What matters in school choice tie-breaking?
How competition guides design

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Abstract

Many school districts apply the student-proposing deferred acceptance algorithm after ties among students are broken exogenously. We compare two common tie-breaking rules: one in which all schools use a single common lottery, and one in which every school uses a separate independent lottery. We identify the balance between supply and demand as the determining factor in this comparison.

First we analyze a two-sided matching model with random preferences in over-demanded and under-demanded markets. In a market with a surplus of seats a common lottery is less equitable and there are efficiency trade-offs between the two tie-breaking rules. However, a common lottery is always preferable when there is shortage of seats. The theory suggests that popular schools should use a common lottery to resolve ties. We run numerical experiments with New York City choice data after partitioning the market into popular and non-popular schools. The experiments support our findings.

1 Introduction

A growing number of school districts around the world have adopted student assignment mechanisms based on the student-proposing deferred acceptance (DA) algorithm (see Abdulkadiroğlu and Sönmez (2003)). These mechanisms satisfy properties that make them attractive designs. First, the student-proposing DA is strategyproof for students, which makes it simple for students to rank schools. Second, the outcomes are stable assignments with respect to students’ preferences and

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schools’ priorities over students; this means that if a student prefers some school to her own assignment, then she has a lower priority in that school than all students who were assigned to it. Often, however, schools assign the same priority to many students and thus the manner in which ties are resolved among equivalent students has welfare consequences (Erdil and Ergin (2008b)).

The student-proposing DA algorithm works as follows. Students apply to schools, which tentatively accept the most preferred students (up to their capacity) and reject all others. Rejected students then apply to their next preferred schools, and again schools tentatively accept the most preferred students so far, and reject all others, and the algorithm iterates until convergence. When students can have the same priority at a school, acceptance and rejection decisions involve tie-breaking decisions.

Typically school districts consider the following two natural tie-breaking rules: the multiple tie-breaking rule (MTB), under which every school independently assigns to each applicant a random lottery number that is used to break ties, and the single tie-breaking rule (STB), under which each student receives a single lottery number to be used for tie-breaking by all schools. A separate lottery in each school, i.e. MTB, seems fairer as students with bad draws at some schools may still have good chances at other schools, but may lead to unnecessary inefficiency (Abdulkadiroğlu and Sönmez (2003)).

These tie-breaking rules were considered, for example, in the school choice reforms in New York City and Amsterdam. While assisting with the reform, Abdulkadiroğlu et al. (2009) and De Haan et al. (2015), also conducted numerical experiments using school choice data from these cities. Both studies find very similar patterns in the data, which verifies intuitive tradeoffs. More students are assigned to their top choice under STB than under MTB; at the same time, STB also results in more students being assigned to their lower-rank choices or even ending up unassigned. In other words, neither students’ rank distributions under these lotteries first-order stochastically dominates the other. These tradeoffs have led to different choices in practice: NYC adopted STB since policymakers favored that many more students are assigned to their first choice, while policymakers in Amsterdam adopted MTB citing equity as a major reason. This paper argues that much of the ambiguity in the selection of a tie-breaking rule can be alleviated using certain

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1 Observe that under both tie-breaking rules DA remains strategyproof.
2 The difference in the assignments to the top choice is approximately 4% in favor of STB.
3 In fact Abdulkadiroğlu et al. (2009) document that prior to the numerical experiments, NYC policymakers favored MTB due to its fairness.
4 After the first year of using MTB (2014), however, Amsterdam switched to STB, following a lawsuit by a couple of families who were interested in switching their assigned schools.
market characteristics.

The key idea of this paper is that the tradeoffs observed in previous studies do not spread through the entire market. To understand better how tie-breaking rules affect students’ assignments, we propose to consider separately two sub-markets of schools based on their demand or their popularity. Distinguishing more popular from non-popular schools depends on the preference model as well as other market characteristics and is admittedly not an obvious task. We define here the *popularity* of a school to be the ratio of the expected number of students for which the school is their first choice to the capacity of the school (this is further discussed in Section 2). This simple definition allows to separate popular from non-popular schools using a certain popularity threshold. Loosely speaking, we find that the tradeoffs between the tie-breaking rules only hold in the sub-market with non-popular schools but disappears in the sub-market with popular schools, in which a single tie-breaking rule is always preferred to the multiple tie-breaking rule.

We first consider a stylized model that allows to convey the main insights. Consider two tiers of schools, with an aggregate of $p$ and $q$ seats, respectively. There are $n$ students, and all students prefer any school from the first tier to any school from the second tier, and each student independently ranks schools within each tier uniformly at random. Further assume that there are not enough seats in the first tier, but enough seats overall. The popularity threshold is set to be 1 and thus only top tier schools are popular.

We compare the impact of STB and MTB on students’ assignments under the DA mechanism using three measures of efficiency and fairness. Central to this study is the *rank distribution* of an assignment, which counts for each $r$ the number of students who are assigned to their $r$-th choice. We ask whether, and when, one rank distribution *stochastically dominates* the other. Second, we measure the number of *Pareto improving pairs* under MTB, which is the number of pairs of students who are better off by exchanging their seats. (With no priorities the assignment under a single lottery is Pareto efficient.) This notion is motivated by a lawsuit filed after the first centralized assignment in Amsterdam, in which parents sued to exchange their children’s school assignments (see also De Haan et al. (2015)). Third, we compare the *variance of rank distributions*. Intuitively, the larger the variance, the larger the range of potential matches the student is faced with, thus the larger the uncertainty. When preferences are identical and independently distributed the variance can be interpreted as a measure for *ex post* equal treatment of *ex ante* equals.

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5There may be Pareto improving cycles with more than two students. However, we limit ourselves to pairs of students since it is arguably much simpler for a student to find one other student who is interested in exchanging seats than to identify an indirect exchange through a cycle with at least two other students.
We find that the balance between demand and supply is the determining factor for all the three measures.

- For students assigned to popular schools:

  The rank distribution of students under STB (almost) stochastically dominates the rank distribution under MTB. Moreover, MTB generates many Pareto improving pairs, and the variance of rank distribution is higher under MTB than under STB.

- For students assigned to non-popular schools:

  Neither rank distribution stochastically dominates the other. Moreover, MTB generates very few Pareto improving pairs, and the variance of rank distribution is higher under STB.

These results imply that within the set of popular schools there is essentially no tradeoff between our notions of efficiency and fairness when choosing a tie-breaking rule, since a single lottery generates better assignments than separate lotteries with respect to all measures. This stands in contrast to the intuition that MTB is (ex post) fairer. For non-popular schools the decision remains ambiguous since separate lotteries are more equitable than a single lottery but more students are assigned to higher choices under the latter (Ashlagi et al. (2015); Arnosti (2015)) and less students are assigned to their lower choices under the former rule.

Next, we apply our approach and examine the insights generated from the model using school choice data from New York City public high school assignment during the year of 2007-2008. In the main round of assignment, students submitted rank-ordered lists of at most 12 programs and the deferred acceptance algorithm was used to assign students. First, consistent with Abdulkadiroğlu et al. (2009), neither the rank distribution under STB nor the rank distribution under MTB stochastically dominates the other. However, when we limit the attention to students who are assigned to “popular” schools, which are schools with popularity above a certain threshold, we find that STB stochastically dominates MTB. In contrast to the stylized model, there is no clear separation between popular and non-popular schools in the NYC data. However, stochastic dominance holds when the threshold for popularity is at least 1. We find in the data similar qualitative results as in the stylized model regarding Pareto improving pairs and the variance of rank distribution.

Importantly many of our assumptions in the stylized model do not hold in the data (e.g., the market is not perfectly tiered, schools have different capacities, schools assign students to various priority classes prior to breaking ties), hinting that our qualitative predictions are robust. Various
potential extensions are discussed in Section 3.1.1. Moreover, Appendix F provides a simple model with a continuum of students, in which similar insights hold when preferences are generated from a multinomial logit model.

### 1.1 Related Work

Closely related are papers that investigate the tradeoffs between STB and MTB empirically and theoretically. Abdulkadiroğlu et al. (2009) and De Haan et al. (2015) find empirically that students’ rank distributions under STB and MTB have a single crossing point, and thus neither stochastically dominates the other. Theoretical studies shed light on these tradeoffs. Ashlagi et al. (2015) explain why STB assigns many more students to top choices than MTB in a model with random preferences (even in a slightly under-demanded market). Independent to this work, Arnosti (2015) explains the single crossing point pattern using a cardinal utility model. His model, which assumes students’ preference lists are short, is essentially equivalent to analyzing a market with a large surplus of seats.\(^6\) The novel approach taken in this paper, which distinguishes between over-demanded schools and under-demanded schools, explains the source of these tradeoffs both theoretically and empirically.

Our theoretical findings complement results by Ashlagi et al. (2017), who analyze the average student rank in unbalanced two-sided random markets. Their results, together with those of Knuth (1995), imply that the average rank of students is significantly better under STB than under MTB in an over-demanded market (with more students than seats), but these average ranks are essentially the same in a market with a surplus of seats (we elaborate on this in Section A.) These papers limit attention to students’ average rank and do not study the rank distributions.

Also related is a strand of literature that studies economic properties in large random matching markets (Immorlica and Mahdian (2005), Kojima and Pathak (2009)). Closest to this work are studies on agents’ ranks under DA (Pittel (1989), Ashlagi et al. (2017)) and on the inefficiency under DA (Lee and Yariv (2014), Che and Tercieux (2014)). Che and Tercieux (2014) further find that in an over-demanded market, the assignment under MTB is not Pareto optimal with high probability. Their argument relies on finding a Pareto improving cycle involving many students and this does not imply the existence of many Pareto improving pairs. This paper further contributes to this literature by studying the variance of rank, the frequency of Pareto improving pairs, and proving

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\(^6\) Abdulkadiroğlu et al. (2015b) analyze the cutoffs that clear the market in continuum model and establish that STB is ordinally efficient (see also Che and Kojima (2010), Liu and Pycia (2012) and Ashlagi and Shi (2014)).
concentration results for previously studied random variables (such as the number of proposals made by a fixed agent).

The tradeoff between incentives and efficiency when preferences contain indifferences has led to papers suggesting several novel tie-breaking approaches, among which are the stable improvement cycles of Erdil and Ergin (2008a), the efficiency-adjusted DA of Kesten (2011), the choice-augmented DA of Abdulkadiroğlu et al. (2015a), and the circuit tie-breaker by Che and Tercieux (2014).

Finally, a few papers study tie-breakings under the top trading cycles algorithm, which finds Pareto efficient outcomes. Pathak and Sethuraman (2011) and Carroll (2014) extend results by Abdulkadiroğlu and Sönmez (1998) to show that under the top trading cycles algorithm (Shapley and Scarf (1974)), there is no difference between a single tie-break (equivalently, random serial dictatorship) and multiple tie-breaks (top trading cycles with random endowments). Che and Tercieux (2015) show that all Pareto efficient mechanisms (and not only top trading cycles) are asymptotically payoff equivalent under certain assumptions.

1.2 Organization of the paper

Section 2 presents the general framework and the notions of comparison. Section 3 presents a specific model and theoretical findings. Section 4 provides an analysis of choice data from NYC for public high schools and Section 5 concludes. All proofs are deferred to the appendices.

2 The framework

In a school choice problem there are \( n \) students, each of whom can be assigned to one seat at one of \( m \) schools. Each school \( c \) has fixed capacity \( q_c > 0 \). Denote the set of students by \( S \) and the set of schools by \( C \). Each student \( s \) has a strict preference list of all schools. Let the rank of a school \( c \) for student \( s \) be the number of schools that \( s \) weakly prefers to \( c \). Thus the most preferred school for \( s \) has rank 1. It is assumed that each student draws her preferences over schools independently from a commonly known distribution \( F \).

Each school \( c \in C \) has a ranking over students that is used to break ties between students in the same priority class. We assume each school has a single priority class containing all students (but dismiss this assumption when experimenting with the data). We refer to a school’s ranking over students as its preference list.

An assignment of students to schools assigns each student to at most one school such that no
school is over capacitated. We sometimes use the term *matching* instead of assignment. We say that the *rank of a student* is \( r \) in an assignment if she is assigned to her \( r \)-th choice in that assignment. An assignment is said to be *unstable* if there is a student \( s \) and a school \( c \) such that \( s \) prefers to be assigned to \( c \) over his current assignment, and \( c \) either has a vacant seat or an assigned student whose priority is lower than \( s \). A assignment is said to be *stable* if it is not unstable.

**Popularity of schools.** We define the *popularity* of a school by

\[
\alpha_c = \frac{\hat{p}_1(c)}{q_c},
\]

where \( \hat{p}_1(c) \) denotes the expected number of students for which school \( c \) is their top choice. We say that a school is *popular* if it has popularity level at least 1 and otherwise it is said to be *non-popular*.

In Appendix F, we point out to a connection between our notion for popularity and multinomial logit preference models. Our measure for popularity is a heuristic and one can possibly think of other reasonable measures. Observe that under the student-proposing deferred acceptance algorithm popular schools will be filled under the deferred acceptance algorithm regardless of how schools rank students. (This does not suggest that a school with a popularity lower than 1 will not be filled under the student-proposing DA.) We examine different thresholds for popularity (ranging from smaller to larger than 1) with the NYC data in Section 4.

**Tie-breaking rules.** We consider two common tie-breaking rules that school districts use to determine rankings of schools over students. Under a *multiple tie-breaking rule* (MTB), each school independently selects a ranking over all students uniformly at random. Under a *single tie-breaking rule* (STB), all schools use the same ranking, which is selected uniformly at random. We study properties of the assignments under the deferred acceptance mechanism when paired with each of these tie-breaking rules. For brevity we refer to these assignments as the outcomes under MTB and STB.

The following simple example illustrates how tie-breaking can affect students’ assignments.

**Example 1.** There are 3 students \( s_1, s_2, s_3 \), and 3 schools, \( c_1, c_2, c_3 \) each with one seat. Schools are

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7When students’ preferences are generated from a multinomial logit model, then the popularity of a school \( c \), \( \alpha_c \), is the “logit weight” (popularity level) of school \( c \) in the logit model normalized by the capacity of \( c \).

8One way of implementing STB is by assigning each student a lottery number that is drawn independently and uniformly at random from \([0,1]\). Similarly, MTB can be implemented in a similar fashion by using a separate lottery (and thus different lottery numbers) for each school.

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indifferent between all students. Students’ preference lists are:

\[ \begin{align*}
 s_1 &: c_1 \succ c_2 \succ c_3, \\
 s_2 &: c_1 \succ c_3 \succ c_2, \\
 s_3 &: c_2 \succ c_1 \succ c_3.
\end{align*} \]

(So, \( s_1 \) prefers \( c_1 \) to \( c_2 \) and \( c_2 \) to \( c_3 \)). Suppose ties are broken as follows:

\[ \begin{align*}
 c_1 &: s_3 \succ s_2 \succ s_1, \\
 c_2 &: s_2 \succ s_1 \succ s_3, \\
 c_3 &: s_2 \succ s_3 \succ s_1.
\end{align*} \]

The (student-proposing) DA then would assign all students to their second choice (i.e., it produces the assignment \( s_1 \)-c2, \( s_2 \)-c3, \( s_3 \)-c1). Suppose instead all schools use the ranking \( s_2 \succ s_1 \succ s_3 \) to break ties. Then DA produces the assignment \( s_1 \)-c2, \( s_2 \)-c1, \( s_3 \)-c3, and so \( s_1 \), \( s_2 \), \( s_3 \) obtain their second, first, and third choices, respectively.

Before we proceed it is worth making a couple of comments that further motivate the focus on STB and MTB.

Remark 1. With a single priority class, no assignment will result in a Pareto improvement for students over STB because DA with STB is equivalent to a serial dictatorship mechanism, in which students select in sequential order their seats according to the tie-breaking order. This is not necessarily the case with multiple priority classes. It is worth noting, however, that any gains from finding an “optimal” stable matching would require the use of a non-strategyproof mechanism (Abdulkadiroğlu et al. (2009)). Observe that under both STB of MTB, the student-proposing DA mechanism remains strategyproof since ties are resolved independently of students’ preferences.

2.1 Notions of comparison

We next present a few definitions which will allow us to compare STB and MTB. The first, natural notion, is stochastic dominance.

Definition 2.1 (stochastic dominance). The rank distribution of students is a function \( R : [1, m] \rightarrow [0, n] \) where \( R(i) \) denotes the number of students who are assigned to their (top) \( i \)-th choice in their
preference list. Fix a constant $\epsilon > 0$. We say that a rank distribution $\mathcal{R}$ stochastically dominates rank distribution $\mathcal{R}'$ if, for any integer $i \in [1, m]$, $\sum_{j=1}^{i} \mathcal{R}(j) \geq \sum_{j=1}^{i} \mathcal{R}'(j)$.

**Definition 2.2 (Pareto improving pairs).** Consider an assignment $\gamma$. Let $\gamma(x)$ be the agent to which $x$ is matched, and suppose $\gamma(x) = \emptyset$ if $x$ is unmatched under $\gamma$. A pair of students $s, s' \in S$ is called a Pareto improving pair in $\gamma$ if $\gamma(s') \succ_s \gamma(s)$ and $\gamma(s) \succ_{s'} \gamma(s')$. We use $\tilde{\gamma}(s)$ to denote the number of Pareto improving pairs in $\gamma$ that contain student $s$.

Next we define the variance of a student rank, and for this need a few notations. Consider an assignment $\gamma$. For any student $s$, $\gamma^\#(s)$ denotes the rank of school $\gamma(s)$ on the preference list of $s$; this notion is defined similarly for schools. Denote the average rank of students who are assigned under $\gamma$ by $\mathcal{A}r(\gamma) = \frac{1}{|\gamma(C)|} \cdot \sum_{s \in \gamma(C)} \gamma^\#(s)$, where $\gamma(C)$ is the set of all assigned students under $\gamma$. When $\gamma$ is clearly known from the context we will simply write $r_s$ for $\gamma^\#(s)$. Denote by $\mu_\pi$ and $\eta_\pi$ respectively the student-optimal and the school-optimal stable matching for a preference profile $\pi$. Finally, given students’ preferences, let $\mu_{STB}$ and $\mu_{MTB}$ be the random variables that denote the student-optimal stable matchings under STB and MTB, respectively.

**Definition 2.3 (Variance of the rank).** The expected variance of the rank of a student $s$ is defined as

$$
\text{Var}[r_s] \triangleq \mathbb{E}_{\{\pi(c) : c \in C\}} \left[ (\mathcal{A}r(\mu_\pi) - \mu_\pi^\#(s))^2 \mid \mu_\pi(s) \neq \emptyset \right],
$$

where the expectation is taken taken over schools’ preferences, which are generated by either the STB or MTB rule.

A different interpretation for (2) could be given by defining the social inequity in a matching $\mu$ to be

$$
\text{Si}(\mu) = \frac{1}{|\mu(C)|} \cdot \sum_{s \in \mu(C)} (\mathcal{A}r(\mu) - \mu^\#(s))^2.
$$

It can be verified that $\mathbb{E}_\pi[\text{Si}(\mu_\pi)]$ is indeed equal to (2) under either of the tie-breaking rules whenever students’ preference are drawn i.i.d. (see Lemma A.4).

We use the notion of variance in the remainder of the paper. When $n, m$ are clearly known from the context, we use $\text{Var}[\mu_{MTB}]$ and $\text{Var}[\mu_{STB}]$ to denote the value of (2) under MTB and STB, respectively.

**Hypothesis**
For students assigned to popular schools: the rank distribution under STB stochastically dominates the rank distribution of students under MTB. Moreover, the variance of rank distribution is higher under MTB than under STB, and MTB generates “many” Pareto improving pairs.

For students assigned to non-popular schools: there is no stochastic dominance relationship between the rank distributions under STB and MTB. The variance of rank distribution is larger under STB, and MTB generates “significantly” fewer Pareto improving pairs among these students than among students assigned to popular schools.

In the next section we establish (most of) the hypothesis under some stylized assumptions. The reader who is interested only in the empirical findings can skip directly to Section 4, though it is useful to first skim the intuition for our results, given in Section 3.3.

3 A stylized model and results

3.1 Model

We consider a school choice problem in which each school has a single seat. We assume that there are two tiers of schools, top and bottom, and there is a shortage of seats in popular schools but there are enough seats for all students in all schools together. Each student prefers top schools to bottom schools. Furthermore, students’ preferences for each type of schools are drawn uniformly at random. Observe that in this model, top schools are popular and bottom schools are non-popular.

For our purposes it will be sufficient to consider a school choice problem, in which students’ preferences are drawn uniformly at random over all schools and assume the market either has a shortage of or a surplus of seats. This is because the rank distributions of students in each of these smaller markets are identical to the rank distributions of students assigned to popular and non-popular schools in the two-tiered market, respectively.\textsuperscript{9}

In particular, we consider a \textit{random school choice problem}, in which students’ preferences are generated by drawing a complete preference list over schools independently and uniformly at random.

\textsuperscript{9}This holds since the outcome of the DA mechanism in the two-tiered market can be generated by first running DA while ignoring the non-popular schools, and then running DA with the remaining unassigned students and non-popular schools.
3.1.1 Discussion of assumptions

We briefly discuss some of the assumptions and limitations of the model. First, we assume that every school has a single seat. This assumption in fact strengthens our result since the more seats schools have, the more information one learns about the student’s lottery number under STB. For example, when schools have identical capacities, the more seats each school has, the less likely a student that is not accepted at her first choice under STB will be accepted at her second choice.

The assumption that there is a single priority class in each school is a limitation since multiple priority classes often affect students’ assignments. But we believe that adding multiple priority classes to the model would not affect our qualitative insights. This will be evident in the experimental results with NYC data, where we take into consideration the different priority classes that are adopted by different schools. It is worth noting, however, that multiple priority classes is another reason for studying the case of small schools. Indeed, schools, especially popular ones, fill many seats with students who belong to top-priority classes (for example schools may assign many students due to their proximity to the school or due their enrolled siblings). This means that competition for seats is essentially over remaining seats.

Finally, agents are assumed to have preferences that are drawn uniformly at random. One natural extension is that students’ preferences are induced from a utilities that follow a multinomial logit (MNL) model.\footnote{That is, the utility of a student $i$ for school $j$ is $u_{ij} = q_j + \epsilon$ where $q_j$ is a commonly known quality factor of school $j$ and $\epsilon$ is a idiosyncratic shock drawn from a Gumbel distribution.} We believe that similar results will hold true also for this model. We conducted extensive simulations for the Logit model, which suggest that STB stochastically dominates MTB in a subset of schools that have a sufficiently (relatively) large quality factor. See also Appendix F, where we illustrate such a result in a continuum model with few large schools.

3.2 A comparison of STB and MTB in over-demanded and under-demanded markets

This section presents our main theoretical results, which establish the sharp difference between markets with a shortage of seats (over-demanded) and markets with a surplus of seats (under-demanded), representing popular and non-popular schools, respectively. First we need the following definition, which is slight weakening of stochastic dominance.

**Definition 3.1** (almost stochastic dominance). A rank distribution $\mathcal{R}$ almost stochastically dominates a rank distribution $\mathcal{R}'$ if, for any integer $i \in [1,m]$, either $\sum_{j=1}^{i} \mathcal{R}(j) \geq \sum_{j=1}^{i} \mathcal{R}'(j)$ or
\[ \sum_{j=1}^{m} R(j) \leq (\log n)^{1+\epsilon}. \]

An intuitive way to think about almost stochastic dominance is as follows. First remove the bottom \((\log n)^{1+\epsilon}\) students (students who are assigned to their lowest preferences) from \(R'\); let \(R''\) denote the resulting rank distribution. Then, \(R\) stochastically dominates \(R''\) if and only if \(R\) almost stochastically dominates \(R'\).

The result is given for markets with a surplus or a shortage of a single seat and we subsequently we discuss the cases of larger imbalances. The proof is provided in Appendix A.

**Theorem 3.2.** Consider a sequence of random school choice problems with \(n\) students and \(m\) school. When \(m = n - 1\), (over-demanded market):

(i) With high probability, \(R_{STB}\) almost stochastically dominates \(R_{MTB}\),

(ii) For any student \(s\), \(\lim_{n \to \infty} \mathbb{P} [\mu_{STB}(s) \geq 1] \rightarrow 1\), and \(\lim_{n \to \infty} \mathbb{E} [\mu_{MTB}(s)] \rightarrow \infty\),

(iii) \(\lim_{n \to \infty} \frac{\text{Var}[\mu_{STB}]}{\text{Var}[\mu_{MTB}]} = 0\).

When \(m = n + 1\), (under-demanded market):

(i) With high probability, \(R_{STB}\) does not almost stochastically dominate \(R_{MTB}\),

(ii) For any student \(s\), \(\lim_{n \to \infty} \mathbb{P} [\bar{\mu}(s) \geq 1] \rightarrow 0\), and \(\lim_{n \to \infty} \mathbb{E} [\bar{\mu}(s)] \rightarrow 0\),

(iii) \(\lim_{n \to \infty} \frac{\text{Var}[\mu_{STB}]}{\text{Var}[\mu_{MTB}]} = \infty\).

Even when the imbalance is larger than 1, all parts in the main theorem still hold exactly as written, except for part (iii) in the case for under-demanded markets. In an under-demanded market, one should expect the variance under STB to decrease as the surplus of seats grows larger, since students will be assigned to better choices. Although the ratio between the variances may not approach infinity in this case, we show in Appendix A that the variance under STB remains strictly larger than the variance under MTB, even when the surplus of seats is of the same order as the number of students (Theorem A.7).

While part (i) in the over-demanded case shows almost stochastic dominance, we conjecture that \(R_{STB}\) stochastically dominates \(R_{MTB}\). To support this conjecture, we present computational

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11. Our findings hold for any (arbitrary small) constant \(\epsilon > 0\).
12. For a sequence of events \(\{E_n\}_{n \geq 0}\), we say this sequence occurs with high probability (whp) if \(\lim_{n \to \infty} \mathbb{P} [E_n] = 1\).
13. When the surplus of seats is \(\lambda n\), the variance under STB remains larger when \(\lambda\) is a constant, but converges to the variance under MTB when \(\lambda\) grows with \(n\).
experiments in the Online Appendix. Also, in Figure 1 we plot the rank distributions in a market with 1000 students and a shortage of one seat. We emphasize that in the under-demanded case, neither of the rank distributions (almost) stochastically dominate the other.\footnote{In fact, in the under-demanded case, a stronger statement than part (i) is proved, showing that the rank distribution under STB does not stochastically dominate the one under MTB even if we remove the bottom $\frac{n}{\log^2 n}$ students from the rank distribution under MTB. (See Theorem A.2)}

![Figure 1: The cumulative rank distribution under MTB and STB in a random market with 1000 students and 999 seats (averaged over 1000 iterations). The dashed and solid lines indicate the rank distributions under MTB and STB, respectively. The y-axis represents the fraction of students that are assigned to one of their top x ranked schools.](image)

The results state that any given student is involved in “many” Pareto improving pairs when the market is over-demanded, and in almost none when the market is under-demanded. Explicit upper and lower bounds are established in Appendix D.

In Section 4 we confirm that the qualitative predictions from the stylized model hold in the NYC data.

### 3.3 Intuition for results

To gain some intuition for our results it will be useful to compare the student-proposing DA outcomes under STB and MTB in a model with a continuum of students and 2 schools.

Consider a mass of $2N$ students and 2 schools $c_1$ and $c_2$ each with capacity 1. Each student prefers $c_1$ over $c_2$ with probability $1/2$ and schools are indifferent between all students. To implement STB, a lottery number $l_s \sim \text{Uniform}[0, 1]$ is drawn for each student $s$ and schools prefer students with lower lottery numbers over students with higher lottery numbers. To implement MTB, a lottery number $l_{s,c} \sim \text{Uniform}[0, 1]$ is drawn for each pair $(s, c)$ of a student and a school. Hence, school $c$ prefers student $s$ to student $s'$ if $l_{s,c} < l_{s',c}$.

Consider next the two cases: over-demanded and under-demanded markets. Observe that in
an under-demanded market ($N < 1$), due the uniform preferences all students will be assigned to their first choice since no student will be rejected under DA.

Suppose the market is over-demanded ($N > 1$). Under STB a mass 1 of students are assigned to each school, all of whom are assigned to their first choice. To see why, run the student-proposing DA iteratively. In the first round, a mass $N$ of students apply to each school and each school rejects a mass of $N - 1$ students. Since students have uniform preferences, a student is accepted in the first round iff she has a lottery number lower than $1/N$. Therefore, no student that applies in the second round will be accepted to her second choice. Under MTB, however, a student that is rejected from her first choice may still be accepted to her second choice if her (new) lottery number in that school is sufficiently high. So a fraction of students will be assigned to their second choice under MTB.$^{15}$

Observe that the average rank for assigned students under MTB is 1 when the market is under-demanded and larger than 1 when the market is over-demanded. It is worth noting that Ashlagi et al. (2017) establish a similar insight in the discrete setting. They find that in a random marriage market the short side has a much lower average rank than the long side; translated to our setting the average rank of students is significantly better when there is a surplus of seats than when there is a shortage of seats.

The simple example above demonstrates the following properties of the continuum model (whose counterparts are proved in Theorem 3.2 in the discrete model). In an over-demanded market: (i) the rank distribution under STB stochastically dominates the rank distribution under MTB.$^{16}$ (ii) There are no Pareto improving pairs under STB since all students are assigned to their first choice while but there is a positive mass of Pareto improving pairs under MTB (in both schools there are many students who are assigned to their second choice). (iii) Variance of the rank distribution is 0 under STB, but it is a strictly positive number under MTB.$^{17}$

Theorem 3.2 establishes these insights in a discrete setting with small capacities; our model assumes many schools with small capacities and therefore rejections from schools reveal very little information about students lottery numbers.

$^{15}$The exact mass of students who are assigned to their first choice under MTB is $x \equiv N - \sqrt{N^2 - N}$, which can be derived using the “cut-off representation” of stable matchings introduced in Azevedo and Leshno (2016).

$^{16}$Recall that the number of unassigned students is the same under MTB and STB; therefore, for the purpose of comparing the two tie-breaking rules, we can safely ignore the unassigned students in the definitions for rank distribution and variance.

$^{17}$We leave as an exercise for the reader to show that even if capacities are not identical, STB stochastically dominates MTB (even though not all students will obtain their their first choice under STB); Note that under STB, a fraction of the rejected students from the smaller school will obtain a seat in the larger school but not vice versa (this is not true under MTB).
Note that the continuum example does not capture the second part of Theorem 3.2 about under-demanded markets since both tie-breaking rules generate the same assignments. In the discrete market, the same happens when the ratio of the number of seats to the number of students converges to infinity (which is a less interesting case).

3.4 Proof ideas

We provide here further intuition and the main ideas behind the proofs. Unless specified otherwise, we focus on the over-demanded case. The proof leverages a result by Ashlagi et al. (2017), which, transplanted to our setting, says there is an almost unique stable matching under MTB when the market is imbalanced. (More precisely, “almost” any student is assigned to a unique school in all stable matchings.) We use this result to analyze the school-proposing DA rather than the student-proposing DA, which turns out to follow a simpler stochastic process for our purpose. The proofs for all parts of the theorem more or less follow the same pattern: for each part, first we show it suffices to prove the claim for the school-proposing DA instead of the student-proposing DA. Second, we simplify the stochastic process corresponding to the school-proposing DA using coupling techniques. In the last step, we prove the claim for the simplified process.

The main idea to establish almost stochastic dominance (part (i)) is showing that a much smaller fraction of students are assigned to one of their top choices under MTB than under STB. In particular, we show that there exists some rank \( r \) for which, with high probability, all but \( (\log n)^{1+\epsilon} \) students are assigned to a rank better than \( r \) under STB, whereas at most half of the students are assigned to a rank better than \( r \) under MTB. Furthermore, roughly half of the students are assigned to their top choice under STB.

For the results about Pareto improving pairs (part (ii)), we first provide a intuition through a simple back-of-the-envelope calculation using a result of Ashlagi et al. (2017), who studied the average rank of agents on one side in a random two-sided matching market (for the proof, however, we need to take a quite different approach by analyzing the school-proposing DA). Let \( \pi \) represent a preference profile. If a student \( s \) is assigned to her average rank, namely \( z + 1 \), in \( \mu_\pi \), then there are \( z \) students that can potentially form a Pareto improving pair with \( s \). Note that for every other student \( s' \),

\[
P[\mu_\pi(s) \succ s', \mu_\pi(s')] = \frac{z}{n}
\]

\(^{18}\)That is, by coupling the stochastic process to a simpler stochastic process with a similar behavior.
holds if one assumes that the preference list of \( s' \), \( \pi(s') \), is selected independently and uniformly at random after the match, conditioned on having \( \mu_\pi(s') \) in the position \( z \) of \( \pi(s') \). Under this simplifying assumption (which does not hold in general), the chance that \( s \) cannot find a Pareto improving pair is roughly \( (1 - \frac{z}{n})^z \). By Ashlagi et al. (2017), in an over-demanded market, the average rank of students, \( z \), is almost \( \frac{n}{\log n} \), implying that this chance converges to 0. Similarly, in an under-demanded market the average rank of students is close to \( \log n \), implying that this chance converges to 1.

While the above calculation is straightforward, we cannot apply their finding (regarding the average rank) since the independence assumption fails to hold due to the correlations in the stochastic process corresponding to the DA algorithm (and independence is crucial for the above calculations). Instead, the proof analyzes the school-proposing DA and computes the number of Pareto improving pairs in the school-optimal matching. Then, using the fact that there is an almost unique stable matching (Ashlagi et al. (2017)), we show that the number of pairs remains almost the same in the student-optimal matching.

Next we discuss the high level idea behind proving the result about the variance of the rank under MTB in the over-demanded case (part (iii)), i.e., showing that the variance is “large”. Again, we apply the small core results to show that the variance of student rank is almost the same under the student-optimal and school-optimal stable matchings. Then, we show that each student, with high probability, receives at least \( d \approx \frac{\log n}{2} \) proposals in the school-optimal assignment. Finally, we use the fact that the variance of the first order statistic of \( d \) i.i.d draws from the uniform distribution over the interval \([1, n]\) is of the order of \( \frac{n^2}{d^2} \) to complete the argument.

4 NYC school choice

Every year in New York City, approximately 90,000 students are assigned to roughly 700 public high school programs through a centralized matching mechanism. Until 2010 the matching process included three rounds of assignments; we focus on the main (second) round, in which about 80,000 students were assigned to schools using the deferred acceptance algorithm.\(^{19}\)

Each student who participated in this round submitted a rank-ordered list that included at most 12 schools. Different programs assigned different priorities to students, and ties were broken exogenously using the STB rule. In particular, every student was assigned a single lottery number,\(^{19}\)

\(^{19}\)The first round assigns students only to specialized exam schools.
and whenever a school had to reject a subset of students from the set of students with the lowest priority, the lottery numbers were used to break the ties.

For our analysis we consider the main round during the 2007-2008 school year, in which 79,694 students and 670 programs took part. In our experiments we will run DA with STB and MTB using the rankings of students and the real priorities schools assign to students.\footnote{Since preferences lists are bounded, the NYC mechanism is not strategyproof. For simplicity, however, we assume that students’ observed preferences are sincere.} In particular, if student $s$ belongs to a lower priority class than student $s'$ in school $c$, then $c$ always prefers $s$ over $s'$. So, as done in practice, schools use lottery numbers generated by STB and MTB only to break ties between students who belong to the same priority class.

4.1 A measure for school popularity

Recall that the popularity of a school $c$ is defined to the ratio of the expected number of students for which school $c$ is their top choice to the capacity of the school that rank the school (1). For the simple exercise in this section, we replace numerator with the actual number of students who ranked school $c$ first in the data.

Formally, let $p_1(c)$ denote the (empirical) number of students who list $c$ as their top choice and recall that $q_c$ is the capacity of school $c$. The popularity of a school for this section is redefined to be

$$\alpha_c = \frac{p_1(c)}{q_c}.$$ 

It is worth noting that when students’ preferences are drawn from a multinomial logit model, this measure is an unbiased estimator for the “weight” of a school normalized by its capacity (see Appendix F for further details). A popularity threshold $\alpha$ will determine a set $P_\alpha$ of “popular” schools, containing all schools with a popularity of at least $\alpha$, i.e., $P_\alpha = \{ c : c \in C, \alpha_c > \alpha \}$. Note that schools with a popularity higher than 1 will be filled under the deferred acceptance algorithm regardless of the tie-breaking rule used (and other schools may or may not be filled). Figure 2 reports the distribution of schools’ popularity in the NYC data.

4.2 Stochastic dominance

The rank distribution over a set of schools $C'$ describes, for each rank $i$, how many students who were assigned to a school in $C'$ were assigned to their $i$-th rank. Formally, the rank distribution of a set of schools $C' \subseteq C$ in a matching $\mu$ is a function $\mathcal{R}_{\mu}^{C'} : [1, m] \rightarrow [0, n]$ where $\mathcal{R}_{\mu}^{C'}(i)$ denotes the...
number of students in $\mu(C')$ who are assigned to their $i$-th choice. When $\mu$ is generated under MTB or STB, we simply denote the rank distribution by $R_{MTB}^{C'}$ or $R_{STB}^{C'}$, respectively. The cumulative rank distribution determines for each $i$ the number (or when specified, percentage) of students who are assigned to one of the top $i$ choices on their list.

We run 50 iterations of the deferred acceptance algorithm under STB and MTB, and for each iteration, we calculate the cumulative rank distributions in popular and non-popular schools for a range of popularity thresholds. We emphasize that we include both popular and non-popular schools in each iteration, but report the average cumulative rank distribution for the sets of popular and non-popular schools separately for certain popularity thresholds. Figure 3 reports the average of cumulative rank distributions of the set of popular schools $P_\alpha$, for $\alpha \in \{1, 1.5, 2\}$. Observe that for each popularity threshold, the rank distribution under STB stochastically dominates the rank distribution under MTB. Naturally, increasing the popularity threshold increases the gap between the two rank distributions.

Figure 4 reports the cumulative rank distributions (under STB and MTB) of non-popular schools with a popularity of at most $\alpha \in \{0.75, 1, 2\}$. Since the number of students in non-popular schools may differ under STB and MTB, we normalize by the total number of students rather than by the number of students assigned to non-popular schools. Observe that for each $\alpha$, neither rank distribution stochastically dominates the other. (Even though for each popularity threshold $\alpha$ the plots for non-popular schools seem to be close to each other, the total number of students assigned to non-popular schools is much larger than the number of students assigned to popular schools.\(^{21}\))

Note that in every school $c$, the number of students assigned to $c$ that ranked it as their first choice is larger under STB than under MTB.\(^{22}\) So for any popularity level, the rank distributions

\(^{21}\)For example, for $\alpha = 2$, 8,500 and 64,000 students are assigned to popular and non-popular schools, respectively.

\(^{22}\)This is intuitive since a student who is tentatively assigned to her first choice is less likely to be rejected under STB than under MTB.
Figure 3: The cumulative rank distributions under MTB and STB for popular schools with different popularity thresholds $\alpha$ (schools with popularity above $\alpha$ are popular). The solid and dashed lines indicate the rank distributions under MTB and STB, respectively. The x-axis represents the rank and the y-axis the fraction of students in popular schools that are assigned at most their x rank.

of non-popular schools cross each other at a rank of at least 2. Moreover, the higher the popularity threshold, the harsher the competition in non-popular schools, hence the larger the rank at which the distributions cross.

Figure 4: The average cumulative rank distributions under MTB and STB of non-popular schools with different popularity thresholds (schools with popularity below $\alpha$ are considered non-popular). The solid and dashed lines indicate the rank distributions under MTB and STB, respectively. The x-axis represents the rank and the y-axis the normalized fraction of students in non-popular schools that are assigned at most their x rank.
4.2.1 School-by-school comparison

Our experiments further reveal that when the popularity $\alpha_c$ of a single school $c$ is high enough, then $R^c_{STB}$ stochastically dominates $R^c_{MTB}$. For all schools with a popularity of at least 1.51 (126 schools) $R^c_{STB}$ either stochastically dominates $R^c_{MTB}$ or does if we increase $R^c_{STB}(1)$ by only one.

Figure 5 reports, for a variety of popularity ranges, the following two measures: (i) the percentage of schools for which $R^c_{STB}$ stochastically dominates $R^c_{MTB}$, and (ii) the percentage of schools for which $R^c_{STB}$ stochastically dominates $R^c_{MTB}$ if we increase $R^c_{STB}(1)$ by one.

Figure 5: Stochastic dominance in each school. The x-axis is the popularity range. The blue (left-hand) bar in each range is the percentage of schools for which the rank distribution under STB in that school stochastically dominates its counterpart under MTB. The red (right-hand) bar stands for the same percentage, but assuming $R^c_{STB}(1)$ is increased by 1 for all schools $c$.

For schools with low popularity, the rank distribution under MTB often almost dominates the rank distribution under STB. More students, on average, are assigned to their first choice under STB than under MTB. However, by increasing $R^c_{MTB}(1)$ by only 1, in 94% out of overall 217 unfilled schools, $R^c_{MTB}$ stochastically dominates $R^c_{STB}$.

4.3 Pareto improving pairs

We provide several statistics for Pareto improving (PI) pairs. Each statistic is calculated by taking an average over 50 iterations of the DA algorithm under MTB. We say that a PI pair is popular if both its students are assigned to popular schools. Define the number of PI students in popular

---

23 For a singleton $C' = \{c\}$ we simply write $R^c_{\mu}$ instead of $R^{C'}_{\mu}$.
24 In fact, this holds even if $R^c_{STB}(1)$ is increased only by 0.5. It is remarkable that stochastic dominance holds strictly in 85% of these 126 schools.
25 The experiments further show that in 82%, 49% and 35% of schools with popularity levels of $[0, 0.5]$, $(0.5, 0.75]$, and $(0.75, 1]$, the shifted $R^c_{MTB}$ stochastically dominates $R^c_{STB}$.
schools to be the number of students that are assigned to popular schools and involved in at least one popular PI pair. Similarly we define PI students in non-popular schools.

Figure 6a reports the empirical probability of a student being involved in a PI pair for various α’s between 1 and 2.5. The following three cases are considered for which we describe the reported statistic: (i) Overall: the number of PI students divided by the total number of assigned students, (ii) In popular schools: the number of PI students in popular schools divided by the total number of students assigned to popular schools, and (iii) In non-popular schools: the number of PI students in non-popular schools divided by the total number of students assigned to non-popular schools. There exists a clear gap between the fraction of PI students in popular and non-popular schools. At α = 1, these fractions are 20% and 1%, respectively. As the popularity threshold α increases, this gap naturally shrinks (at α = 2.5, these fractions change to 16% and 7%, respectively), since as α increases, so does the number of non-popular schools.

Figure 6b plots the average degree, which, roughly speaking, is the average number of PI pairs that a student is involved in. The average degree is computed for the following 5 cases: (i) Overall: twice the total number of PI pairs divided by total number of students, (ii) and (iii) In (non-)popular schools: twice the number of PI pairs in (non-)popular schools divided by the total number of students assigned to (non-)popular schools, (iv) and (v) Crossing (non-)popular schools: the number of PI pairs with a student in a popular school and a student in non-popular school divided by the total number of students in (non-)popular schools.

Table 1 reports similar but unnormalized statistics. It shows the average number of PI students and the average number of PI pairs in popular and non-popular schools, reported for a range of

![Graphs](image-url)
### Table 1: The total number of students, the average number of students in PI pairs, and the average number of PI pairs in popular and non-popular schools for a variety of popularity thresholds.

<table>
<thead>
<tr>
<th>α</th>
<th>#students</th>
<th>#students in PI pairs</th>
<th>#PI pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>popular</td>
<td>27019</td>
<td>5431.2</td>
</tr>
<tr>
<td></td>
<td>non-popular</td>
<td>45663.4</td>
<td>455.8</td>
</tr>
<tr>
<td>1.25</td>
<td>popular</td>
<td>19929</td>
<td>4297.7</td>
</tr>
<tr>
<td></td>
<td>non-popular</td>
<td>52753.4</td>
<td>1021.9</td>
</tr>
<tr>
<td>1.5</td>
<td>popular</td>
<td>14886</td>
<td>3536.8</td>
</tr>
<tr>
<td></td>
<td>non-popular</td>
<td>57796.4</td>
<td>1524.7</td>
</tr>
<tr>
<td>2</td>
<td>popular</td>
<td>8500</td>
<td>1788.1</td>
</tr>
<tr>
<td></td>
<td>non-popular</td>
<td>64182.4</td>
<td>3159.1</td>
</tr>
<tr>
<td>2.5</td>
<td>popular</td>
<td>2443</td>
<td>722.5</td>
</tr>
<tr>
<td></td>
<td>non-popular</td>
<td>68249.4</td>
<td>4938.9</td>
</tr>
</tbody>
</table>

popularity thresholds. Note that these quantities are decreasing in popular schools and increasing in non-popular schools with the popularity threshold $\alpha$.

### 4.4 Variance of rank

Table 2 reports the variance of rank in popular schools. That is the average, over all students assigned to popular schools, of the squares of the differences between the rank of a student and the average rank in popular schools. This definition adapts the notion of variance in Section 3, which was defined for a single-tiered market (where schools are either all popular or all non-popular). Similarly, Table 3 presents variance of the rank distribution in non-popular schools. Consistent with Theorem 3.2, we observe that MTB results in a higher variance than STB in popular schools, but a lower variance in non-popular schools.

### Table 2: Popular schools. Variance of the rank and average rank of students assigned to popular schools.

<table>
<thead>
<tr>
<th>α</th>
<th>Variance</th>
<th>Mean</th>
<th>Variance</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.10</td>
<td>1.83</td>
<td>2.99</td>
<td>2.21</td>
</tr>
<tr>
<td>1.5</td>
<td>1.47</td>
<td>1.65</td>
<td>2.87</td>
<td>2.18</td>
</tr>
<tr>
<td>2</td>
<td>1.27</td>
<td>1.59</td>
<td>2.99</td>
<td>2.19</td>
</tr>
<tr>
<td>2.5</td>
<td>1.09</td>
<td>1.51</td>
<td>2.81</td>
<td>2.21</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Variance</td>
<td>Mean</td>
<td>Variance</td>
<td>Mean</td>
</tr>
<tr>
<td>--------</td>
<td>----------</td>
<td>------</td>
<td>----------</td>
<td>------</td>
</tr>
<tr>
<td>1</td>
<td>4.22</td>
<td>2.52</td>
<td>3.69</td>
<td>2.50</td>
</tr>
<tr>
<td>1.5</td>
<td>3.90</td>
<td>2.41</td>
<td>3.58</td>
<td>2.44</td>
</tr>
<tr>
<td>2</td>
<td>3.76</td>
<td>2.34</td>
<td>3.52</td>
<td>2.42</td>
</tr>
<tr>
<td>2.5</td>
<td>3.64</td>
<td>2.31</td>
<td>3.47</td>
<td>2.40</td>
</tr>
</tbody>
</table>

Table 3: Non-popular schools. Variance of the rank and average rank of students assigned to non-popular schools.

5 Discussion

This paper revisits the impact of two common tie-breaking rules on students’ assignments in school choice. Splitting the market into popular and non-popular schools proved useful in explaining the source of the tradeoffs between tie-breaking rules; in non-popular schools (under-demanded market), neither random assignment rule stochastically dominates the other, and a separate lottery for each school results in lower variance of rank. Importantly, in popular schools (over-demanded market), there is essentially no tradeoff as a common lottery proves to be the superior choice: STB stochastically dominates MTB, the variance of rank is surprisingly smaller under STB, and MTB creates many Pareto improving pairs.

Our theoretical predictions have been established in a stylized model and we presented empirical findings using New York City choice data that supports these insights by restricting attention, separately, to popular and non-popular schools.\(^{26}\) (See also the Online Appendix for simulations that test the robustness of our theoretical findings with regard to correlation in preferences, imbalance in the market, large capacities, and short preference lists.) It is important to note that in practice, determining which schools are popular is not straightforward and an interesting research direction is to define a well-grounded empirical measure for popularity.

We believe that even when students’ preferences are generated from a multinomial-logit model, a stochastic dominance relation holds in a sub-market of popular schools\(^{27}\), and furthermore, that stochastic dominance holds in each popular school separately. Appendix F provides an analysis of special cases with 2 and 3 schools in a model with a continuum of students.

This study provides another rationale for selecting a single lottery for breaking ties (see also

\(^{26}\)Thus reducing much of the ambiguity and tradeoff documented in previous studies Abdulkadiroğlu et al. (2009); De Haan et al. (2015).

\(^{27}\)A school is popular if the measure of students that rank it as first choice is larger than its capacity.
When fairness, however, is a major consideration, a hybrid tie-breaking rule (HTB) may be attractive. Under such a tie-breaking rule all popular schools will use a single lottery and each non-popular school will use a separate lottery. When schools are perfectly tiered, HTB results in a Pareto improvement over MTB and even obtains the same rank distribution as STB in popular schools. However, typically we would not observe a perfectly tiered market and the rank distribution under HTB is likely to lie in between the rank distributions under MTB and STB. We observe this in simulations and provide some intuition in Appendix G. These initial findings motivate further applied work to better understand the relation between priorities and students’ assignments.

References


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## A Roadmap

We break Theorem 3.2 into several smaller (sub-)theorems and prove each one separately. This is done in sections A.1, A.2, and A.3, which state and discuss the theorems about stochastic dominance, Pareto improving pairs, and variance, respectively. The theorems stated in these sections are then proved separately in the later sections. Section B contains the proofs for stochastic dominance. Section C contains some preliminary results that will be used in Sections D and E, which contain the proofs about Pareto improving pairs and variance, respectively.

### A.1 Stochastic dominance

We focus on the over-demanded case in the next theorem, and on the under-demanded market in the theorem that comes after that.

**Theorem A.1.** Consider a sequence of random school choice problems with \( n \) students and \( m \) schools where \( n = m + 1 \). Then, with high probability, \( \mathcal{R}_{STB} \) almost stochastically dominates \( \mathcal{R}_{MTB} \).\(^{28}\)

\(^{28}\)For a sequence of events \( \{E_n\}_{n \geq 0} \), we say this sequence occurs with high probability (whp) if \( \lim_{n \to \infty} P[E_n] = 1 \).
Theorem A.2. Consider a sequence of random school choice problems with \( n \) students and \( m \) schools with \( n = m - 1 \). Then, with very high probability (wvhp), \( R_{STB} \) does not almost stochastically dominate \( R_{MTB} \).\(^{29}\) Furthermore, wvhp \( R_{STB} \) does not stochastically dominate \( R_{MTB}[k] \) for any \( k = o(n/\ln^2 n) \), where \( R_{MTB}[k] \) is the rank distribution resulting from the removal of the bottom \( k \) students from \( R_{MTB} \).\(^{30}\)

The proofs for both theorems are given in Section B. The same proofs imply that the theorem holds when the imbalance is larger than one. (i.e. the theorems hold for \( n > m \) and \( m < n \) respectively in the over-demanded and under-demanded cases)

A.2 Pareto improving pairs

In the next theorem, we show that deferred acceptance paired with MTB generates many Pareto improving pairs when there is a shortage of seats, and very few Pareto improving pairs when there is a surplus of seats. The proof is given in Section D.

Theorem A.3. Consider a sequence of random school choice problems with \( n \) students and \( m \) schools, let \( \mu = \mu_{MTB} \), and let \( s \) be an arbitrary student.

1. If \( n > m \),

\[
\lim_{n \to \infty} \Pr[\mu(s) \geq 1] \to 1, \quad \lim_{n \to \infty} \mathbb{E}[\mu(s)] \to \infty.
\]

2. If \( n < m \),

\[
\lim_{n \to \infty} \Pr[\mu(s) \geq 1] \to 0, \quad \lim_{n \to \infty} \mathbb{E}[\mu(s)] \to 0.
\]

A.3 Variance

The next lemma says that, under either tie-breaking rule, if students’ preference are i.i.d, then the expected social inequity is equal to the expected variance of the rank of an arbitrarily fixed student. The proof appears in Appendix E.1.

---

\(^{29}\)For a sequence of events \( \{E_n\}_{n \geq 0} \), we say that the sequence occurs with very high probability (wvhp) if

\[
\lim_{n \to \infty} \frac{1 - \Pr[E_n]}{\exp\left(-\frac{1}{(\log n)^{2/4}}\right)} = 0.
\]

\(^{30}\)For any two functions \( f, g : \mathbb{Z}_+ \to \mathbb{R}_+ \) we adopt the notation \( g = o(f) \) when \( \lim_{n \to \infty} \frac{g(n)}{f(n)} = 0 \), \( g = O(f) \) when \( f \neq o(g) \), \( g = \Theta(f) \) when \( f = O(g) \) and \( g = O(f) \), and finally, \( g = \Omega(f) \) when \( f = O(g) \).
Lemma A.4. For any student $s \in S$

$$E_{\pi(s') : s' \in S, s' \neq s} \left[ \text{Var}[r_s] \right] = E_{\pi} \left[ Si(\mu_{\pi}) \right] ,$$

where expectation on the left-hand side is taken over all students’ preferences except $s$, and expectation on the right-hand side is taken over all students’ and schools’ preferences with schools’ preferences generated by either the STB or the MTB rule.

The lemma further shows that the expected variance of rank of a student $s$ is equal to the expected variance of rank of a student $s'$, for any $s, s' \in S$. Therefore, we sometimes refer to this notion as the variance of student rank, without specifying $s$.

The next theorem shows that the imbalance in the market determines whether MTB or STB results in a larger variance.

Theorem A.5. Consider a sequence of random school choice problems with $n$ students and $m$ schools.

1. If $n = m$ or $n = m - 1$, then $\lim_{n \to \infty} \frac{\text{Var}[\mu_{\text{STB}}]}{\text{Var}[\mu_{\text{MTB}}]} = \infty$.

2. If $n = m + 1$, then $\lim_{n \to \infty} \frac{\text{Var}[\mu_{\text{STB}}]}{\text{Var}[\mu_{\text{MTB}}]} = 0.31$

Theorem A.5 follows directly from the next result, which quantifies the social inequities in our model.

Lemma A.6. Consider a sequence of random school choice problems with $n$ students and $m$ schools.

1. If $n = m + 1$, the expected social inequity under MTB is $\Omega(\frac{n^2 \log n}{n})$ and under STB is $\Theta(n)$.

2. If $n = m$, the expected social inequity under MTB is $O(\log^4 n)$, and under STB is $\Theta(n)$.

3. If $n = m - 1$, the expected social inequity under MTB is $O(\log^2 n)$ and under STB is $\Theta(n)$.

The proof for Lemma A.6 is given in Appendix E.

We briefly discuss how our results on variance are affected by varying the size of the imbalance, length of preference lists, and correlation in preferences. Before this, we note that our empirical findings using NYC data support the theoretical findings (see Section 4).

In an over-demanded market with $m$ schools, for any $n > m + 1$, the variance under STB remains the same as when $n = m + 1$; however, the variance under MTB remains at least as high as

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Footnote 31: Expectations are taken over students’ preferences and the tie-breaking lotteries.
in the case \( n = m + 1 \) due to the harsher competition (this is implied by the proof of the first part of Lemma A.6). Thus, part 2 of Theorem A.5 always holds as long as \( n > m \). In an under-demanded market with \( n \) students, one should expect the variance under STB to decrease as the surplus of seats grows larger, since an increasing number of students will be assigned to their top choices. Nevertheless, we show in the next theorem that the variance under STB remains strictly larger than the variance under MTB, even when the surplus of seats is of the same order as the number of students. (The proof is given in Appendix E.3.)

**Theorem A.7.** Suppose \( m = n + \lambda n \) for any positive \( \lambda \leq 0.01 \). Then, \( \lim_{n \to \infty} \frac{\mathbb{E}[\mathcal{S}(\mu_{\text{STB}})]}{\mathbb{E}[\mathcal{S}(\mu_{\text{MTB}})]} > 1 \), where the expectations are taken over preferences and the tie-breaking rules.

We conjecture that this theorem holds for any positive fixed \( \lambda \). To avoid unnecessary technicalities, we only prove it for \( \lambda \leq 0.01 \). We quantify the ratio between social inequities for different values of \( \lambda \) in our computational experiments in the Online Appendix. (For instance, we see that this ratio is around 3 for \( \lambda = 0.1 \).)

In another set of experiments (see the Online Appendix) we show that the gap between the variances persists even when the preference lists are short. To test how correlation in preferences affects our results, we conduct experiments (see the Online Appendix), in which students’ preferences are drawn independently from a discrete choice model (one may think of these preferences as drawn proportionally with respect to publicly known schools’ qualities). We see that in an under-demanded market, the variance under STB is larger than the variance under MTB, unless students’ preferences are extremely correlated, in which case the rank distributions will become similar.

**B Proofs for Section A.1**

We will use the following definitions in the proofs. Denote by \( \mathcal{A} = S \cup C \) the set of schools and students. We often refer to a school or a student by an *agent*. Consider a matching \( \gamma \). Let \( \gamma(x) \) be the agent to which \( x \) is matched to and for any subset of agents \( A \in \mathcal{A} \), let \( \gamma(A) \) be the set of agents matched to agents in \( A \). Therefore, \( \gamma(C) \) is the set of students who are assigned under \( \gamma \).

For any student \( s \), \( \gamma^#(s) \) denotes the rank of school \( \gamma(s) \) for \( s \), and similar notions are used for schools. Denote the average rank of students who are assigned under \( \gamma \) by \( \mathcal{A}r(\gamma) = \frac{1}{|\gamma(C)|} \sum_{s \in \gamma(C)} \gamma^#(s) \). When it is clear from the context we will simply write \( r_s \) for \( \gamma^#(s) \), and \( r \) for \( \mathcal{A}r(\gamma) \).
Denote by $\mu_\pi$ and $\eta_\pi$ the student-optimal and the school-optimal stable matching for a preference profile $\pi$, respectively. Finally, given students’ preferences, let $\mu_{STB}$ and $\mu_{MTB}$ the random variables that denote the student-optimal stable matchings under STB and MTB, respectively.

For any rank distribution $\mathcal{R}$, let $\mathcal{R}^+$ denote the corresponding cumulative rank distribution, i.e. $\mathcal{R}^+(k) = \sum_{i=1}^{k} \mathcal{R}(i)$ is the number of students who are assigned to one of their top $k$ choices under $\mathcal{R}$.

### B.1 Proof of Theorem A.1

We need the following lemmas before proving this theorem.

#### B.1.1 Computing $\mathcal{R}_{MTB}$

**Lemma B.1.** When $n = m + 1$, wvhp there at most $\frac{3n \log n}{t}$ students who receive more than $t$ proposals in the school-proposing DA.

**Proof.** The proof is a direct consequence of the following result by Pittel (1989): When $n = m + 1$, the school-proposing DA takes no more than $3n \log n$ proposals, wvhp.

**Definition B.2.** Let $t = 3\theta \log m$, where $\theta > 1$ is a large constant that we set later.

**Proposition B.3.** At most $\frac{n}{\theta}$ students receive more than $t$ offers in school-proposing DA wvhp. This is a direct consequence of B.1.

**Lemma B.4.** Suppose a student $s$ receive $t$ proposals in the school-proposing DA such that $1 \leq t \leq \bar{t}$. Then, for any constant $\alpha > 2$

$$\mathbb{P}\left[\eta^\#(s) > \frac{m}{\alpha t}\right] \geq \exp\left(-\frac{2m}{\alpha(m-t)}\right)$$

**Proof.** By the principle of deferred decisions, we can assume that students rank proposals upon receiving them. Upon receiving each proposal, the student assigns a (yet unassigned) rank to the school who offers the proposal. The probability that the first school is ranked worse than $\frac{m}{\alpha t}$ is $1 - \frac{m/\alpha t}{m}$. In general, the probability that $i$-th school who proposes to $s$ gets ranked worse than $\frac{m}{\alpha t}$
is \(1 - \frac{m/\alpha t}{m-t}\). Thus, we have

\[
\Pr[\eta^\#(s) > \frac{m}{\alpha t}] = \prod_{i=1}^{t} \left(1 - \frac{1}{\alpha t(1-i/m)}\right) \\
\geq \exp\left(-\sum_{i=1}^{t} \frac{2}{\alpha t(1-i/m)}\right) \geq \exp\left(-\frac{2m}{\alpha(m-t)}\right)
\]

where in the first inequality we have used the fact that \(1 - x \geq e^{-2x}\) for any \(x < 1/2\).

**Lemma B.5.** For any constant \(\alpha > 4\), \(\mathcal{R}_{MTB}(\lfloor \frac{m\alpha}{\alpha t} \rfloor) \leq 0.4n + o(n)\), wvhp.

**Proof.** To compute \(\mathcal{R}_{MTB}\), first we run the school-proposing DA and prove the lemma statement for the school-optimal matching. Then, using the fact that almost every student has the same match in the student-optimal matching Ashlagi et al. (2017), we establish the lemma statement (which holds for the student-optimal matching).

For any student \(s\), let \(x_s\) be a binary random variable that is 1 iff \(\eta^\#(s) > \frac{m}{\alpha t}\). Also, let \(S'\) denote the subset of students who received at least one but no more than \(t\) offers. For any \(s \in S'\), Lemma B.4 implies

\[
\Pr[\eta^\#(s) > \frac{m}{\alpha t}] \geq e^{-1/2},
\]

since \(\alpha > 4\). This means \(\mathbb{E} \left[\sum_{s \in S'} x_s\right] \geq e^{-1/2} \cdot |S'| \geq 0.606m\). Now, applying a standard Chernoff concentration bound implies that wvhp \(\sum_{s \in S'} x_s \geq 0.6n\). This fact, together with the fact that \(|S\setminus S'| = o(n)\) (which holds by B.3, there are at most \(0.4n + o(n)\) students \(s\) for whom \(\eta^\#(s) \leq \frac{m}{\alpha t}\).

It is straightforward to imply a similar result for the student-optimal matching, \(\mu\). Note that the number of students who have different matches in \(\mu\) and \(\eta\) is at most \(n/\sqrt{\log n}\), wvhp Ashlagi et al. (2017). Consequently, there are at most \(0.4n + o(n) + n/\sqrt{\log n}\) students \(s\) for whom \(\mu^\#(s) \leq \frac{m}{\alpha t}\). □

**B.1.2 Computing \(\mathcal{R}_{STB}\)**

**Lemma B.6.** Suppose student \(s \in S\) has priority number \(n-x\). Then, the probability that \(s\) is not assigned to one of her top \(i\) choices is at most \((1 - \frac{x}{n})^i\)

**Proof.** The probability that \(s\) is not assigned to his top choice is \(1 - \frac{x}{n}\). The probability that \(s\) is not assigned to his second top choice is \((1 - \frac{x}{n})(1 - \frac{x}{n-1})\), which is at most \((1 - \frac{x}{n})^2\). Similarly, it is straightforward to see that the probability that \(s\) is not assigned to her \(i\)-th top choice is at most \((1 - \frac{x}{n})^i\). □
Lemma B.7. A student \( s \) who has priority number \( n - x \) is assigned to one her top \( \frac{2n \log(n)}{x} \) choices with probability at least \( 1 - \frac{1}{n} \).

Proof. Set \( i = \frac{2n \log(n)}{x} \) and apply Lemma B.6. Noting that \( (1 - \frac{x}{n})^i \leq e^{-\frac{x}{n}} \) proves the claim. \( \square \)

Lemma B.8. For any positive constant \( \alpha > 1 \), \( R_{\text{STB}}^+ \left( \left\lfloor \frac{m}{\alpha t} \right\rfloor \right) \geq n - O(\log n \cdot \log \log n) \)

Proof. Define \( x = \frac{\alpha x n \log n}{m} \). Let \( S' \) be the subset of students who have priority numbers better than \( n - x \). First, we apply Lemma B.7 on each student in \( S' \). Lemma B.7 implies that a student with priority number \( n - x \) or better, gets assigned to one of her top \( m \alpha t \) choices with probability at least \( 1 - \frac{1}{n} \). Taking a union bound over all students with priority number no worse than \( n - x \), implies that at least \( n - x \) students are assigned to one of their top \( m \alpha t \) choices, with probability at least \( 1 - \frac{1}{n} \). This means \( R_{\text{STB}}^+ \left( \frac{m}{\alpha t} \right) \geq n - x = n - O(\log^2 n) \) holds with probability at least \( 1 - 1/n \). To prove the sharper bound in the lemma statement, we need to take the students in \( S \setminus S' \) into account.

Let \( S'' \subset S \setminus S' \) denote the subset of students who have priority number between \( n - \beta t \cdot \log \log n \) and \( n - x \), where \( \beta = \frac{2\alpha^2 t}{\log n} \). Lemma B.6 implies that for any \( s \in S'' \),

\[
P[\mu^\#(s) > \frac{m}{\alpha t}] \leq \exp\left(-\frac{\beta}{\alpha} \cdot \log \log n\right).
\]

Having \( \beta = \frac{2\alpha^2 t}{\log n} \) implies

\[
P[\mu^\#(s) > \frac{m}{\alpha t}] \leq (\log n)^{-\frac{2\alpha^2 t}{\log n}}.
\]

Now, we use the above bound to write a union bound over all \( s \in S'' \):

\[
P\left[ \max_{s \in S''} \mu^\#(s) > \frac{m}{\alpha t} \right] \leq |S''| \cdot (\log n)^{-\frac{2\alpha^2 t}{\log n}} \leq O(1/\log^4 n),
\]

where in the last inequality we have used the fact that \( x = \frac{\alpha^2 n \log n}{m} = O(\log^2 n) \).\(^{32}\)

Taking a union bound over the students in \( S', S'' \) implies that

\[
P\left[ \max_{s \in S' \cup S''} \mu^\#(s) > \frac{m}{\alpha t} \right] \leq 1/n + O(1/\log^4 n).
\]

\(^{32}\)The convergence rate \( O(1/\log^4 n) \) can be easily improved to \( O(1/n) \) in the expense of changing \( \log n \cdot \log \log n \) to \( (\log n)^{1+\epsilon} \) in the lemma statement. Note that we already proved this fact for \( \epsilon = 1 \) in the current proof.

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Consequently, $\mathcal{R}_{\text{STB}}^+ \left( \left\lfloor \frac{m}{1.5t} \right\rfloor \right) \geq n - |S\setminus(S' \cup S'')|$ holds whp. To finish the proof, just note that $|S\setminus(S' \cup S'')| = \beta \cdot \log \log n = O(\log n \cdot \log \log n)$.

Lemma B.9. Let $\epsilon > 0$ be an arbitrary constant. Then, whp, at least $(1-\epsilon)n/2$ students are assigned to their top choice in STB.

Proof. Consider the following implementation of STB. Student with the highest priority number chooses her favorite school, then the student with the next highest priority number chooses, and so on. We call the student with the $i$-th highest priority number student $i$. Let $X_i$ be a binary random variable which is 1 iff student $i$ is assigned to her first choice, and let $X = \sum_{i=1}^{n} X_i$. Observe that $\Pr[X_i = 1] = (i - 1)/n$. Therefore, $E[X] = \sum_{i=1}^{n} \frac{i-1}{n} = \frac{n-1}{2}$. A standard application of Chernoff bound then implies that for any $\epsilon > 0$, we have

$$\Pr[X < (1 - \epsilon) \cdot E[X]] \leq \exp \left( -\frac{\epsilon^2 E[X]}{2} \right).$$

This proves the claim.

Now, we are ready to prove Theorem A.1.

Proof of Theorem A.1. Lemma B.5 says that $\mathcal{R}_{\text{MTB}}^+ \left( \left\lfloor \frac{m}{1.5t} \right\rfloor \right) \leq 0.4n + o(n)$ whp. This and Lemma B.9 together imply that $\mathcal{R}_{\text{MTB}}^+ \left( \left\lfloor \frac{m}{1.5t} \right\rfloor \right) < \mathcal{R}_{\text{STB}}^+ (1)$ whp. On the other hand, Lemma B.8 says that $\mathcal{R}_{\text{STB}}^+ \left( \left\lfloor \frac{m}{1.5t} \right\rfloor \right) \geq n - (\log n)^{1+\epsilon}$ with high probability. The two latter facts, by definition, imply that $\mathcal{R}_{\text{STB}}$ almost stochastically dominates $\mathcal{R}_{\text{MTB}}$.

B.2 Proof of Theorem A.2

Lemma B.10. When $n = m - 1$, at least $\frac{n(1-\epsilon)}{46\log^2 n}$ students are not assigned to one of their top $3\log^2 n$ choices in STB, whp, for any $\epsilon > 0$.

Proof. Let $x = 3\log^2 n$ and $t = \frac{n}{4\log^2 n}$. Also, let $X_s$ be a binary random variable which is 1 iff student $s$ is not assigned to one of her top $x$ choices. By the principle of deferred decisions, we can assume that $\{X_s\}_{s \in S}$ are independent random variables.

Applying Lemma B.11 implies that any student with priority number below $n - t$ is assigned to one of her top $x$ choices with probability at most $3/4$; in other words, it implies $\Pr[X_s = 1] \geq 1/4$. Now, let $S_t$ denote the set of students with lowest $t$ priority numbers in STB. A standard application of Chernoff bound implies that $\sum_{s \in S_t} X_s \geq |S_t|(1 - \epsilon)/4$, whp, for any $\epsilon > 0$. 

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Lemma B.11. A student with priority number $n - t$ in STB is assigned to one of her top $x$ choices with probability at most \( \frac{tx}{n - x + 1} \).

Proof. The probability that $s$ is not assigned to her top choice is $1 - \frac{t}{n}$. The probability that $s$ is not assigned to her top two choices is $(1 - \frac{t}{n})(1 - \frac{t}{n - 1})$. Similarly, the probability that $s$ is not assigned to her top $i$ choices is $\prod_{j=1}^{i} (1 - \frac{t}{n - j + 1})$. To complete the proof, it is enough to see that $\prod_{j=1}^{x} (1 - \frac{t}{n - j + 1}) \geq 1 - \frac{tx}{n - x + 1}$. \(\square\)

C Preliminary findings: concentration lemmas

Lemma C.1. Suppose $n = m + 1$, and fix a student $s$. Then, under MTB, in the school-proposing DA, the number of offers received by $s$ is whp at most $(1 + \epsilon) \log n$ for any constant $\epsilon > 0$.

Proof. The proof idea is defining another stochastic process that we denote by $B$. Process $B$ is defined by a sequence of binary random variables $X_1, \ldots, X_k$, where $k = (1 - \delta)n \log n$ for some arbitrary small constant $\delta > 0$. Each random variable in this sequence takes the value 1 with probability $\frac{1}{n - 3 \log^2 n}$, and 0 otherwise. For convenience, we also refer to these random variables by coins, and the process that determines the value of a random variable by coin-flip.

Define $X = \sum_{i=1}^{k} X_k$. The goal is to show that $X$ is a good upper bound on the number of proposals that are received by $s$. The high-level idea is based on two facts: First, the number of total proposals is stochastically dominated by the coupon-collector problem, and so is wvhp at most $k$. Second, by Pittel (1989), we know that wvhp, each school makes at most $3 \log^2 n$ proposals, and so, each proposal is made to $s$ with probability at most $\frac{1}{n - 3 \log^2 n}$. Consequently, the number of proposals made to $s$ cannot be more than $\frac{k(1 + \delta)}{n - 3 \log^2 n}$ whp, for any constant $\delta > 0$. (The latter fact is a direct consequence of the Chernoff bound which is applicable since the coin flips are independent).

The problem with this argument is that the proposal-making processes of schools are not independent of each other, and we have to account for the dependencies. We have to define a new random process, $B$, which is a simple coin-flipping process: it flips a number of coins independently, all with success probabilities $\frac{1}{n - 3 \log^2 n}$. Then, we define a new random process $(DA, B)$, which is a coupling of the random processes $DA, B$. The coupled process would have two components, one for each of the original random processes. Each component behaves (statistically) identical to its corresponding original process, but there is no restriction on the joint behavior of the components. It is straightforward to define a simple coupling in which in almost all sample paths (i.e. wvhp), the
number of successful coin flips is an upper bound on the number of proposals made to $s$. Whenever a school wants to make a proposal during the DA, process $B$ flips the next coin. Then:

1. If $c$ has made a proposal to $s$ before, ignore the coin flip, and let $c$ pick a school uniformly at random from the set of students whom it has not proposed to yet.

2. If $c$ has made a proposal to $s$ before, then let $c$ make a proposal to the rest of the students that she has not proposed to yet, uniformly at random.

   (a) Suppose $c$ has made $d \leq \log^2 n$ proposals so far. (Otherwise, ignore this sample path)

   (b) With probability $\frac{n-3 \log^2 n}{n-d}$, let $c$ make a proposal to $s$, otherwise, let $c$ make a proposal to the rest of the students that she has not proposed to yet, uniformly at random.

It is straightforward to verify that this defines a valid coupling of DA, $B$. Now, note that the total number of successful coin flips in $B$ is an upper bound on the total number of proposals made to $s$ in the coupled DA process, in almost all sample paths (i.e. wvhp). Therefore, we can apply the argument that we mentioned in the beginning of the proof to conclude the lemma.

**Lemma C.2.** Suppose $m = n + \lambda n$. Then, for any positive constant $\epsilon$, the number of proposals received by a fixed student in the school-proposing DA is wvhp at least $(1-\epsilon)\kappa$, where $\kappa = \frac{n}{2(1+K)} + \frac{\lambda n}{2}$ and $K = (1 + \lambda) \log(1 + 1/\lambda)$.

**Proof.** The proof idea is defining another stochastic process that we denote by $B$. Process $B$ is defined by a sequence of binary random variables $X_1, \ldots, X_k$. Each random variable in this sequence is 1 with probability $1/n$, and is 0 otherwise. For convenience, we also refer to these random variables by *coins*. We describe the process $B$ in a high level and then define it formally. First, we set the number of coins ($k$) and then we start flipping them. Based on the outcome of each coin-flip, we might decrease the number of remaining coin-flips (by dismissing some of the coins). The process is finished when there are no coins left. We define the process formally below.

1. Fix a small constant $\delta > 0$.
2. Let $k = 2\kappa n(1 - \delta)$.
3. Let $i = 1$.
4. While $i \leq k$ do
(a) Flip coin $i$.

(b) If the outcome is 0 then $i \leftarrow i + 1$, otherwise $k \leftarrow k - n$.

Next, we would like to use the number of successful coin-flips, defined by $X = \sum_{i=1}^{k} X_i$, as a lower bound for the number of proposals made to $s$, which we denote by $d_s$. To this end, we couple the process $\mathcal{B}$ with the school-proposing DA, and denote the coupled process by $(\text{DA}, \mathcal{B})$. Our coupling has the property that in almost all of its sample paths (except for a negligible fraction), $X \leq d_s$. In other words, if we pick a sample path of $(\text{DA}, \mathcal{B})$ uniformly at random (from the space of all sample paths), then $X \leq d_s$ holds in that sample path wvhp.

**Claim C.3.** In $(\text{DA}, \mathcal{B})$, wvhp we have $d_s \geq X$.

**Claim C.4.** For any constant $\delta' > 0$, $X \geq (1 - \delta')(1 - \delta)\kappa$ holds wvhp.

The proofs of these claims are stated after the proof of the lemma. First, we verify that if we are given a valid coupling and the above claims, then proof of the lemma is almost complete: In Claim C.4, we show that for any constant $\delta' > 0$, the inequality

$$X \geq (1 - \delta')(1 - \delta)\kappa$$

holds wvhp. Therefore by Claim C.3, $d_s \geq (1 - \epsilon)\kappa$ holds wvhp for any constant $\epsilon > 0$.

To complete the proof, it remains to define our coupling. As mentioned before, this involves defining a new process, $(\text{DA}, \mathcal{B})$, which is in fact a coupling of the processes DA, $\mathcal{B}$. First, we define the coupling formally, and after that we prove Claim C.3.

**Definition of the Coupling**

Recall that we fixed a student $s$, with the purpose of providing a lower bound on the number of proposals made to $s$ during the DA algorithm. We define the process $(\text{DA}, \mathcal{B})$ by running both of DA and $\mathcal{B}$ simultaneously. The results of coin-flips in $\mathcal{B}$ would be used to decide whether each proposal in DA is made to $s$ or not.

Suppose we are running the school-proposing DA. Let $S_c$ denote the set of students that $c$ has proposed to them so far. In the coupled process, each school could have 3 possible states: active, inactive, and idle. In the beginning, all schools are active. We will see that as the process evolves, schools might change their state from active to inactive or idle and from inactive to idle.
In the coupled process, a coin-flip corresponds to a new proposal. If there are no coins left to flip (in \( B \)), or no proposals left to make (in DA), then (DA, \( B \)) stops. Suppose it is the turn of a school \( c \) to make a new proposal. This will be done by considering the following cases:

1. If \( c \) is active, then use a coin-flip to decide whether \( c \) proposes to \( s \) in her next move. This is done as it follows: Flip one of the unflipped coins. If it is a successful flip (with probability \( 1/n \)), then \( c \) will propose to \( s \); make \( c \) idle, and dismiss \( n \) of the unflipped coins. Otherwise, if the coin-flip is not successful then: with probability \( 1 - \frac{1}{1-1/n} \), propose to \( s \) and make \( c \) inactive, and with probability \( \frac{1}{1-1/n} \) propose to one of the students in \( S \setminus (S_c \cup \{ s \}) \) uniformly at random (without changing the state of \( c \)).

2. If \( c \) is inactive, then flip one of the unflipped coins. If it is a successful flip, make \( c \) idle, and dismiss \( n \) of the unflipped coins; otherwise, do not change the state of \( c \). Propose to one of the students in \( S \setminus S_c \) uniformly at random.

3. If \( c \) is idle, then do not flip any coins. Propose to one of the students in \( S \setminus S_c \) uniformly at random.

This completes the description of (DA, \( B \)).

**Proof of Claim C.3.** For any school \( c \) who has made a proposal to \( s \), there is at most one successful coin-flip corresponding to \( c \). This holds since

(i) A successful coin-flip that corresponds to school \( c \) happens when \( c \) is either active or inactive. In both of these cases, \( c \) must have made a proposal to \( s \).

(ii) After a successful coin-flip that corresponds to school \( c \), \( n \) coins are removed (which account for the next proposals from \( c \)). So, there will be no two successful coin-flips both of which correspond to a proposal from \( c \) to \( s \).

Consequently, the number of successful coin-flips is no larger than the number of proposals made to \( s \).

**Proof of Claim C.4.** First, we show that wvhp (DA, \( B \)) terminates with no coins left. To see this, note that in (DA, \( B \)), the number of proposals that are made is at most equal to the number of flipped or dismissed coins. On the other hand, by the results of Ashlagi et al. (2017), the number of proposals made by the school-proposing DA is at least \( k = (\frac{n^2}{1+K} + \lambda n^2)(1 - \delta) \), wvhp (To see
why, note that the number of proposals made by empty schools and the number of proposals made by non-empty schools respectively are at least $\lambda n^2(1 - \delta)$ and $(\frac{n^2}{1 + \kappa})(1 - \delta)$, whp. Since $\mathcal{B}$ starts with $k$ coins, then, whp, $(\mathcal{D}, \mathcal{B})$ ends when there are no coins left.

We are now ready to prove the lemma. Partition the set of $k$ coins into two subsets with equal size, namely subsets $A, B$. Correspond the operation $k \leftarrow k - n$ (in the process $\mathcal{B}$) to the operation of removal of $n$ coins from the subset $B$ (as long as $B$ is non-empty). One way of running $\mathcal{B}$ would be flipping the coins in $A$ one by one and removing $n$ coins from $B$ whenever a coin-flip is successful. This will be continued until $B$ is empty. Suppose $X'$ denotes the number of successful coin-flips in this process. Since $X \geq X'$ in each sample path of the process, it is enough to prove the lemma statement for $X'$ (instead of $X$). A standard application of Chernoff bound implies that $X' \geq \frac{|A|}{n} \cdot (1 - \delta')$ whp. This proves the lemma since $|A| \geq n\kappa(1 - \delta)$, by definition of $k$.

**Lemma C.5.** Suppose $n = m + 1$. Then, for any positive constant $\epsilon$, the number of proposals received by a fixed school in the student-proposing DA is whp at least $(1 - \epsilon)\kappa$, where $\kappa = \frac{n}{2\log n}$.

**Proof.** The proof is similar to the proof of Lemma C.2. The only adjustments are swapping the roles of schools and students and using the new definition of $\kappa$ stated in this lemma.

**Lemma C.6.** Suppose $n = m + 1$. Fix an arbitrary small constant $\epsilon > 0$. Then, in the school-proposing DA, the number of proposals received by a fixed student in the school-proposing algorithm is whp at least $(1 - \epsilon) \cdot \kappa$, where $\kappa = \frac{\log n}{2}$.

**Proof.** The proof is similar to our proof for Lemma C.2, with the exception that we should use the new definition of $\kappa$ that we state in this lemma.

For notational convenience in this section, we adopt the following definition.

**Definition C.7.** Let $\underline{r}, \overline{r}$ respectively denote $n/(\log n)^{1+\epsilon}, n/(\log n)^{1-\epsilon}$.

**Lemma C.8.** Suppose $n = m + 1$ and fix a student $s \in S$. Then, for any constant $\epsilon > 0$ we have

$$\mathbb{P} \left[ \mu^\#(s) \not\in [\underline{r}, \overline{r}] \right] = o(1).$$

**Proof.** Instead of proving the claim directly, we will show that

$$\mathbb{P} \left[ \eta^\#(s) \not\in [\underline{r}, \overline{r}] \right] = o(1).$$

(3)
Ashlagi et al. (2017) show that $P[\mu(s) \neq \eta(s)] \leq \frac{\sqrt{\log n}}{n}$. Therefore it is sufficient to show that (3) holds.

We use Lemma C.6 to prove (3). Let $d$ denote the number of proposals received by $s$. Lemma C.6 implies that

$$P[d < \alpha \log n] = o(1),$$

(4)

where $\alpha$ is a positive constant. So, we can safely assume that $d \geq \alpha \log n$. Let $X_1, \ldots, X_d$ be random variables that denote the utility of $s$ from the $j$-th proposal she receives. Note that $\eta^\#(s) = \min\{X_1, \ldots, X_d\}$.

Since students preferences are drawn uniformly at random, we can write

$$P[\eta^\#(s) \geq r] = \prod_{i=1}^{d} \left(1 - \frac{r}{m - i + 1}\right) \leq \left(1 - \frac{r}{m}\right)^d \leq e^{-\frac{dr}{m}} \leq \exp(-\alpha (\log n)^\epsilon) = o(1).$$

(5)

In the other hand, we have

$$P[\eta^\#(s) \leq r] = 1 - P[\eta^\#(s) > r]$$

$$\leq 1 - \prod_{i=1}^{d} \left(1 - \frac{r}{m - i + 1}\right)$$

$$\leq 1 - \left(1 - \frac{r}{m - d}\right)^d \leq 1 - \left(1 - \frac{dr}{m - d}\right) \leq O\left(\frac{\alpha}{(\log n)^\epsilon}\right) = o(1).$$

(6)

Taking a union bound over the bounds (4), (5), and (6) completes the proof.

\[\square\]

D Proofs for Section A.2

Consider a matching $\gamma$. Let $\gamma(x)$ be the agent to which $x$ is matched, and for any subset of agents $A \subseteq S \cup C$, let $\gamma(A)$ be the set of agents matched to agents in $A$. Therefore, $\gamma(C)$ is the set of students who are assigned under $\gamma$. 

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Theorem D.1. Suppose \( n = m + 1 \) and fix a student \( s \in S \). Then, under MTB, we have

\[
\lim_{n \to \infty} \mathbb{P}\left[ \hat{\mu}(s) \geq \frac{n}{(\log n)^{2+\epsilon}} \right] \to 1
\]

for any constant \( \epsilon > 0 \).

Next, we will define a random variable \( \Pi(s) \), which we will use in the proof of Theorem D.1. Recall that \( r, \bar{r} \) respectively denote \( n/(\log n)^{1+\epsilon}, n/(\log n)^{1-\epsilon} \). For a fixed student \( s \), we will define the random variable \( \Pi(s) \), which represent a preference profile that is constructed by fixing the the interval \([r, \bar{r}]\) of the preference list of \( s \), while letting the rest of the preference profile be constructed randomly. This notion is formally defined below.

Definition D.2. For a fixed student \( s \), we define a random variable \( \Pi(s) \), which is a subset of preference profiles. We define \( \Pi(s) \) by constructing it, this would implicitly define the corresponding support and probability mass function (PMF); we denote the PMF by \( \mathcal{P}(s) \). We define \( \Pi(s) \) by first defining a partial preference profile \( \hat{\pi} \), as follows:

1. For all students \( s' \neq s \), let \( \hat{\pi}(s') \) be drawn independently uniformly at random.
2. Positions \( r, \ldots, \bar{r} \) in \( \hat{\pi}(s) \) are filled with schools \( r, \ldots, \bar{r} \), respectively.

\( \Pi(s) \) contains the set of all preference profiles \( \pi \) who are consistent with \( \hat{\pi} \) (i.e. agree with \( \hat{\pi} \) on the positions where \( \hat{\pi} \) is defined). Given a realization \( \Pi(s) \), let \( \mathcal{U}(\Pi(s)) \) denote the uniform distribution over the elements of \( \Pi(s) \).

Lemma D.3. Suppose \( \Pi(s) \sim \mathcal{P}(s) \). Also, suppose \( \pi, \pi' \) are preference profiles that are drawn independently uniformly at random from \( \Pi(s) \). Then, whp \( \mu_{\pi} = \mu_{\pi'} \). (i.e., almost all student-optimal matchings in \( \Pi(s) \) are identical, whp)

Proof. By definition, \( \pi, \pi' \) are selected so that they are identical everywhere except on a fixed student, namely \( s \). So, \( \pi, \pi' \) coincide on the interval \([r, \bar{r}]\) of the preference list of \( s \), but they are constructed independently (and uniformly at random) everywhere else in the preference list of \( s \). (In other words, the schools listed in the interval \([r, \bar{r}]\) of \( \pi'(s) \) are the same as \( \pi(s) \), but in all other schools in \( \pi'(s) \) are shuffled randomly)
Using lemma C.8 and a simple union bound we obtain that

\[
\mathbb{P}\left[\mu^\#(s) \notin [r, \bar{r}] \lor \mu'^\#(s) \notin [r, \bar{r}]\right] \\
\leq \mathbb{P}\left[\mu^\#(s) \notin [r, \bar{r}]\right] + \mathbb{P}\left[\mu'^\#(s) \notin [r, \bar{r}]\right] = o(1).
\]

(7)

The preference list of each student \(s' \neq s\) is the same in \(\pi, \pi'\); also, whp, \(\mu(\pi)(s)\), \(\mu(\pi')(s)\) are both in the interval \([r, \bar{r}]\) of the preference list of \(s\). If this holds, then since the preference lists \(\pi(s), \pi'(s)\) are identical in this interval, we get \(\mu(\pi) = \mu(\pi')\) (It is straight-forward to verify this). Therefore, \(\mu(\pi) = \mu(\pi'),\) whp.

Proof of Theorem D.1. For a preference profile \(\pi\), define \(B(\pi)(s)\) to be the subset of students \(s'\) for which \(\mu(\pi)(s) \succ s' \mu(\pi')(s)\). Define \(A(\pi)(s)\) to be the subset of students \(s'\) for which \(s' \in B(\pi)(s)\), and moreover, \(\mu(\pi)(s') \succ s \mu(\pi)(s)\). The proof is done in two steps. In Step 1, we show that \(|B(\pi)(s)|\) is “large”, whp. In Step 2, we show that \(|A(\pi)(s)|\) is “large”, whp; this would prove the lemma.

**Step 1.** Consider an arbitrary school \(c \in C\). We will show that whp, there are “many” students who rank \(c\) above their match in the student-optimal matching. Then, taking a union bound over all schools \(c \in C\) would show that whp, many students rank \(\mu(\pi)(s)\) above their current match, implying that \(|B(\pi)(s)|\) is large. Instead of showing that many students rank \(c\) above their match in the student-optimal matching, we can equivalently show that \(c\) receives many proposals in the student-proposing DA. This is what we proved in Lemma C.5.

We now formalize this idea. By Lemma C.5, for any constant \(\epsilon > 0\), each school receives at least \(\frac{n(1-\epsilon)}{2\log n}\) proposals whp, which also implies that all schools receive at least \(\frac{n(1-\epsilon)}{2\log n}\) proposals whp. Thus, \(\mu(\pi)(s)\) receives at least \(\frac{n(1-\epsilon)}{2\log n}\) proposals whp, which means for any constant \(\epsilon > 0\), whp we have \(|B(\pi)(s)| > \frac{n(1-\epsilon)}{2\log n}\). This completes Step 1.

Observe that in Step 1 we showed that

\[
\mathbb{P}_{\pi \sim \mathcal{P}} \left[|B(\pi)(s)| > \frac{n(1-\epsilon)}{2\log n}\right] \geq 1 - o(1),
\]

(8)

where \(\mathcal{P}\) denotes the uniform distribution over all preference profiles. Next, we write an alternative version of (8), which will be used later in Step 2.

Recall Definition D.2, by which \(\Pi(s)\) is a random variable containing the set of all the possible
placements of schools \([m \mid r, \overline{r}]\) in positions \([m \mid r, \overline{r}]\). Note that, without loss of generality, we can assume that schools listed on positions \(r, \ldots, \overline{r}\) of \(\pi(s)\) are schools \(r, \ldots, \overline{r}\), respectively. Thus, we can rewrite (8) as

\[
P_{\Pi(s) \sim \mathcal{P}(s), \pi \sim \mathcal{U}(\Pi(s))} \left[ |B_{\pi}(s)| > \frac{n(1 - \epsilon)}{2 \log n} \right] \geq 1 - o(1). \tag{9}
\]

**Step 2** Lemma D.3 shows that, when \(\Pi(s) \sim \mathcal{P}(s)\), almost all student-optimal matchings in \(\Pi(s)\) (i.e. a fraction \(1 - o(1)\) of them) are the same whp. Let \(\mu\) denote this matching. Suppose that, for \(\pi, \pi' \in \Pi(s)\), we have \(\mu_{\pi} = \mu_{\pi'} = \mu\). Then, see that by the definition of \(\Pi(s)\), we have \(B_{\pi}(s) = B_{\pi'}(s)\). Thus, we let \(B(s)\) denote \(B_{\pi}(s)\) for any \(\pi \in \Pi(s)\) for which \(\mu_{\pi} = \mu\). Now, (9) implies that \(|B(s)|\) is large, whp. This means, if \(\pi \sim \mathcal{U}(\Pi(s))\), then, both of the events \(\mu_{\pi} = \mu\) and \(|B_{\pi}(s)| \geq \frac{n(1 - \epsilon)}{2 \log n}\) hold whp. We use this fact to prove that \(|A_{\pi}(s)|\) is large, whp. This would conclude Step 2.

Let \(\pi \sim \mathcal{U}(\Pi(s))\). We show that whp, a large number of schools in \(B(s)\) have a rank better than \(r\) in \(\pi(s)\). This would imply that \(|A_{\pi}(s)|\) is large, whp. First note that we can safely assume that \(\mu_{\pi} = \mu\) (and so \(B_{\pi}(s) = B(s)\)), since \(\mu_{\pi} \neq \mu\) is a low-probability event (has probability \(o(1)\)) by Lemma D.3. Therefore, we assume that the event \(\mu_{\pi} = \mu\) holds in the rest of this analysis.

Let \(X(c)\) be a binary random variable which takes the value 1 iff school \(c\) has a rank \(r\) or better in \(\pi(s)\). Also, let \(X = \sum_{c \in \mu(B(s))} X_c\). For any \(c \in \mu(B(s))\), we have

\[
P[X_c = 1] \geq \frac{r}{n} = \frac{1}{(\log n)^{1+\epsilon}}.
\]

Thus, \(\mathbb{E}[X] \geq \frac{|B(s)|}{(\log n)^{1+\epsilon}}\). A standard application of Chernoff bounds imply that for any \(\delta > 0\), we have

\[
P[X < (1 - \delta) \cdot \mathbb{E}[X]] \leq \exp\left(-\frac{\delta^2 \mathbb{E}[X]}{2}\right).
\]

Thus, \(|A_{\pi}(s)|\) is at least \(\frac{(1 - \delta) \cdot |B(s)|}{(\log n)^{1+\epsilon}}\) whp. In Step 1, (9) shows that \(|B(s)|\) is large whp. Consequently,

\[
P_{\Pi(s) \sim \mathcal{P}(s), \pi \sim \mathcal{U}(\Pi(s))} \left[ |A_{\pi}(s)| \geq \frac{n(1 - \epsilon)(1 - \delta)}{2(\log n)^{2+\epsilon}} \right] \geq 1 - o(1)
\]

for any constants \(\epsilon, \delta > 0\). This concludes Step 2 and completes the proof. \(\square\)
**Theorem D.4.** Fix a student $s$. Under MTB, if $n < m$

$$\lim_{n \to \infty} \mathbb{P}[\bar{\mu}(s) \geq 1] \to 0.$$  

**Proof.** Let $l = 3 \log^2 n$. Pittel (1989) proves that wvhp, every student is assigned to one of her top $l$ choices. Let $L(s)$ denote the top $l$ schools listed by student $s$. We show that for any student $s' \neq s$,

$$\mathbb{P}[|L(s) \cap L(s')| \geq 2] \leq O\left(\frac{\log^4 n}{n^2}\right). \quad (10)$$

That is, the probability that $(s, s')$ is a Pareto improving pair is very small. Assuming (10) holds the proof is completed by taking a union bound over all $s' \neq s$ since the union bound implies that

$$\mathbb{P}[\bar{\mu}(s) \geq 1] \leq n \cdot O\left(\frac{\log^4 n}{n^2}\right) = o(1).$$

It remains to show that (10) holds. First fix $L(s)$ and then start constructing $L(s')$ randomly (we are using the principle of deferred decisions). It is straightforward to verify that

$$\mathbb{P}[|L(s) \cap L(s')| \geq 2] \leq \left(\frac{l}{2}\right) \cdot \frac{(l/m)^2}{l^4/m} \leq l^4/m$$

$$= O\left(\frac{\log^4 n}{n^2}\right).$$

$\square$
E Proofs for Section A.3

E.1 Equivalence of social inequity and variance

Proof of Lemma A.4. Let \( q = \min\{m, n\} \) be the number of assigned students, which is the same in all stable matchings. Then,

\[
E[Si(\mu)] = E\left[ \frac{1}{|\mu(C)|} \cdot \sum_{t \in \mu(C)} (\text{Ar}(\mu) - \mu^#(t))^2 \right]
\]

\[
= E\left[ \frac{1}{q} \cdot \sum_{t \in \mu(C)} \text{Ar}(\mu) + \mu^#(t)^2 - 2\text{Ar}(\mu)\mu^#(t) \right]
\]

\[
= \sum_{t \in S} P_{\mu} \left[ t \in \mu(C) \right] \cdot \sum_{t \in \mu(C)} \text{Ar}(\mu) + \mu^#(t)^2 - 2\text{Ar}(\mu)\mu^#(t) \right] t \in \mu(C)
\]

\[
= \frac{q}{n} \cdot \sum_{t \in S} \sum_{t \in \mu(C)} \text{Ar}(\mu) + \mu^#(t)^2 - 2\text{Ar}(\mu)\mu^#(t) \right] t \in \mu(C)
\]

\[
= \frac{1}{n} \cdot \sum_{t \in S} \sum_{t \in \mu(C)} \text{Ar}(\mu) + \mu^#(s)^2 - 2\text{Ar}(\mu)\mu^#(s) \right] s \in \mu(C)
\]

(11)

\[
= E_{\mu} \left[ \text{Ar}(\mu) + \mu^#(s)^2 - 2\text{Ar}(\mu)\mu^#(s) \right] s \in \mu(C)
\]

(12)

\[
= E_{\mu} \left[ \text{Var} \left[ r_s \right] \right]
\]

(13)

\[
= E_{\mu} \left[ \text{Var} \left[ r_s \right] \right]
\]

(14)

In the above inequalities, (12) holds because the term inside the expectation in (11) is equal for all students by symmetry. (14) holds since, by symmetry, the inner expectation in (13) is equal for all preference profiles of \( s \).

E.2 Proof of Lemma A.6

E.2.1 preliminaries

Proposition E.1. Suppose \( d \leq n \), and define the random variable \( X = \min\{X_1, \ldots, X_d\} \), where \( X_1, \ldots, X_d \) respectively represent the first \( d \) elements of a permutation over \( [n] \) that is chosen uniformly at random. Then, \( E[X^2] = \frac{d(n+1)(n-d)}{(d+1)^2(d+2)} + \frac{(n+1)^2}{(d+1)^2} \).

Proof. It is known that \( E[X] = \frac{n+1}{d+1} \) and \( \text{Var}[X] = \frac{d(n+1)(n-d)}{(d+1)^2(d+2)} \) (see Arnold et al. (1992), Page 55).

Plugging these equations into \( \text{Var}[X] = E[X^2] - E[X]^2 \) proves the claim. \( \square \)
Lemma E.2. Suppose \( n \leq m \). Then, a student \( s \) with priority number \( n - t \) is assigned to one of her top \( \frac{n \log(n)}{t} \) choices with probability at least \( 1 - 1/n \).

**Proof.** The probability that \( s \) is not assigned to his top choice is \( 1 - \frac{1}{n} \). The probability that \( s \) is not assigned to his second top choice is \( (1 - \frac{1}{n})(1 - \frac{1}{n-t}) \), which is at most \( (1 - \frac{1}{n})^2 \). Similarly, it is straightforward to see that the probability that \( s \) is not assigned to her \( i \)-th top choice is at most \( (1 - \frac{1}{n})^i \), which is at most \( e^{-\frac{n}{t}} \). Setting \( i = \frac{n}{t} \log(n) \) proves the claim. \( \square \)

Lemma E.3. Suppose \( |n - m| = 1 \). Then, under STB, for any student \( s \),

\[
\mathbb{E}_\pi \left[ \mu^\#_\pi(s)^2 \big| \mu_\pi(s) \neq \emptyset \right] = O(n).
\]

**Proof.** We prove this assuming that \( m \geq n \). The proof for \( m < n \) is identical to the proof for \( m = n \): To see this, suppose that \( n = m \), and note that the expected social inequity does not change when one more student is added to the market.

Let \( t = \sqrt{n} \log n \) and let \( p_s \) be the “priority number” of \( s \) in the corresponding random serial dictatorship. We consider two cases: either \( p_s \leq n - t \) or not. Note that

\[
\mathbb{E}_\pi \left[ \mu^\#_\pi(s)^2 \big| \mu_\pi(s) \neq \emptyset \right] = \mathbb{P}[p_s \leq n - t] \cdot \mathbb{E} \left[ \mu^\#_\pi(s)^2 \big| p_s \leq n - t \right] + \mathbb{P}[n - t < p_s] \cdot \mathbb{E} \left[ \mu^\#_\pi(s)^2 \big| n - t < p_s \right]. \tag{15}
\]

We provide an upper bound for each of the terms in the right-hand side of (15).

By Lemma E.2, we have:

\[
\mathbb{E} \left[ \mu^\#_\pi(s)^2 \big| p_s \leq n - t \right] \leq (1 - \frac{1}{n}) \cdot (n \log(n)/t)^2 + \frac{1}{n} \cdot (n^2) \leq 2n,
\]

which implies that

\[
\mathbb{P}[p_s \leq n - t] \cdot \mathbb{E} \left[ \mu^\#_\pi(s)^2 \big| p_s \leq n - t \right] \leq 2n. \tag{16}
\]
Also, we have that

\[ P[n - t < p_s] \cdot \mathbb{E} \left[ r_s^2 \mid n - t < p_s \right] \leq \frac{t}{n} \cdot \sum_{i=1}^{t} \frac{1}{t} \mathbb{E} \left[ r_s^2 \mid p_s = n - i + 1 \right] \]

\[ \leq \frac{1}{n} \cdot \sum_{i=1}^{t} 2(n/i)^2. \tag{17} \]

\[ \leq n \cdot \frac{\pi^2}{3}. \tag{18} \]

where (17) holds since for a geometric random variable \( X \) with mean \( p \) we have \( \mathbb{E}[X] = \frac{2-p}{p^2} \).

Finally, putting (16) and (18) together implies

\[ \mathbb{E}_\pi \left[ \mu_p^n(s)^2 \mid \mu_p(s) \neq \emptyset \right] \leq n(2 + \frac{\pi^2}{3}). \]

\[ \square \]

### E.2.2 Proof of Lemma A.6 - Part 1

The proof for Part 1 of Lemma A.6 is directly implied by Lemmas E.4 and E.5.

**Lemma E.4.** When \( n = m + 1 \), expected social inequity in MTB is \( \Omega\left( \frac{n^2}{\log^2 n} \right) \).

**Proof.** The proof has two steps. In Step 1, we show that if we run the school-proposing DA, then the variance of the rank of each student is high. In Step 2, we show that even when we move from the school-optimal matching to the student-optimal matching, the variance remains high. The rough intuition behind Step 2 is that only \( o(n) \) of the students would have a different match under the school-optimal and the student-optimal matchings.

**Step 1.** Since the social inequity and the expected variance in the rank of a fixed student are equal by Lemma A.4, there is no harm in analyzing the latter notion (we switch to the former notion in Step 2). We are interested in providing a lower bound on \( \mathbb{E} \left[ (r_s - r)^2 \right] \), where \( r_s \) is a random variable denoting the rank for student \( s \) and \( r = \mathcal{A}(\eta) \) (note that \( r \) is also equal to the average rank of \( s \), conditioned on being assigned). Since \( \mathbb{E} \left[ (r_s - r)^2 \right] = \mathbb{E} \left[ r_s^2 \right] - r^2 \), we can instead provide a lower bound on the RHS of the equality.

Fix an arbitrary small constant \( \epsilon > 0 \). Let \( E_s \) denote the event in which student \( s \) receives at
most \((1 + \epsilon) \log n\) proposals. Then
\[
\mathbb{E} \left[ r_s^2 \right] \geq \Pr[E_s] \cdot \mathbb{E} \left[ r_s^2 | E_s \right] + \left( 1 - \Pr[E_s] \right) \cdot 0. \tag{19}
\]

To give a lower bound on the RHS of (19), we provide a lower bound on \(\mathbb{E} \left[ (r_s - r)^2 | E_s \right]\). If student \(s\) receives \(d_s\) proposals in school-proposing DA, then it chooses the best out of these \(d_s\) proposals, which means its rank is the first order statistic among the proposals that she had received. In Proposition E.1, we calculate \(\mathbb{E} \left[ r_s^2 | d_s \right]\) (which is the expected rank squared for \(s\) conditioned on receiving \(d_s\) proposals).

Using Proposition E.1 and (19) together we can write
\[
\mathbb{E} \left[ r_s^2 | E_s \right] \geq \Pr[E_s] \cdot \mathbb{E} \left[ r_s^2 | E_s \right] + \left( 1 - \Pr[E_s] \right) \cdot 0 \geq \left( 1 - o(1) \right) \cdot \frac{3n^2}{2 \log^2 n} + o(1) \cdot 0, \tag{20}
\]
where (20) follows from Lemma C.1, which shows event \(E_s\) happens whp.

It is known that, \(r \in \left[ \frac{(1-\delta)n}{\log n}, \frac{(1+\delta)n}{\log n} \right]\) for any constant \(\delta > 0\) and large enough \(n\) (see Ashlagi et al. (2017)). Therefore, together with (20),
\[
\mathbb{E} \left[ (r_s - r)^2 \right] = \mathbb{E} \left[ r_s^2 \right] - r^2 \geq (1 - o(1)) \cdot (3/2 - (1 + \delta)^2) \cdot \frac{n^2}{\log^2 n} = \Theta\left( \frac{n^2}{\log^2 n} \right).
\]

This finishes Step 1.

**Step 2.** In this step, instead of working with the notion of expected variance in the rank of a fixed student, we switch to its equivalent notion, expected social inequity. Step 1 and Lemma A.4 together imply that \(\mathbb{E}_\pi [Si(\eta_\pi)]\) is \(\Omega(n^2/\log^2 n)\). In this step, we show that moving from the school-optimal matching to the student-optimal matching does not change the social inequity much in expectation, and as the result, we would prove that \(\mathbb{E}_\pi [Si(\mu_\pi)]\) is also \(\Omega(n^2/\log^2 n)\). This is done as follows.

\[
m \cdot \mathbb{E}_\pi [Si(\mu_\pi) - Si(\eta_\pi)] = \mathbb{E}_\pi \left[ \sum_{s \in \mu_\pi(C)} \mu_\pi^#(s)^2 + Ar(\mu_\pi)^2 - 2\mu_\pi^#(s)Ar(\mu_\pi) \right.

\[
- \left. \sum_{s \in \eta_\pi(C)} \eta_\pi^#(s)^2 + Ar(\eta_\pi)^2 - 2\eta_\pi^#(s)Ar(\eta_\pi) \right] \]

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\begin{align*}
= m \cdot E_{\pi} \left[ A_r(\mu_\pi)^2 - A_r(\eta_\pi)^2 \right] + E_{\pi} \left[ \sum_{s \in \mu_\pi(C)} \mu_\pi^\#(s)^2 - \eta_\pi^\#(s)^2 \right] \\
- 2E_{\pi} \left[ \sum_{s \in \mu_\pi(C)} \mu_\pi^\#(s)A_r(\mu_\pi) - \sum_{s \in \eta_\pi(C)} \eta_\pi^\#(s)A_r(\eta_\pi) \right].
\end{align*}

We can rewrite the above inequality by simplifying (21) as

\begin{align*}
2E_{\pi} \left[ \sum_{s \in \mu_\pi(C)} \mu_\pi^\#(s)A_r(\mu_\pi) - \sum_{s \in \eta_\pi(C)} \eta_\pi^\#(s)A_r(\eta_\pi) \right] = 2m \cdot E_{\pi} \left[ A_r(\mu_\pi)^2 - A_r(\eta_\pi)^2 \right],
\end{align*}

which together with the previous equation implies that

\begin{align*}
m \cdot E_{\pi} [S_i(\mu_\pi) - S_i(\eta_\pi)] &= \quad (22) \\
- m \cdot E_{\pi} [A_r(\mu_\pi)^2 - A_r(\eta_\pi)^2] \quad (23) \\
+ E_{\pi} \left[ \sum_{s \in \mu_\pi(C)} \mu_\pi^\#(s)^2 - \eta_\pi^\#(s)^2 \right]. \quad (24)
\end{align*}

To prove the lemma, we provide lower bounds for (23) and (24). When we move from the school-optimal matching to the student-optimal matching, each student gets assigned to a school at least as good as before. Let \( \Delta_\pi(s) = \eta_\pi^\#(s) - \mu_\pi^\#(s) \), and \( \Delta_\pi = \sum_{s \in \mu_\pi(C)} \Delta_\pi(s) \).

**A Lower Bound for (23).** First, we note that \( m \cdot (A_r(\eta_\pi) - A_r(\mu_\pi)) = \Delta_\pi \) is "small" wvhp. This is a direct consequence of Theorem 5 in Ashlagi et al. (2017); they show that there exist constants \( n_0, \delta > 0 \) such that for \( n > n_0 \), we have

\begin{align*}
\mathbb{P}_{\pi \sim \Pi} [\Delta_\pi \geq \delta n \log n] < \exp \{- (\log n)^{0.4} \}. \quad (25)
\end{align*}

According to this bound, we have that

\[ m \cdot (A_r(\mu_\pi)^2 - A_r(\eta_\pi)^2) = m \cdot ((A_r(\eta_\pi) - \Delta_\pi/m)^2 - A_r(\eta_\pi)^2) = \Delta_\pi^2/m - 2\Delta_\pi A_r(\eta_\pi). \]
By taking expectation from both sides of the above equation, we can write

\[
m \cdot \mathbb{E}_\pi [\mathcal{A}r(\mu_\pi)^2 - \mathcal{A}r(\eta_\pi)^2] = \mathbb{E}_\pi [\Delta_\pi^2/m - 2\Delta_\pi \mathcal{A}r(\eta_\pi)] \\ \leq (\bar{\delta} n \log n)^2/m,
\]

where the last inequality follows from (25), for any constant \( \bar{\delta} > \delta \) and sufficiently large \( n \). This implies a lower bound of \(- (\bar{\delta} n \log n)^2/m \) for (23).

**A Lower Bound for (24).** First, we rewrite (24) as follows.

\[
\mathbb{E}_\pi \left[ \sum_{s \in \eta_\pi(C)} \mu_{\pi}^{\#}(s)^2 - \eta_{\pi}^{\#}(s)^2 \right] = \mathbb{E}_\pi \left[ \sum_{s \in \mu_\pi(C)} (\eta_{\pi}^{\#}(s) - \Delta_\pi(s))^2 - \eta_{\pi}^{\#}(s)^2 \right] \\ \geq -2\mathbb{E}_\pi \left[ \sum_{s \in \mu_\pi(C)} \eta_{\pi}^{\#}(s)\Delta_\pi(s) \right].
\]

(27)

We proceed by providing a lower bound on (27). First, we use the Cauchy-Schwarz inequality to write

\[
\sum_{s \in \eta_\pi(C)} \eta_{\pi}^{\#}(s)\Delta_\pi(s) \leq \left( \sum_{s \in \eta_\pi(C)} (\eta_{\pi}^{\#}(s))^2 \cdot \sum_{s \in \eta_\pi(C)} (\Delta_\pi(s))^2 \right)^{1/2} \\
\leq m^{3/2} \cdot \left( \sum_{s \in \eta_\pi(C)} (\Delta_\pi(s))^2 \right)^{1/2}.
\]

Taking expectation from both sides of the above inequality implies

\[
\mathbb{E}_\pi \left[ \sum_{s \in \eta_\pi(C)} \eta_{\pi}^{\#}(s)\Delta_\pi(s) \right] \leq m^{3/2} \cdot \mathbb{E}_\pi \left[ \left( \sum_{s \in \eta_\pi(C)} (\Delta_\pi(s))^2 \right)^{1/2} \right].
\]

Using (25), we can rewrite the above upper bound:

\[
\mathbb{E}_\pi \left[ \sum_{s \in \eta_\pi(C)} \eta_{\pi}^{\#}(s)\Delta_\pi(s) \right] \leq m^{3/2} \cdot n(\bar{\delta} \log n)^{1/2},
\]

which holds for any constant \( \bar{\delta} > \delta \). According to (27), this upper bound can be directly translated into a lower bound \(-2m^{3/2} \cdot n(\bar{\delta} \log n)^{1/2} \) for (24).
Using the lower bounds that we provided for (23) and (24), we can rewrite equation (22) as follows:

\[ m \cdot \mathbb{E}_\pi [S_i(\mu_{\pi}) - S_i(\eta_{\pi})] \geq -\left(\delta n \log n\right)^2 / m - 2m^{3/2} \cdot n(\delta \log n)^{1/2}. \]

In the other hand, In Step 1 we established that \( \mathbb{E}_\pi [S_i(\eta_{\pi})] \geq \Omega(n^2 / \log^2 n) \). The two latter inequalities together imply that

\[ \mathbb{E}_\pi [S_i(\mu_{\pi})] = \mathbb{E}_\pi [S_i(\eta_{\pi})] + \mathbb{E}_\pi [S_i(\mu_{\pi}) - S_i(\eta_{\pi})] \geq \Omega(n^2 / \log^2 n). \]

This completes the proof.

Lemma E.5. Suppose \(|n - m| = 1\). Then, under STB, the expected social inequity is \( \Theta(n) \).

Proof. First, we compute a lower bound on the expected social inequity in STB. With probability at least \( 1/2 \), the student with the lowest priority number in STB gets assigned to a school that she has ranked on lower half of her preference list. So, for any student \( s \in S \) we can write:

\[ \mathbb{E} [S_i(\mu_{\pi})] = \mathbb{E} [\text{Var} [r_s]] \geq \frac{1}{n} \cdot \left( \text{Ar}(\mu_{\#STB}(s)) - n \right)^2. \]

It is proved by Knuth (1995) that \( \text{Ar}(\mu_{\#STB}(s)) = \Theta(\log n) \). Plugging this into the above inequality implies that \( \mathbb{E} [S_i(\mu_{\pi})] \geq \Omega(n) \). On the other hand, by Lemma E.3 we have that

\[
S_i(\mu_{\pi}) = \mathbb{E}_\pi \left[ (\text{Ar}(\mu_{\pi}) - \mu_{\#}(s))^2 | \mu_{\pi}(s) \neq \emptyset \right] \\
= \mathbb{E}_\pi \left[ \mu_{\#}(s)^2 | \mu_{\pi}(s) \neq \emptyset \right] - \mathbb{E}_\pi [\text{Ar}(\mu_{\pi})]^2 \\
\leq \mathbb{E}_\pi \left[ \mu_{\#}(s)^2 | \mu_{\pi}(s) \neq \emptyset \right] = O(n),
\]

which completes the proof.

E.2.3 Proof of Lemma A.6 - Part 2

Pittel (1989) shows that wvhp, \( \max_{s \in S} \mu_{\#MTB}(s) \leq 3 \log^2 n \). Therefore, wvhp

\[ \frac{1}{n} \cdot \sum_{s \in S} (\text{Ar}(\mu_{\#MTB}) - \mu_{\#MTB}(s))^2 \leq 9 \log^4 n. \]
This implies that the expected social inequity under MTB is $O(\log^4 n)$. On the other hand, Lemma E.5 implies that the expected social inequity under STB is $\Theta(n)$.

### E.2.4 Proof of Lemma A.6 - Part 3

First note that Part 2 implies a weaker version of Part 3. That is, if $n = m - 1$, the expected social inequity under MTB is still $O(\log^4 n)$, by the same analysis for $n = m$. On the other hand, by Lemma E.5 the expected social inequity under STB is $\Theta(n)$. This gap is large enough that Theorem A.5 still holds, even with this weaker version of Part 3.

We prove here that the gap is even larger, by showing how the bound on the expected social inequity under MTB can be improved to $O(\log^2 n)$. The proof follows the same steps as the proof of Lemma E.7, where we provide an upper bound on $\mathbb{E}[S_i(\mu_{MTB})]$ when the imbalance is linear. During the proof, we will also use Lemma C.5, which was proved in Section D.

The proof is done in 2 Steps. In Step 1, we show that that the variance of the rank of student $s$ in the student-proposing DA is approximately equal to the variance of its rank in the school-proposing DA. Then, in Step 2, we provide an upper bound on the variance of rank in the school-proposing DA. Steps 1,2 then together will prove the claim.

**Step 1.** First, we rewrite the following equality from the proof of Lemma E.4.

\[
m \cdot \mathbb{E}_\pi [S_i(\mu_\pi) - S_i(\eta_\pi)] = \tag{28}
\]

\[
- m \cdot \mathbb{E}_\pi \left[ Ar(\mu_\pi)^2 - Ar(\eta_\pi)^2 \right] \tag{29}
\]

\[
+ \mathbb{E}_\pi \left[ \sum_{s \in \mu_\pi(C)} \mu_\pi^#(s)^2 - \eta_\pi^#(s)^2 \right]. \tag{30}
\]

To complete Step 1, we need to provide upper bounds for (29) and (30).

**An upper bound for (29)** We will use the following relation between average ranks, provided by Theorem 3 of Ashlagi et al. (2017): wvhp we have

\[
Ar(\eta_\pi) \leq Ar(\mu_\pi)(1 + o(1)).
\]

Consequently, $m \cdot o(1) \cdot \mathbb{E}_\pi [Ar(\mu_\pi)]$ is a valid upper bound for (29).
An upper bound for (30) is a valid upper bound since, by the definition of \( \mu, \eta \), we always have \( \mu^{\#}(s) \leq \eta^{\#}(s) \).

Plugging the provided upper bounds into (28) implies

\[
E_\pi [S_i(\mu_\pi) - S_i(\eta_\pi)] \leq o(1) \cdot E_\pi [\mathcal{A}r(\mu_\pi)].
\]

When there are linearly more seats, \( E_\pi [\mathcal{A}r(\mu_\pi)] = O(1) \). This implies

\[
E_\pi [S_i(\mu_\pi) - S_i(\eta_\pi)] \leq o(1),
\]

which concludes Step 1.

\textbf{Step 2.} Suppose we are running the school-proposing DA. First, see that

\[
E_\pi [S_i(\eta_{MTB})] = E_\pi [(\mathcal{A}r(\eta_\pi) - \eta^{\#}_{\pi}(s))^2| \eta_\pi(s) \neq \emptyset] = E_\pi [\eta^{\#}_{\pi}(s)^2| \eta_\pi(s) \neq \emptyset] - E_\pi [\mathcal{A}r(\eta_\pi)]^2 \\
\leq E_\pi [\eta^{\#}_{\pi}(s)^2| \eta_\pi(s) \neq \emptyset].
\]

For notational simplicity, let \( r_s \) denote the rank of student \( s \). Note that since \( s \) is always assigned, then \( r_s \in [m] \). We can write the above bound as

\[
E_\pi [S_i(\eta_{MTB})] \leq E \left[ r_s^2 \right].
\]

Next, we provide an upper bound on \( E \left[ r_s^2 \right] \). Fix an arbitrary small constant \( \epsilon > 0 \). Let \( E_s \) denote the event in which student \( s \) receives at least \( \kappa = \frac{(1-\epsilon)n}{2\log n} \) proposals. Lemma C.5 shows that \( E_s \) happens wvhp. Consequently,

\[
E \left[ r_s^2 | E_s \right] \leq P [E_s] \cdot E \left[ r_s^2 | E_s \right] \leq O(\log^2 n),
\]

where we used Proposition E.1 to bound \( E \left[ r_s^2 | E_s \right] \).

Now we are ready to finish the proof of Part 3. See that (32) and (33) together imply that

\[
E_\pi [S_i(\eta_{MTB})] \leq O(\log^2 n).
\]
Therefore, together with Step 1, we have that

\[ \mathbb{E}_\pi [S_i(\mu_\pi)] \leq \mathbb{E}_\pi [S_i(\mu_\pi) - S_i(\eta_\pi)] + \mathbb{E}_\pi [S_i(\eta_\pi)] \]
\[ \leq o(1) + \mathbb{E}_\pi [S_i(\eta_\pi)] \approx O(\log^2 n). \]

### E.3 Proof of Theorem A.7

We first prove a weaker version of Theorem A.7 (Theorem E.6) and at the end of this section, we explain how our proof for Theorem E.6 can be adapted to work for Theorem A.7.

**Theorem E.6.** Suppose \( m = n + \lambda n \) for any positive \( \lambda \leq 0.008 \). Then, \( \lim_{n \to \infty} \frac{\mathbb{E}[S_i(\mu_{STB})]}{\mathbb{E}[S_i(\mu_{MTB})]} > 1 \), where the expectations are taken over preferences and the tie-breaking rules.

To prove this theorem, we need the following lemmas, the proofs for which appear after the proof of the theorem.

**Lemma E.7.** Suppose \( m = n + \lambda n \). Then, under MTB we have

\[ \lim_{n \to \infty} \mathbb{E}_\pi [S_i(\mu_\pi)] \leq T(2T - 1) - K^2, \]
where \( K = (1 + \lambda) \log(1 + 1/\lambda) \) and \( T = \frac{2(1+\lambda)}{\lambda+1/(1+1)} \).

**Lemma E.8.** Suppose \( m = n + \lambda n \). Then, under STB we have

\[ \mathbb{E}_\pi [S_i(\mu_\pi)] \geq \frac{2(1+\lambda)}{\lambda} - (1 + \lambda) \log(1 + 1/\lambda) - (1 + \lambda)^2 \log(1 + \frac{1}{\lambda})^2. \]

**Proof of Theorem E.6.** The proof is directly implied by Lemmas E.7 and E.8 below.

\[ \lim_{n \to \infty} \frac{\mathbb{E}[S_i(\mu_{STB})]}{\mathbb{E}[S_i(\mu_{MTB})]} \geq \frac{2(1+\lambda)}{\lambda} - (1 + \lambda) \log(1 + 1/\lambda) - (1 + \lambda)^2 \log(1 + \frac{1}{\lambda})^2}{T(2T - 1) - K^2}. \]

where \( K = (1 + \lambda) \log(1 + 1/\lambda) \) and \( T = \frac{2(1+\lambda)}{\lambda+1/(1+1)} \). For \( \lambda \leq 0.008 \), RHS of the above inequality is strictly greater than one.

Next, we prove the two lemmas that we used in the proof of this theorem. To simplify algebraic calculations, we use the notions \( \approx, \geq \) which respectively mean equality and inequality up to vanishingly small terms.
Proof of Lemma E.7. We use Lemma A.4, by which the expected social inequity and the expected variance of the rank of a fixed student are equal. So, to prove the lemma, we fix a student \( s \) and show that

\[
\lim_{n \to \infty} \mathbb{E}_{\pi(s') : s' \in S, s' \neq s} \left[ \text{Var} \left[ r_s \right] \right] \leq T(2T - 1) - K^2. \tag{34}
\]

We prove (34) in 2 Steps. In Step 1, we show that the variance of the rank of student \( s \) in the student-proposing DA is approximately equal to the variance of its rank in the school-proposing DA. Then, in Steps 2, we provide an upper bound \( T(2T - 1) - K^2 \) on the variance of rank in the school-proposing DA. Steps 1,2 then together will imply that (34) holds.

To prove the lemma, it remains to prove each of the steps separately.

**Step 1.** This step is identical to Step 1 in the proof of Part 3 of Lemma A.6, which was presented in Section E.2.4).

**Step 2.** This Step is similar to Step 1 in the proof of Lemma E.4.

Since the expected social inequity and the expected variance of the rank of a fixed student are equal by Lemma A.4, in this step we use the latter notion. We will switch to the former notion in Step 2. We are interested in providing an upper bound on \( \mathbb{E} \left[ (r_s - r)^2 \right] \), where \( r_s \) is a random variable denoting the rank for student \( s \) and \( r = \mathcal{A} r(\eta) \) (note that \( r \) is also equal to the average rank of \( s \), conditioned on being assigned). Since \( \mathbb{E} \left[ (r_s - r)^2 \right] = \mathbb{E} \left[ r_s^2 \right] - r^2 \), we can instead provide an upper bound on the RHS of the equality.

Fix an arbitrary small constant \( \epsilon > 0 \). Let \( E_s \) denote the event in which student \( s \) receives at least \( (1 - \epsilon)\kappa \), where \( \kappa = \frac{n}{2(1 + K)} + \frac{\lambda n}{2} \). (recall that \( K = (1 + \lambda) \log(1 + 1/\lambda) \)) Therefore

\[
\mathbb{E} \left[ r_s^2 \right] \leq \mathbb{P} \left[ E_s \right] \cdot \mathbb{E} \left[ r_s^2 \mid E_s \right] + (1 - \mathbb{P} \left[ E_s \right]) \cdot (n + \lambda n)^2. \tag{35}
\]

We proceed by providing an upper bound on the RHS of (35). Lemma C.2 implies \( E_s \) happens wvhp, and so, we can ignore the second term in the RHS of (35) since it is a lower order term. We provide an upper bound on the first term in the RHS of (35), i.e. on \( \mathbb{E} \left[ r_s^2 \mid E_s \right] \). If student \( s \) receives \( d_s \) proposals in school-proposing DA, then it chooses the best out of these \( d_s \) proposals, which means its rank is the first order statistic among the proposals that she had received. In Proposition E.1, we calculate \( \mathbb{E} \left[ r_s^2 \mid d_s \right] \) (which is the expected rank squared for \( s \) conditioned on
receiving $d_s$ proposals).

Using Proposition E.1 and (35) together we can write

$$
\mathbb{E} \left[ r_s^2 \mid E_s \right] \lesssim \mathbb{P} [ E_s ] \cdot \mathbb{E} \left[ r_s^2 \mid E_s \right] \\
\lesssim \left( \frac{n(1 + \lambda)}{\kappa} \right) \left( \frac{2n(1 + \lambda)}{\kappa} - 1 \right) \\
= \left( \frac{1 + \lambda}{\frac{1}{2(1 + K)} + \frac{\lambda}{2}} \right) \left( \frac{2(1 + \lambda)}{\frac{1}{2(1 + K)} + \frac{\lambda}{2}} - 1 \right) = T(2T - 1). \tag{37}
$$

Now, (37) implies that

$$
\lim_{n \to \infty} \mathbb{E}_\pi \left[ S_i(\eta_\pi) \right] = \lim_{n \to \infty} \mathbb{E}_\pi \left[ r_s^2 - r^2 \right] = T(2T - 1) - K^2. \tag{38}
$$

This completes Step 2.

Now we are ready to finish the proof of the lemma. Note that

$$
\mathbb{E}_\pi \left[ S_i(\mu_\pi) \right] \leq \mathbb{E}_\pi \left[ S_i(\mu_\pi) - S_i(\eta_\pi) \right] + \mathbb{E}_\pi \left[ S_i(\eta_\pi) \right] \\
\leq o(1) + \mathbb{E}_\pi \left[ S_i(\eta_\pi) \right] \tag{39} \\
\approx T(2T - 1) - K^2, \tag{40}
$$

where (39) follows from Step 1, and (40) follows from (38).

Next, we show how the proof works for Lemma E.8.

**Proof of Lemma E.8.** Suppose students indexed with respect to their priority number in STB, i.e. the student with the highest priority number is indexed 1, and the student with the lowest priority number is indexed with $n$. Fix a student $s$. Using Lemma A.4, we can write

$$
\mathbb{E}_\pi \left[ S_i(\mu_\pi) \right] = \text{Var}[r_s] = \mathbb{E} \left[ r_s^2 \right] - \mathbb{E} \left[ r_s \right]^2, \tag{41}
$$

where $r_s$ denotes the rank assigned to student $s$.

To provide a lower bound for (41), we lower bound $\mathbb{E} \left[ r_s^2 \right]$ and upper bound $\mathbb{E} \left[ r_s \right]^2$.

**Upper bound for $\mathbb{E} \left[ r_s \right]^2$.** First, we state the following claim.

**Claim E.9.** Suppose $m = (1 + \lambda)n$. Then, $\mathbb{E} \left[ r_s \right] \approx (1 + \lambda) \log(1 + \frac{1}{\lambda})$. 55
Proof. This follows from Ashlagi et al. (2017).

By Claim E.9, we have that
\[ \mathbb{E}[r_s]^2 \approx (1 + \lambda)^2 \log(1 + \frac{1}{\lambda})^2. \]

**Lower bound for** \( \mathbb{E}[r_s^2] \) First, see that
\[ \mathbb{E}[r_s^2] = \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[r_s^2 | s \text{ has priority } i + 1]. \]

Next, we state the following claim; its proof comes after the proof of this lemma.

**Claim E.10.** Suppose \( m = (1 + \lambda)n \). Then, \( \mathbb{E}[r_{k+1}^2] \geq \frac{2-p}{p} - O\left(\frac{\log^5 m}{m}\right), \) where \( p = \frac{m-k}{m} \).

Now, we use Claim E.10 to calculate an upper bound on the RHS of the above inequality:
\[ \mathbb{E}[r_s^2] \geq \frac{1}{n} \sum_{i=0}^{n-1} \frac{2}{\left(\frac{m-i}{m}\right)^2} - \frac{1}{\left(\frac{m-i}{m}\right)^2} - (1 + \lambda) \log(1 + 1/\lambda). \]

Now, using the inequality \( \frac{1}{x^2} \geq \frac{1}{x} - \frac{1}{x+1} \) we can write
\[ \mathbb{E}[r_s^2] \geq \frac{2m^2}{n} \cdot \sum_{i=0}^{n-1} \frac{1}{(m-i)^2} - (1 + \lambda) \log(1 + 1/\lambda). \]
\[ \geq \frac{2m^2}{n} \cdot \left( \frac{1}{\lambda n} - \frac{1}{(\lambda+1)n} \right) - (1 + \lambda) \log(1 + 1/\lambda). \]
\[ = \frac{2(1 + \lambda)}{\lambda} - (1 + \lambda) \log(1 + 1/\lambda). \]

By combining the above bounds, we can provide the promised lower bound on (41).
\[ \mathbb{E}[5i(\mu_x)] = \mathbb{E}[r_s^2] - \mathbb{E}[r_s]^2 \]
\[ \geq \frac{2(1 + \lambda)}{\lambda} - (1 + \lambda) \log(1 + 1/\lambda) - (1 + \lambda)^2 \log(1 + \frac{1}{\lambda})^2. \]
Proof of Claim E.10. A straight-forward calculation gives

\[ \mathbb{E}[r_{k+1}^2] = \sum_{j=0}^{k} (j+1)^2 \cdot \left(1 - \frac{k-j}{m-j}\right) \cdot \prod_{l=0}^{j-1} \frac{k-l}{m-l}. \]  

(42)

Define \( \ell = \min\{k, 5 \log_{1+\lambda} n\} \). To provide a lower bound, we only consider the first \( \ell \) summands in the above sum (the sum of the rest of the summands will be very small). Fix an arbitrary \( t \leq \ell \).

We provide a lower bound for the summand corresponding to \( j = t \). This summand contains the term \( \prod_{l=0}^{t-1} \frac{k-l}{m-l} \), which is at least

\[ \prod_{l=0}^{t-1} \frac{k-l}{m-l} \geq \prod_{l=0}^{t-1} \frac{k}{m} - \sum_{l=0}^{t-1} \left| \frac{k-l}{m-l} \right| \geq \prod_{l=0}^{t-1} \frac{k}{m} - \frac{\lambda t^2}{m-t} = \left(\frac{k}{m}\right)^t - \frac{\lambda t^2}{2m}. \]

Now, using the above inequality, we provide the following upper bound on (42):

\[ \mathbb{E}[r_{k+1}^2] \geq \left( \sum_{j=0}^{\ell} (j+1)^2 \cdot \left(1 - \frac{k}{m}\right) \left(\frac{k}{m}\right)^j \right) - \frac{\lambda \ell^2}{2m}. \]

(43)

We are almost done. In the RHS of (43), we bound the first term from below by

\[ \sum_{j=0}^{\ell} (j+1)^2 \cdot \left(1 - \frac{k}{m}\right) \left(\frac{k}{m}\right)^j \geq \frac{2-p}{p} - \lambda \ell^2 m. \]

which holds because of the following well-known fact: \( \mathbb{E}[Z^2] = \frac{2-q}{q} \) where \( Z \) is a geometric random variable with success probability \( q \). Using the above bound, we can rewrite (43) as

\[ \mathbb{E}[r_{k+1}^2] \geq \frac{2-p}{p} - O\left(\frac{\log^5 m}{m}\right), \]

which completes the proof.

E.4 Proof Sketch for Theorem A.7

Finally, we describe how proof of Theorem E.6 can be adapted to prove Theorem A.7. The main difference is in Lemma C.2. By proving a stronger version of Lemma C.2, the same proof would work for \( \lambda > 0 \). Some of the less important details are omitted from this proof.
We define the stronger version of Lemma C.2 simply by using the variable
\[
\kappa' = \left( \frac{n}{1 + K} + \lambda n \right) \cdot \left( 1 - \frac{1}{(1 + K)(1 + \lambda)} \cdot \frac{1}{2} \right)
\]
instead of a variable \( \kappa \). Replacing \( \kappa \) with \( \kappa' \) in the lemma statement would give the stronger version of the lemma. To show why the stronger version holds, we need to consider again the coupling \((DA, B)\) which we defined in the proof of Lemma C.2. There, for each successful coin-flip (a proposal made to \( s \)), we removed \( n \) coins. However, instead of doing that, here we remove \( n - y \) coins, where \( y \) is the number of proposals made by the proposer so far. Everything else in the coupling remains the same, e.g. the number of coins that we flip will remain \( 2n\kappa(1 - \delta) \). We will follow the same proof that we gave for Lemma C.2, with some adjustments. We sketch the proof below.

Let \( X \) be a random variable that denotes the total number of successful coin flips in the coupling. Our goal is showing that \( X \geq \kappa'(1 - \delta) \) holds wvhp.

**Claim E.11.** Wvhp, \( X \geq \kappa'(1 - \delta) \).

First, we verify that the lemma is proved by the above claim, and after that we prove the claim itself. To prove the lemma, we follow the proof of Lemma E.7 by rewriting (36) and (37) as follows. Let \( E_s \) denote the event at which \( s \) receives at least \( \kappa'(1 - \delta) \) proposals. Then,

\[
\mathbb{E} [r_s^2 | E_s] \leq \mathbb{P} [E_s] \cdot \mathbb{E} [r_s^2 | E_s] \\
\leq \frac{n(1 + \lambda)}{\kappa'} \cdot \left( \frac{2n(1 + \lambda)}{\kappa'} - 1 \right) \tag{44}
\]

Now, (44) implies that

\[
\lim_{n \to \infty} \mathbb{E}_{\pi} [S_i(\eta_{\pi})] = \lim_{n \to \infty} \mathbb{E}_{\pi} [r_s^2 - r^2] \leq \frac{n(1 + \lambda)}{\kappa'} \cdot \left( \frac{2n(1 + \lambda)}{\kappa'} - 1 \right) - K^2. \tag{45}
\]

Note that (45) is an improved upper bound. On the other hand, as we showed in Step 1 of the proof of Lemma E.7,

\[
\mathbb{E}_{\pi} [S_i(\mu_{\pi})] \approx \mathbb{E}_{\pi} [S_i(\eta_{\pi})].
\]

Consequently,

\[
\lim_{n \to \infty} \frac{\mathbb{E} [S_i(\mu_{\pi})]}{\mathbb{E} [S_i(\mu_{MT})]} \geq \frac{2(1 + \lambda)}{\lambda} - (1 + \lambda) \log(1 + 1/\lambda) - (1 + \lambda)^2 \log(1 + \frac{1}{\lambda})^2 \cdot \left( \frac{n(1 + \lambda)}{\kappa'} \cdot \left( \frac{2n(1 + \lambda)}{\kappa'} - 1 \right) - K^2 \right). \tag{58}
\]
where $K = (1 + \lambda) \log(1 + 1/\lambda)$. The RHS of the above inequality is strictly greater than one for any positive constant $\lambda \leq 0.01$. (Note that the RHS is only a function of $\lambda$) This would prove the lemma. It remains to prove Claim E.11.

First, we will argue that the claim holds in expectation, i.e. $\mathbb{E}[X] \geq \kappa'(1 - \delta)$. Recall that in the (new) coupling, after each successful coin-flip, i.e. a proposal made to $s$ by a school $c$, only $z_c$ coins are removed where $z_c = n - y_c$ and $y_c$ is the number of proposals that $c$ has made so far. Let $d_c$ be the total number of proposals made by school $c$. Also, let $F_c$ denote the event in which school $c$ makes a proposal to $s$. Conditioning on school $c$ making exactly $d_c$ proposals, we get

$$\mathbb{E}[y_c|d_c, F_c] = \frac{d_c + 1}{2},$$

which holds for any arbitrary school $c \in C$. This holds simply because we can relabel the students (using a consistent permutation of the labels), without changing the student-optimal matching (up to relabeling). This equality, together with

$$\mathbb{E}[y_c|F_c] \approx \frac{n}{2(1 + \lambda)} \cdot \left(\frac{1}{1 + K} + \lambda\right),$$

(46)

which follows from Ashlagi et al. (2017)) imply

$$\mathbb{E}[y_c|F_c] \approx \frac{n}{2(1 + \lambda)} \cdot \left(\frac{1}{1 + K} + \lambda\right).$$

Now, since all of the $2n\kappa(1 - \delta)$ coins will be flipped wvhp, the following holds wvhp as well:

$$\mathbb{E}[X] \cdot n + \mathbb{E}[X] \cdot (n - \mathbb{E}[y_c|F_c]) \approx 2n\kappa,$$

(47)

$$\implies \mathbb{E}[X] \cdot (2n - \mathbb{E}[y_c|F_c]) \approx 2n\kappa,$$

$$\implies \mathbb{E}[X] \approx \frac{2n\kappa}{2n - \mathbb{E}[y_c|F_c]}$$

$$= \frac{2\kappa}{2 - \mathbb{E}[y_c|F_c]/n} = \frac{n}{2(1 + K)} + \lambda n$$

(48)

where (47) holds since, on average, for any $n$ unsuccessful coin flips, we have 1 successful one, which results in removal of $\mathbb{E}[z_c|F_c]$ coins in expectation, and also, (48) holds by (46). So, the weaker version of Claim E.11 that we mentioned holds, i.e. when wvhp is replaced with expectation. Following the same approach, we can prove Claim E.11. We explain the high-level idea here. Note
that if the random variables \( \{y_c\} \) were known to be independent, we could simply apply the Chernoff bound, which would imply that the sum \( \sum_c y_c \) taken over all \( c \) that propose to \( s \) is concentrated around its mean, \( X \cdot \mathbb{E}[y_1|F_1] \). This would let us write a stronger version of (47) (which holds w.v.h.p, and not in expectation), which then proves Claim E.11. Although \( \{y_c\} \) are not independent, they are “almost” independent, roughly speaking, because preferences of schools are constructed independently. A careful treatment of these dependencies let us write the same concentration bounds. We omit the details.

F A continuum model with aligned preferences

Section 3.3 sketched the main insights from our theory using a continuum model with a mass of \( N \) students and 2 schools, in which each student has uniform preferences over schools. Identical arguments hold true for any finite number of schools with the same capacities (i.e, when the market is over-demanded the rank distribution under STB stochastically dominates the rank distribution under MTB).

Consider next a continuum model with a mass of \( N \) students and \( m \) schools with identical capacities but now students’ preferences are based on a symmetric multinomial-logit discrete choice model. Each school \( c \) has a quality factor \( \mu_c > 0 \) and each student’s preference list is generated as follows. A student’s first choice is drawn proportionally to the school quality factors,\(^{33}\) her second choice is drawn in a similar way after her top choice is removed and so on.

We say that a school is popular if the number of students that rank it as their first choice is larger than the school’s capacity. That is, we say a school \( c \) is popular if \( \alpha_c \geq 1 \) (see Section 2 for the definition of \( \alpha_c \)). Observe that in the logit model above, \( \alpha_c \) equals \( \mu_c/q_c \). In the case that \( \mu_c \) are unknown, a refined definition of \( \alpha_c \) could be the empirical version adapted from Section 4.1; under that definition \( \alpha_c \) becomes an unbiased estimator for \( \mu_c/q_c \).

**Example 2.** Consider a school choice problem with \( m = 2 \) schools with quality factors \( \mu_1 \geq \mu_2 \). We argue that STB stochastically dominates MTB regardless of how popular schools are. Observe that every student that is assigned to school 1 under STB obtained her first choice.\(^{34}\) Moreover the same number of students are rejected from school 1 under both STB and MTB. However, if the market is over-demanded, a larger fraction of these students will be able to obtain a seat in school

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\(^{33}\)That is, her top choice is school \( c \) with probability \( \frac{\mu_c}{\sum_{c'} \mu_{c'}} \).

\(^{34}\)Students rejected from school 2 have a worse lottery number than all students that have not been rejected after the first round of DA in school 1.
2 under MTB than under STB. Note that the rank distributions are similar when the market is under-demanded.

Already with 3 schools things are more interesting as the following example illustrate.

**Example 3.** Consider a school choice problem with $m = 3$ schools with quality factors $\mu_1 \geq \mu_2 > \mu_3$. Suppose schools 1 and 2 are popular but the market is under-demanded so school 3 will remain under capacitated. We argue that STB stochastically dominates MTB in the set of popular schools $P = \{1, 2\}$ but not in all schools. Consider running the DA algorithm as follows: in the first round all students apply to their first choice and schools reject students beyond their capacity, in the second round all rejected students apply to their second choice and so forth.

Note that after the first round, the same number of students are rejected from school 1 under both STB and MTB. Moreover, the same number of students apply to school 2 in the second round (by assumption there are no such students from school 3) and more of these students will be accepted to school 2 under MTB than under STB. Therefore, after the second round of DA STB stochastically dominates MTB in $P$. But observe that under STB the assignment in $P$ is almost surely finalized after the second round of DA, while under MTB the rank distribution for students assigned to $P$ only worsens.

Next we show that MTB stochastically dominates STB in school 3. First note that all students who rank school 3 as their first choice will be assigned to it. The same holds true for all rejected students from the first round of DA who rank school 3 as their second choice. Note that STB is finalized almost surely after the third round, in which only students who rank school 3 as their third choice apply to it. Under MTB, however, DA will proceed to more rounds and more students who rank school 3 as their second choice will be assigned to it. This completes the argument since the number of students that are assigned to school 3 is the same under MTB and STB.

We mention two open problems we found interesting. Consider the continuum model with multinomial logit preferences described above. First, we believe that the rank distribution in popular schools under STB stochastically dominates the distribution under MTB for any number of schools $m \geq 2$ and any quality factors $\mu_1, \mu_2, \ldots, \mu_m$. We further believe that STB stochastically dominates MTB not just in the set of popular schools but in every popular school separately.
G A hybrid tie-breaking rule

Consider a school district that uses MTB and let $P$ denote the set of “popular” schools in this city. For instance, the city of Amsterdam first adopted MTB, and as De Haan et al. (2015) note, there were 4 “over-demanded” schools. Theorem 3.2 suggests that there will be many Pareto improving pairs within popular schools, which is consistent with our experiments. The school district may instead adopt a hybrid tie-breaking rule, in which all popular schools use the same lottery and each non-popular school uses an independent lottery. Theorem 3.2 implies that in a perfectly tiered market, using a hybrid tie-breaking rule eliminates Pareto improving pairs. Also the rank distribution in popular schools under the hybrid tie-breaking rule will stochastically dominate the one under MTB.\(^{35}\)

In NYC the market is not perfectly tiered. Therefore, to test the hybrid tie-breaking rule, we select heuristically a set $P$ of popular schools. (A thorough study on classifying schools based on their popularity is an essential prerequisite of using the hybrid rule in practice.) We let $P = P_\alpha$ for $\alpha = 2$, which contains about 12% of the schools (where as, e.g., $\alpha = 1$ would contain more than 33% of the schools). The choice of $\alpha = 2$ is a conservative choice, made to ensure that the schools in $P$ are popular enough.

Figures 7a and 7b report the average cumulative ranks over 50 iterations under STB, MTB, and the hybrid tie-breaking rule (HTB). Observe that the rank distribution in popular schools under HTB stochastically dominates the rank distribution under MTB, while these rank distributions in non-popular schools almost coincide.

Naturally, the lower the popularity threshold, the “closer” the rank distribution under HTB is to the rank distribution under STB in both popular and non-popular schools (plots are omitted). In particular HTB assigns more students to their top choices than MTB, and less students to their low choices than STB. In other words, HTB is not a Pareto-improvement over any of the other two rules in our experiments above. This is a consequence of not having a perfectly tiered market. Few examples are given in the next section to provide some intuition.

G.1 Intuition

As discussed above, in a school choice problem with two perfect tiers, HTB will result in a (ex ante) Pareto improvement over MTB and further coincide with STB in popular schools. The following

\(^{35}\)See online Appendix for more details and simulations.
couple of examples provide intuition for why these predictions need not hold when schools cannot be perfectly tiered. For simplicity we illustrate these argument using continuum models.

Next we provide an example, in which STB stochastically dominates HTB in popular schools.

**Example 4.** There is a continuum of students of mass 3.5, and 3 schools $c_1, c_2$ and $c_3$, each with one unit capacity. There are two types of students, $a$ and $b$, whose masses are 3 and 0.5, respectively. Type a student prefer both $c_1$ and school $c_2$ to school $c_3$ and type $b$ students rank school $c_3$ first. All students rank $c_1$ and $c_2$ uniformly at random.

Consider first DA under STB. After the first round of DA, all students that are rejected from their first choice (mass of 0.5 from school $c_1$ and a mass of 0.5 from school $c_2$) will not get accepted to their second choice as well almost surely. These students apply to their 3rd choice, school $c_3$, and all rejected students from this school will remain unassigned since even type $b$ students will not obtain their second or third choice due their lottery numbers.

Consider next DA under HTB. The first two rounds of DA are similar to the first two rounds under STB. However, some fraction of rejected students from school $c_3$ are students of type $b$ who rank school $c_3$ first. A fraction of these students will be accepted to their second or third choice since their lottery number is in $c_1$ and $c_2$ is different than the one in $c_3$. So STB outperforms HTB in popular schools.

Next we provide an example, in which MTB stochastically dominates HTB in non-popular schools.

**Example 5.** Consider the same school choice problem in example 3. That is, there are 3 schools
such that overall there are sufficiently many seats for students, preferences follow a multinomial logit (MNL) model where schools $c_1$ and $c_2$ are popular and school $c_3$ is non-popular. Observe from that example that the DA algorithm under HTB and STB can be coupled so that the assignments are identical. But as we observed, MTB stochastically dominates STB in school $c_3$ and therefore it also stochastically dominates HTB in that school.

There are other examples, in which students have MNL preferences and the rank distribution under HTB lies strictly in between the rank distributions under MTB and STB both in popular and non-popular schools. In particular, HTB assigns more students to their top choices compared to MTB, and less students to their low choices compared to STB, similar to the patterns in the computation experiments above.\textsuperscript{36} The intuition is similar the intuition behind examples 4 and 5.

\textsuperscript{36}One example we confirmed this using simulations is the following. There are 4300 students, 90 schools, each with 90 seats. 10 schools have a quality factor of 8 and all other schools have a quality factor or 1 (see Appendix F for MNL preferences).