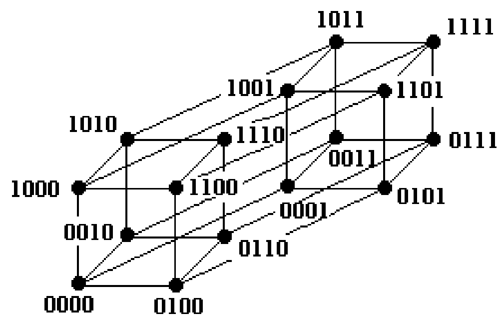
CME 305: Discrete Mathematics and Algorithms

Instructor: Reza Zadeh (rezab@stanford.edu)

Midterm

The hypercube graph Q_h is an undirected regular graph with 2^h vertices, where each vertex corresponds to a binary string of length h. Two vertices labeled by strings x and y are joined by an edge if and only if x can be obtained from y by changing a single bit. As usual, the number of nodes is denoted $n = 2^h$.



The hypercube graph Q_4.

1. (10 points) Prove that Q_h is bipartite.

   Solution: Partition into vertices that have odd and even number of 1s.

2. (5 points) Prove that Q_h has an independent set of size 2^{h-1}.

   Solution: One side of the bipartition forms an independent set of the desired size.

3. (15 points) Consider a simple random walk starting from an arbitrary vertex in Q_h. Show that the cover time of the random walk on Q_h, i.e. the expected time that it takes for the random walk to visit all the vertices is at most $O(nh^3)$.

   Solution: We exhibit a path of length at most h between any two nodes u and v. Starting at any node u, we can get to any other node v by 'correcting' the bits of u from left to right, one by one to match v. Each correction corresponds to traversing a single edge, and there can be at most h corrections. Furthermore, to get from all zeros to all ones we must make at least h corrections, thus the diameter is exactly h.

   By Matthew's bound we know cover time is $O(H \log(n))$, where H is the largest hitting time in the graph. We can trivially upperbound the largest hitting by the largest commute time. The largest commute time is given by 2m times the largest effective resistance. In a graph with diameter h, effective resistance between any two nodes is at most h. Thus the largest hitting time is at most $2mh$. The number of edges in the hypercube is $m = hn/2$ since each node has degree h. Putting all these together:

$$C(G) \leq 2H \log(n) \leq 4mh \log(n) = 2nhh \log(n) = 2nhh \log(2^h) = O(nh^3)$$

4. (5 points) Prove that the maximum flow between any two nodes in Q_h is at most h.

   Solution: Any degree gives this upper bound.

5. (10 points) Prove that the maximum flow between any two nodes in Q_h is at least h.

Solution: We show that in a hypercube graph Q_h there are at least h edge-disjoint paths between any two nodes, thus the max flow between then is at least h.

We want to show that there are n internally vertex disjoint paths between any two distinct vertices u and v of Q_h. By the symmetry of the hypercube we can assume without loss of generality that $u = (0, 0, ..., 0)$ and $v = (\underbrace{0, \ldots, 0}_{k}, \underbrace{1, \ldots, 1}_{h-k})$ for some $k \in \{0, 1, \ldots, n-1\}$. Now for each $i = 1, 2, ..., h$ define a path P_i as follows: start from the vertex u (which is all zeros), flip the i-th bit to a 1 and then correct wrong bits (bits in which the current vertex and the destination vertex v differ) one by one moving leftwards from position $i-1$ to 1 to n and back to i. Observe that the paths P_i are by construction internally vertex disjoint. Thus the vertex-connectivity and the edge-connectivity of Q_h are at least h.