $e = (u, v)$ such that $u \in S$ and $v \in V - S$. Prove that f is submodular.

7. Let T be a spanning tree of a graph G with an edge cost function c. We say that T has the *cycle property* if for any edge $e' \notin T$, $c(e') \geq c(e)$ for all e in the cycle generated by adding e' to T. Also, T has the *cut property* if for any edge $e \in T$, $c(e) \leq c(e')$ for all e' in the cut defined by e. Show that the following three statements are equivalent:

(a) T has the cycle property.

(b) T has the cut property.

(c) T is a minimum cost spanning tree.

Remark 1: Note that removing $e \in T$ creates two trees with vertex sets V_1 and V_2. A *cut* defined by $e \in T$ is the set of edges of G with one endpoint in V_1 and the other in V_2 (with the exception of e itself).

8. Given a sequence p_i of stock prices on n days, we need to find the best pair of days to buy and sell. i.e. find i and j that maximizes $p_j - p_i$ subject to $j \geq i$. Give an $O(n)$ dynamic programming solution.