

Hamiltonian operators and free fermionic lattice models

1) Introduction

2) Heisenberg algebra representations

3) Identities from Hamiltonians

4) Lattice models and Hamiltonian operators

5) q -Fock space and lattice models with charge

1) Introduction

Goal: use Hamiltonian operators to prove identities about symmetric functions

Identity	Hamiltonian analogue
Cauchy	Commutation of Hamiltonians
Pieri	Action of a certain operator
Jacobi-Trudi	Wick's theorem (only for certain Hamiltonians)
Branching rule	Branching rule (adjoint to Pieri rule)

Eventually: we'll explore which lattice models have Hamiltonians, and try to prove identities about them

2) Heisenberg algebra representations

Heisenberg algebra: $\mathcal{H} = \langle B_k \mid k \in \mathbb{Z} \setminus \{0\} \rangle$

$$[B_k, B_l] = \begin{cases} k, & l = -k \\ 0, & \text{else} \end{cases}$$

Uniqueness of representations

(see e.g. Kac - Raina Bombay lectures):

Let v_1, v_2 be nonzero highest weight vectors v_1, v_2 . Then there exists a unique isomorphism of \mathcal{H} -modules $\phi: V_1 \rightarrow V_2$ s.t. $\phi(v_1) = v_2$.

Two important reps:

1) Bosonic Fock space:

Space: algebra Λ of symmetric functions

$$\text{Action: } B_k \mapsto \begin{cases} P_{-k}, & \text{if } k \leq -1, \\ k \frac{\partial}{\partial P_k}, & \text{if } k \geq 1, \end{cases}$$

where P_k is the k -th power sum symmetric function

$$P_k = X_1^k + X_2^k + \dots + X_n^k$$

Highest wt vector: $\underline{1}$

By scaling the generators, and abstracting the power sum symmetric functions:

$P_k \mapsto S_k$, where s_1, s_2, \dots is an algebraically independent of symmetric functions, we can write instead

$$B_k \mapsto \begin{cases} s_{-k}, & \text{if } k \leq -1, \\ k \frac{\partial}{\partial S_k}, & \text{if } k \geq 1, \end{cases}$$

2) Fermionic Fock space

Clifford algebra: $A = \langle \psi_i^*, \psi_i \mid i \in \mathbb{Z} - \frac{1}{2} \rangle$

$$\text{Relations: } \psi_i \psi_j + \psi_j \psi_i = 0$$

$$\psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0$$

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{i,j}$$

$$W := \bigoplus_{i \in \mathbb{Z}} \mathbb{C} \psi_i^* \oplus \bigoplus_{i \in \mathbb{Z}} \mathbb{C} \psi_i$$

$$W_{\text{ann}} := \bigoplus_{i < 0} \mathbb{C} \psi_i^* \oplus \bigoplus_{i > 0} \mathbb{C} \psi_i$$

$$W_{\text{cr}} := \bigoplus_{i > 0} \mathbb{C} \psi_i^* \oplus \bigoplus_{i < 0} \mathbb{C} \psi_i$$

$$\bar{\mathcal{F}} := A / A W_{\text{ann}} \quad \bar{\mathcal{F}}^* := A W_{\text{cr}} \setminus A$$

Cyclic A modules, with generators

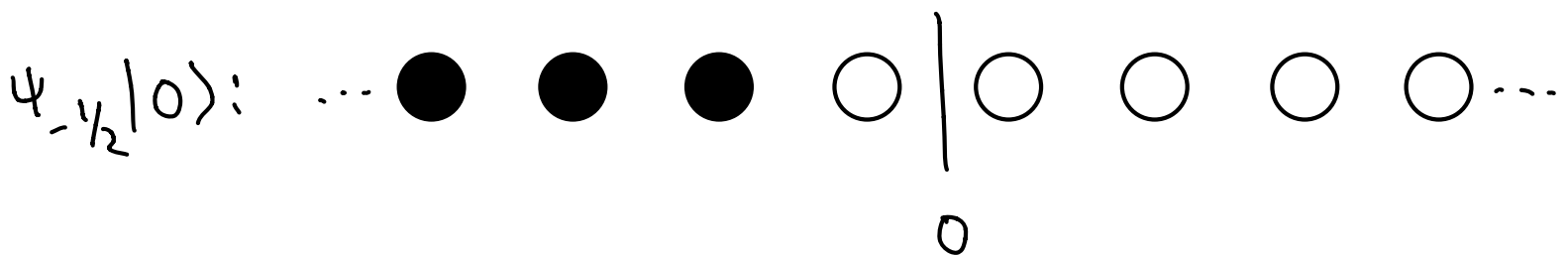
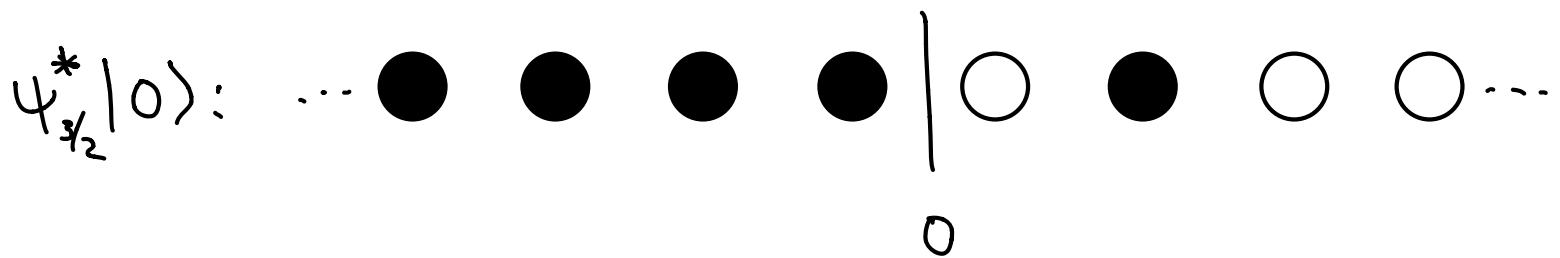
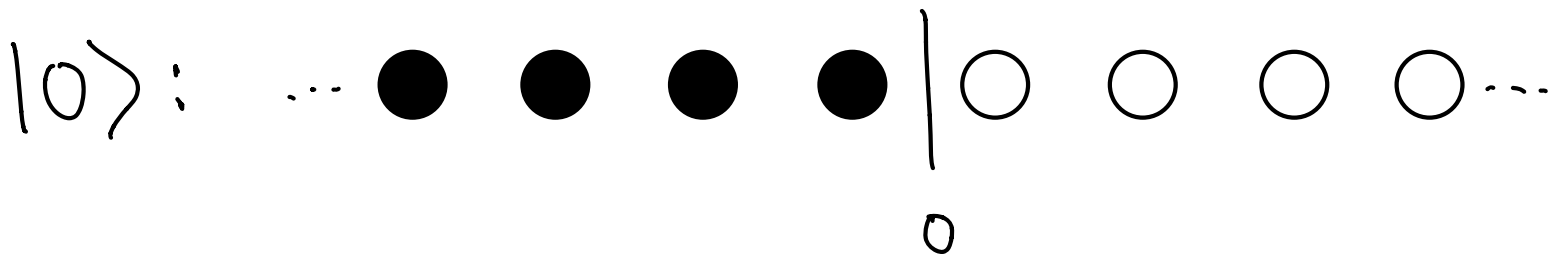
$$|0\rangle \quad \text{and} \quad \langle 0|$$

Symm. bilinear pairing:

$$\langle 0|a \otimes_A b|0\rangle$$

Dirac sea: each basis vector of \mathcal{F} is a state of particles

e.g.



$$\text{Let } \text{cyl}(\infty) := \left\{ \sum_{i,j} a_{i,j} : \psi_i^* \psi_j : \mid \exists N \text{ s.t. } a_{i,j} = 0 \text{ if } |i-j| > N \right.$$

"normal ordering"

\mathcal{F} decomposes into $\text{cyl}(\infty)$ -irreps

$$\mathcal{F} = \bigoplus_{l \in \mathbb{Z}} \mathcal{F}_l$$

where \mathcal{F}_l has highest weight vector

$$|l\rangle = \begin{cases} \psi_{l+\frac{1}{2}} \dots \psi_{-\frac{1}{2}} |0\rangle & , \text{ if } l < 0 \\ |0\rangle & , \text{ if } l = 0 \\ \psi_{l-\frac{1}{2}}^* \dots \psi_{\frac{1}{2}}^* |0\rangle & , \text{ if } l > 0 \end{cases}$$

$$|l\rangle : \dots \bullet \bullet \bullet \bullet \mid \circ \circ \circ \circ \dots$$

l

If λ is a strict partition with l parts:

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l),$$

then

$$|\lambda\rangle := \psi_{\lambda_1}^* \psi_{\lambda_2}^* \dots \psi_{\lambda_l}^* |0\rangle \in \mathcal{F}_l$$

$$\langle\lambda| := \langle 0| \psi_{\lambda_l} \psi_{\lambda_{l-1}} \dots \psi_{\lambda_1} \in \mathcal{F}_l^*$$

Note that $\langle\mu|\lambda\rangle = \begin{cases} 1, & \lambda = \mu \\ 0, & \text{else} \end{cases}$

Current operators:

$$J_n = \sum_{i \in \mathbb{Z} - \frac{1}{2}} : \psi_{i-n}^* \psi_i : \quad (\text{move one particle } n \text{ spaces left})$$

The current operators form a Heisenberg algebra:

$$[J_m, J_n] = m \delta_{m, -n}$$

$$H[s] = \sum_{n=1}^{\infty} s_n J_n, \quad e^{H[s]} = \sum_{k=0}^{\infty} \frac{1}{k!} H[s]^k$$

Boson-Fermion correspondence: Let $B_k = s_k J_k$

The map $a|0\rangle \mapsto \langle l | e^{H[s]} a|0\rangle$

is an isomorphism of \mathcal{H} -modules, In other words,

$$\langle l | e^{H[s]} B_k a|0\rangle = B_k \langle l | e^{H[s]} a|0\rangle$$

$$\text{and } \langle u+p | e^{H[p]} | \lambda+p \rangle = S_{\lambda/\mu}$$

Wick's Theorem:

$$\langle l | \psi_{i_1} \dots \psi_{i_n} e^H \psi_{j_1}^* \dots \psi_{j_n}^* | l \rangle$$

$$= \det_{1 \leq a, b \leq n} \langle l | \psi_{i_a} e^H \psi_{j_b}^* | l \rangle$$

$$\langle \mu | e^H | \lambda \rangle = \det_{1 \leq a, b \leq l} \langle 0 | \psi_{\mu_a} e^H \psi_{\lambda_b}^* | 0 \rangle$$

3) Identities from Hamiltonians

Now, let

$$H := H_+ := H_+[s] = \sum_{k \geq 1} s_k J_k$$

$$\text{where } s_k = \sum_{j=1}^n s_k^{(j)}$$

If $s_k^{(j)} = x_j^k$, then $s_k = p_k$

$$\text{Let } H_- := H_-[s] = \sum_{k \geq 1} s_{-k} J_k, \text{ where } s_{-k} = \sum_{j=1}^n s_{-k}^{(j)}$$

Cauchy identity

$$\text{Let } F_{\lambda/\mu} = \langle \lambda | e^{H_-} | \mu \rangle$$

$$G_{\lambda/\mu} = \langle \mu | e^{H_+} | \lambda \rangle$$

Then,

$$\begin{aligned}\sum_{\nu} G_{\nu/\lambda} F_{\nu/\mu} &= \sum_{\nu} \langle \lambda | e^{H_+} | \nu \rangle \langle \nu | e^{H_-} | \mu \rangle \\ &= \langle \lambda | e^{H_+} e^{H_-} | \mu \rangle \\ &= e^{[H_+, H_-]} \langle \lambda | e^{H_-} e^{H_+} | \mu \rangle \\ &= e^{[H_+, H_-]} \sum_{\nu} \langle \lambda | e^{H_-} | \nu \rangle \langle \nu | e^{H_+} | \mu \rangle \\ &= e^{[H_+, H_-]} \sum_{\nu} F_{\lambda/\nu} G_{\mu/\nu}\end{aligned}$$

Now, let $\eta_k := \langle 0 | e^H | (k) \rangle$

$$U_k = \sum_{\lambda \vdash k} z_{\lambda}^{-1} B_{\lambda_1} B_{\lambda_2} \cdots B_{\lambda_{\ell}},$$

where $z_{\lambda} = 1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! \cdots$

$$m_i(\lambda) = |\{j \mid \lambda_j = i\}|$$

Pieri rule:

$$\eta_{\lambda/\mu} G_{\lambda/\rho} = \sum_{\nu} \langle \mu | U_k | \lambda \rangle G_{\mu/\rho}$$

Jacobi-Trudi identity:

$$G_{\lambda/\mu} = \langle \mu | e^H | \lambda \rangle$$

$$= \det_{|s_a, b, s_n} \langle 0 | \psi_{\mu_a} e^H \psi_{\lambda_b}^* | 0 \rangle$$

$$= \det_{|s_a, b, s_n} \eta_{\lambda_b - \mu_a}$$

Branching rule:

$$\text{Let } \phi^{(j)} = \sum_{k \geq 0} s_k^{(j)} J_k \text{ so that } H_+ = \sum_{j=1}^n \phi^{(j)}$$

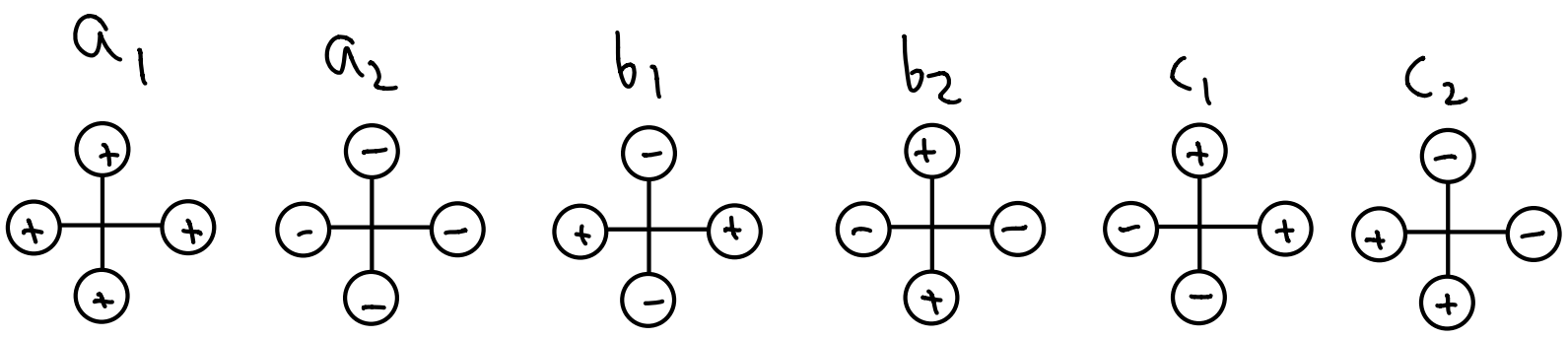
$$G_{\lambda/\mu} = \langle \mu | e^H | \lambda \rangle$$

$$= \langle \mu | e^{\phi^{(n)}} \dots e^{\phi^{(1)}} | \lambda \rangle$$

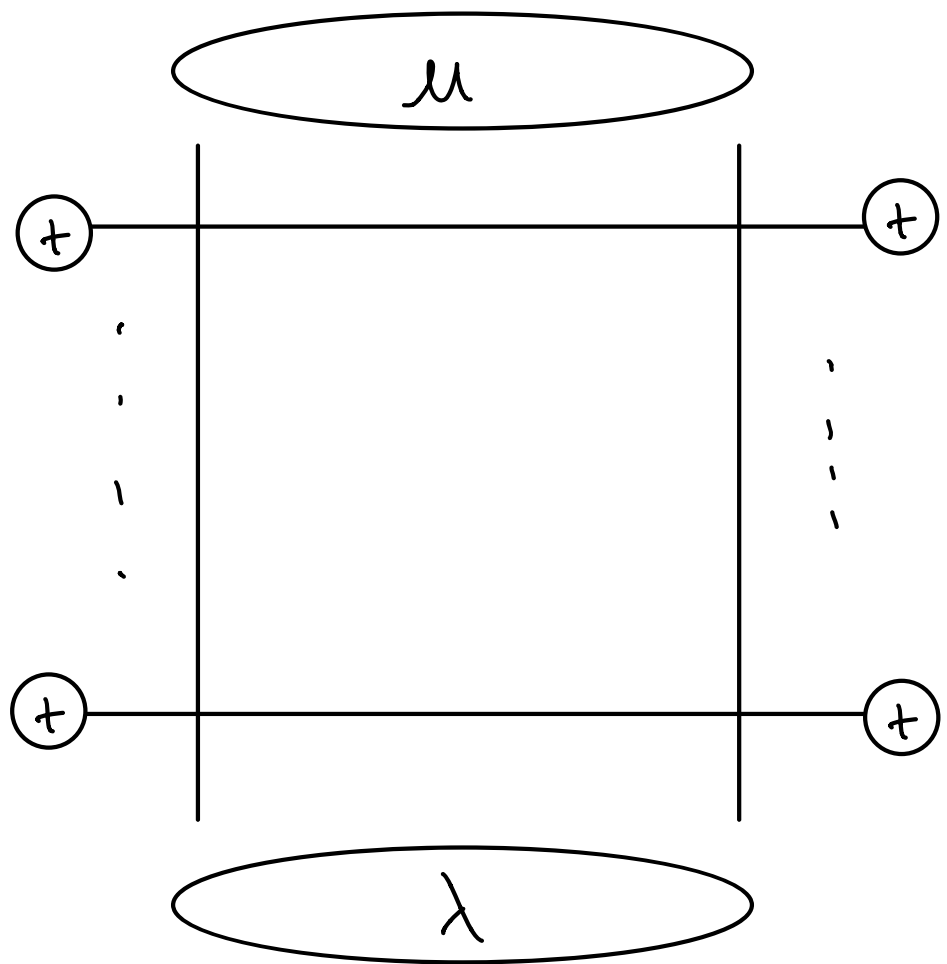
$$= \sum_{\nu} \langle \mu | e^{\phi^{(n)}} | \nu \rangle \langle \nu | e^{\phi^{(n-1)}} \dots e^{\phi^{(1)}} | \lambda \rangle$$

4) Lattice models and Hamiltonian operators

Let $G_{\lambda/\mu}$ be a lattice model with n rows, m columns ($m \geq \lambda_1, \mu_1$), and admissible vertices:



and boundary conditions:



$$(\ell(\lambda) = \ell(\mu) = \ell)$$

When do there exist parameters $s_k^{(j)}$ such that

$Z_{\lambda/\mu} := Z(G_{\lambda/\mu})$ corresponds to a

Hamiltonian operator?

Answer: when the Boltzmann weights are

free fermionic: $a_1 a_2 + b_1 b_2 = c_1 c_2$

$$\text{Let } s_k^{(j)} = \frac{1}{k} \left(\left(\frac{b_2^{(j)}}{a_1^{(j)}} \right)^k + (-1)^{k-1} \left(\frac{a_2^{(j)}}{b_1^{(j)}} \right)^k \right)$$

Then if $a_1 a_2 + b_1 b_2 = c_1 c_2$, we have:

$$Z_{\lambda/\mu} = \left(\prod_{i=1}^n a_1^{(i)^{m-l}} b_1^{(i)^l} \right) \cdot \langle \mu | e^H | \lambda \rangle$$

So free fermionic lattice models correspond to

Hamiltonians of free fermions...

Sketch of Proof: First treat 1-particle systems:
 $l(\lambda) = l(\mu) = 1$. We can prove that the result holds
 for these partitions precisely when $s_j^{(k)}$ is given
 by the expression above.

For multiple particles, do the $n=1$ case first, and
 reduce to the 1-particle case via Wick's
 theorem. The determinant coming from the Hamiltonian
 equals the one-row partition function precisely when
 the weights are free fermionic!

Takeaways:

$$1) \text{ Let } x_j := \frac{b_2^{(j)}}{a_1^{(j)}}, \quad y_j := -\frac{a_2^{(j)}}{b_1^{(j)}}$$

Then we have

$$s_k^{(j)} = \frac{1}{k} (x_j^k - y_j^k) = \frac{1}{k} P_k(x|y),$$

the superalgebra analogue of the power sum symmetric function



(Brubaker-Schultz showed \leftarrow)

2) We obtain Cauchy, Pieri, Jacobi-Trudi identities for supersymmetric Schur functions.

3) Edrei-Thoma theorem: A homomorphism $\Lambda \rightarrow \mathbb{R}$ is Schur-positive if and only if

$$P_1 \mapsto \gamma + \sum_j (x_j + y_j)$$

$$P_k \mapsto \sum_j (x_j^k + (-1)^{k-1} y_j^k), \quad k \geq 2,$$

where $\gamma, x_j, y_j \geq 0$

Corollary: Let $a_1^{(j)}, a_2^{(j)}, b_1^{(j)}, b_2^{(j)}, c_1^{(j)}, c_2^{(j)} \in \mathbb{R}$, and

$$\text{let } a_1^{(j)} a_2^{(j)} + b_1^{(j)} b_2^{(j)} = c_1^{(j)} c_2^{(j)}.$$

Then $Z_{\lambda/\mu} \geq 0$ for all λ, μ



$$\frac{b_2^{(j)}}{a_1^{(j)}} \geq 0, \quad \frac{a_2^{(j)}}{b_1^{(j)}} \geq 0 \quad \text{for all } 1 \leq j \leq n$$

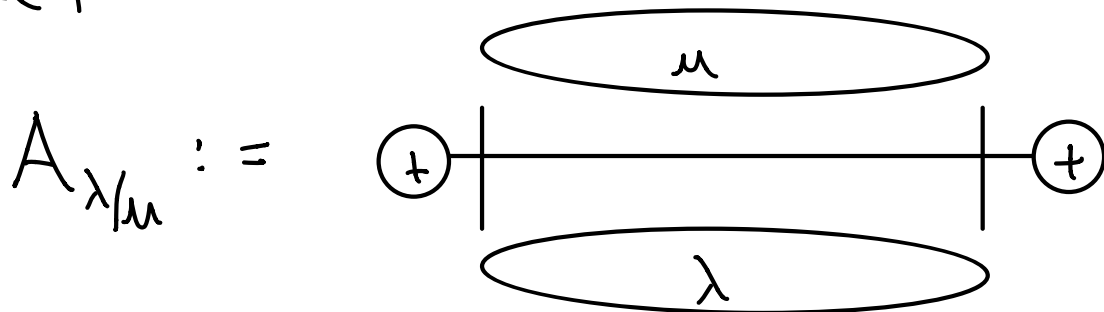
Question: can this be generalized to the non-free fermionic case? Is there a relationship between Δ and γ ?

Boundary conditions

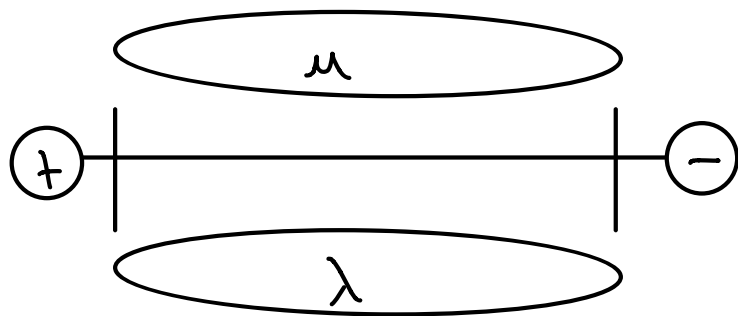
Two ways to deal with them

1) Use creation/deletion operators

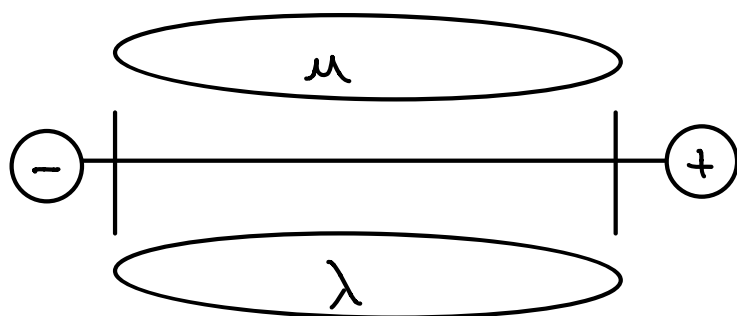
Let:



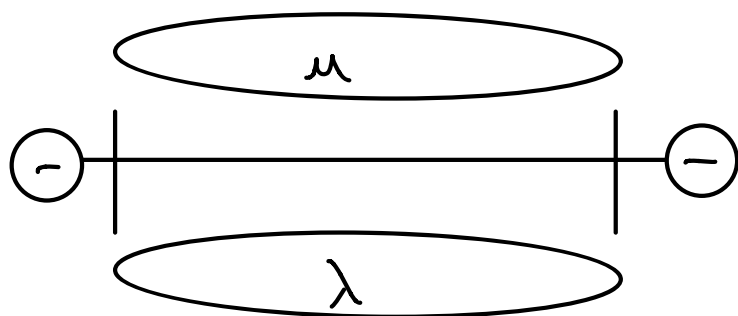
$$B_{\lambda/\mu} :=$$



$$C_{\lambda/\mu} :=$$



$$D_{\lambda/\mu} :=$$



Then if $A_{\lambda/\mu} = \langle \mu | e^\phi | \lambda \rangle$, then

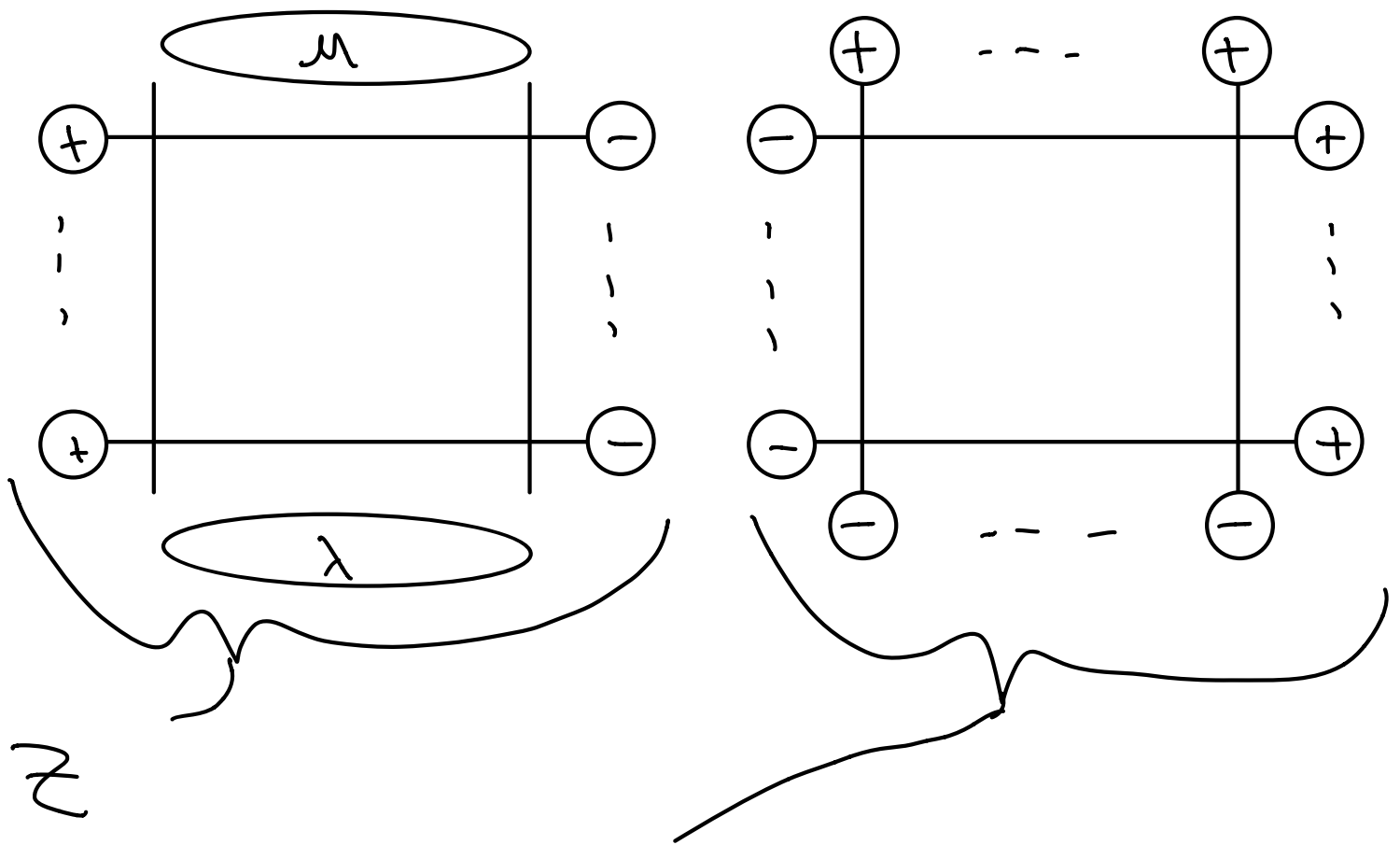
$$B_{\lambda/\mu} = \frac{1}{c_1} \langle \mu | e^\phi \psi_{m+1/2}^* | \lambda \rangle$$

$$C_{\lambda/\mu} = \frac{1}{c_2} \langle \mu | \psi_{-1/2} e^\phi | \lambda \rangle$$

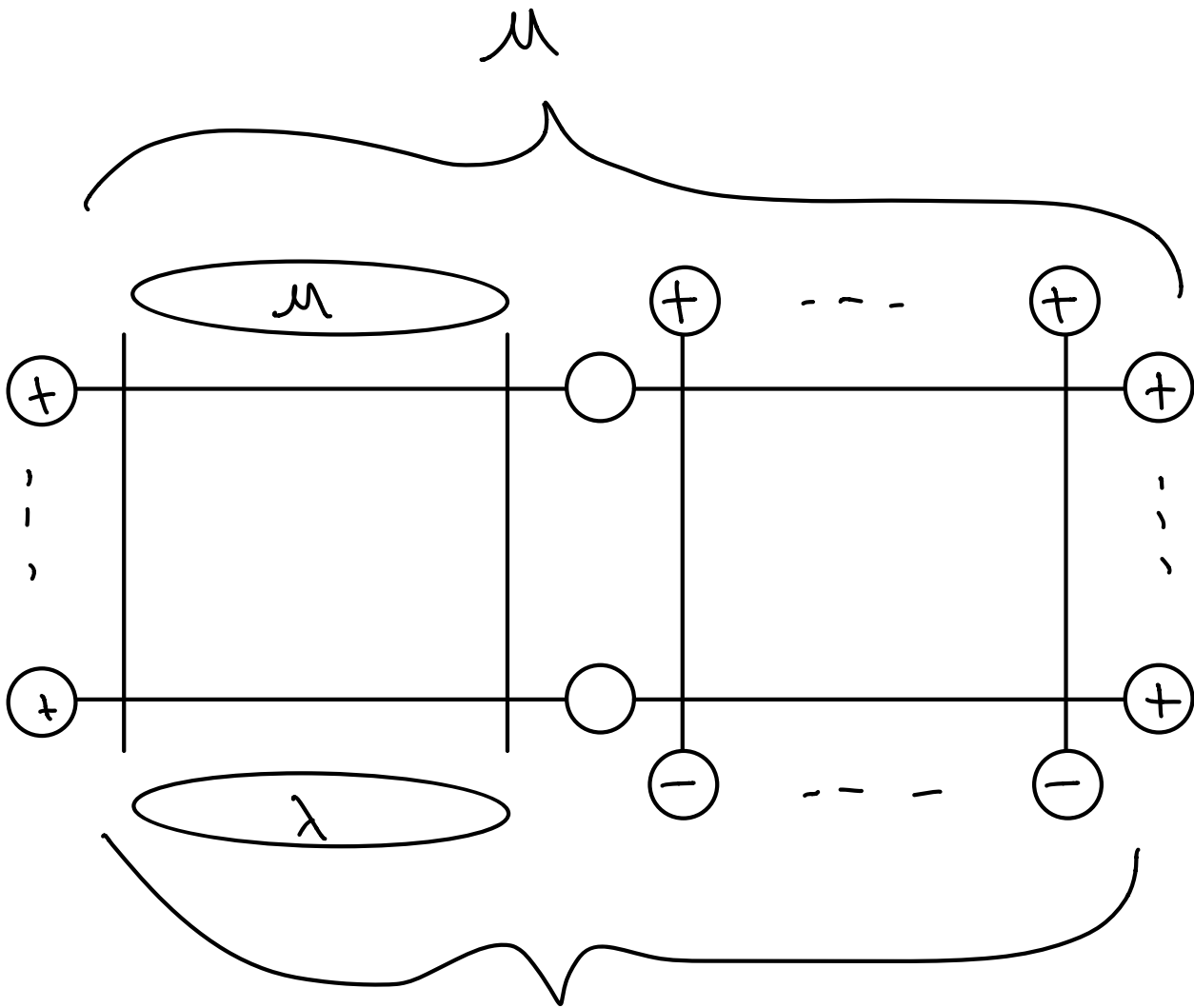
$$D_{\lambda/\mu} = \frac{1}{c_1 c_2} \langle \mu | \psi_{-1/2} e^\phi \psi_{m+1/2}^* | \lambda \rangle$$

"ghost vertices"

2) For "solid" boundary conditions, add on an extra block



$$\prod_{i=1}^n c_2^{(i)} \prod_{i < j} (a_1^{(i)} a_2^{(j)} + b_1^{(j)} b_2^{(i)})$$



$$\lambda' = (m+h, m+h-1, \dots, m+1, \lambda_1, \dots, \lambda_\ell)$$

$$Z = \frac{\left(\prod_{i=1}^n a_1^{(i)m-l} b_1^{(i)l+n} \right) \langle \mu | e^H | \lambda' \rangle}{\prod_{i=1}^n c_2^{(i)} \prod_{i < j} \left(a_1^{(i)} a_2^{(j)} + b_1^{(j)} b_2^{(i)} \right)}$$

5) q -Fock space and lattice models with charge

(Brubaker - Buciumas - Bump - Gustafsson)

KMS Fock space associated to $U_q(\mathfrak{sl}_n)$

$$\mathcal{F}_0 = \left\langle u_{i_m} \wedge u_{i_{m-1}} \wedge \dots \mid i_m = m \text{ for } m \ll 0 \right\rangle$$

$$u_l \wedge u_m = \begin{cases} -u_m \wedge u_l & , \quad \text{if } l \equiv m \pmod{n} \\ g(l-m)u_m \wedge u_l + \dots & , \quad \text{otherwise} \end{cases}$$

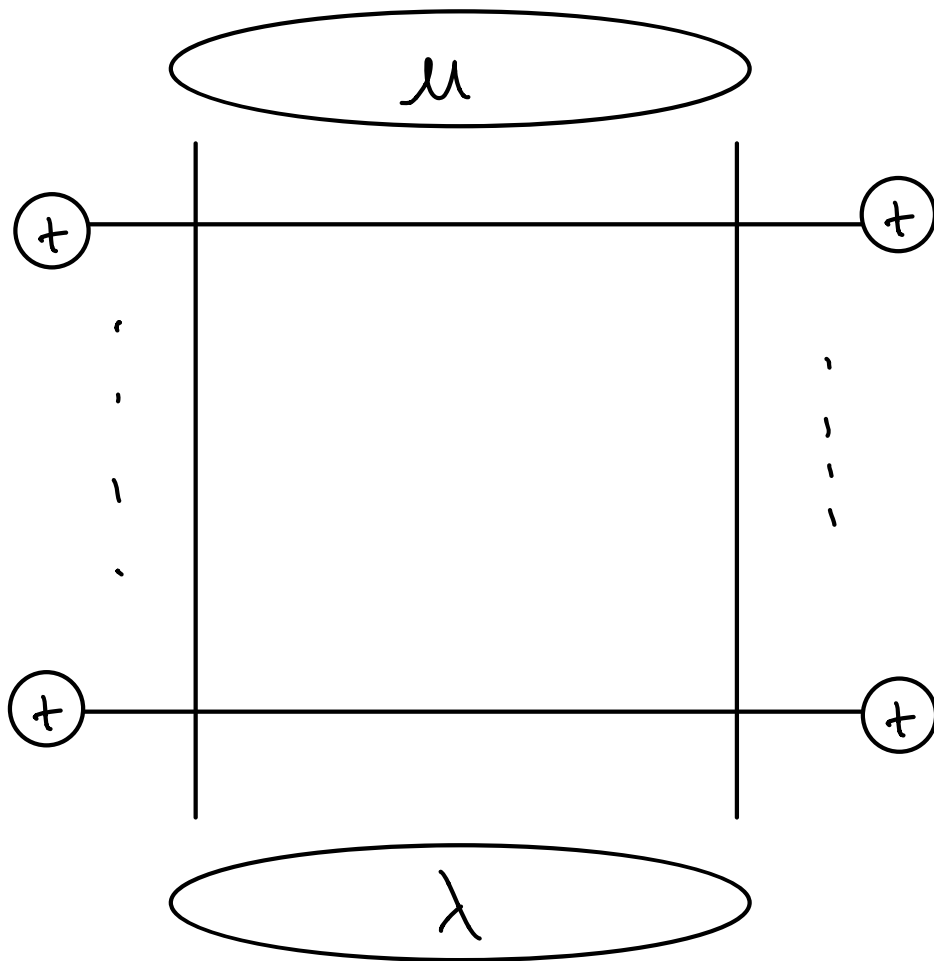
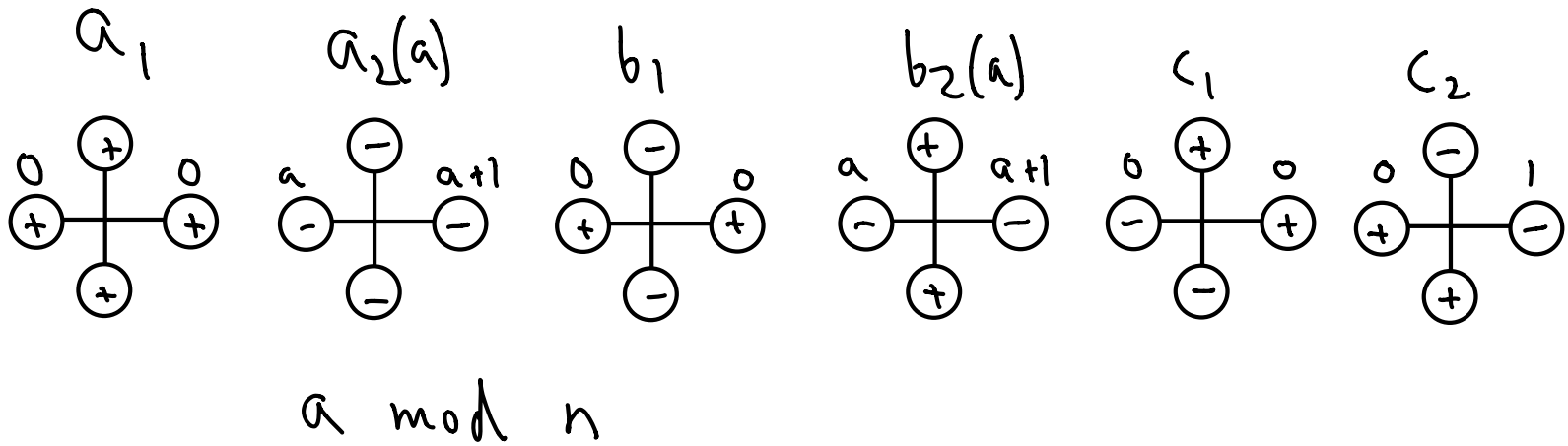
$$g(0) = -v \quad (v = -q^2)$$

$$g(a)g(-a) = v, \quad \text{if } a \not\equiv 0 \pmod{n}$$

$$\mathcal{J}_k(u_{i_m} \wedge u_{i_{m-1}} \wedge \dots)$$

$$= u_{i_{m-h}} \wedge \dots + u_{i_m} \wedge u_{i_{m-1}-h} \wedge \dots + \dots$$

Consider a 6-vertex model with charge:



When does this lattice model match a q -Fock space Hamiltonian?

(Note: we will let v vary. However, the q -Fock space may become degenerate if $v=0$ or a root of $\mathbb{1}$.)

Answer: when the Boltzmann weights satisfy the following conditions:

- $a_1 a_2(0) + b_1 b_2(0) - c_1 c_2 = 0$

- $\frac{a_2(0)}{b_2(0)} \neq 0$ or a root of $\mathbb{1}$

- For any $1 \leq a \leq n-1$, $\frac{a_2(0)}{b_2(0)} = -\frac{a_2(a) a_2(-a)}{b_2(a) b_2(-a)}$

We'll call this the "generalized free fermion condition".

$$\text{Set } s_k^j = \frac{1}{k} \left(\prod_a \left(\frac{b_2^{(j)}(a)}{a_1^{(j)}} \right)^k + (-1)^{k-1} \prod_a \left(\frac{a_2^{(j)}(a)}{b_1^{(j)}} \right)^k \right)$$

Then,

$$Z_{\lambda/\mu} = \langle \mu | e^H | \lambda \rangle$$

and we obtain the supersymmetric LLT polynomials:

$$Z_{\lambda/\mu} = G_{\lambda/\mu}(x|y)$$

$$\text{where } x_j = \prod_a \frac{b_2^{(j)}(a)}{a_1^{(j)}}, \quad y_j = \prod_a \frac{a_2^{(j)}(a)}{b_1^{(j)}}$$

Note that we cannot have $y = 0$

since this would entail setting $v = 0$,

So we cannot obtain the regular LLT polynomial

$$G_{\lambda/\mu}(x) := G_{\lambda/\mu}(x|0)$$

in this way.

Identities: can do Cauchy, Pieri, but not Jacobi-Trudi.

Boundary conditions: can do method 1, but not method 2.