Lattice models as Poincaré pairings
(aka "everything is geometry"
aka "yet another interpretation of
the lattice models for Whittaker
functions")

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Outline

1. Bird's eye view:
   bases of integrable systems = bases in p-adic representation = bases in equivariant cohomology

2. Some equivariant geometry

3. Example: Frozen pipes model for Schubert polynomials
4 Example: Colored & uncolored Whittaker lattice models

\text{dwahou}

\begin{align*}
\lambda + \rho \\
\langle \lambda + \rho, \tilde{\mathcal{M}}^\vee(Y(\omega)^0) \rangle
\end{align*}

\text{Spherical}

\begin{align*}
\lambda + \rho \\
\langle \lambda + \rho, \tilde{\mathcal{M}}(Y(1)) \rangle
\end{align*}

this vector $\leftrightarrow$ Poincaré dual of $\tilde{\mathcal{M}}^\vee(Y(\omega)^0) / \tilde{\mathcal{M}}(Y(1))$

5 Future directions?
1. Bird's eye view of Schubert calculus

- One set of spectral parameters
- Five vertex models
- Over 6 vertex models
- Two sets of spectral parameters

Figure 1. Three "orthogonal" directions to generalize classical Schubert calculus

(not pictured: quantum Schubert calculus)

Source: Rimanyi, "h-deformed Schubert calculus in equivariant cohomology, K-theory, & elliptic cohomology"
Generalized equivariant cohomology \((H^*_T, K_T, E^{1,1}_T)\) \(\leftrightarrow\) Bethe algebra of quantum integrable systems

- fixed point basis (easy) \(\leftrightarrow\) Bethe basis (hard)

- geometric basis (Schubert classes / stable envelopes) (hard) \(\leftrightarrow\) spin basis (easy)

Defined by Maulik & Okounkov for general Nakajima quiver varieties — in particular, for \(X = T^*(G/B)\) (cotangent bundle)
Some equivariant geometry

**Def. (Γ-equivariant cohomology)**

If $\Gamma$ acts freely on $X$,

$$H^*_\Gamma(X) := H^*(X/\Gamma)$$

Otherwise, let $E\Gamma$ be a contractible space w/ free $\Gamma$-action. Then

$$H^*_\Gamma(X) := H^*((X \times E\Gamma)/\Gamma).$$

Note that $H^*_\Gamma(pt) = H^*(E\Gamma/\Gamma)$ is not necessarily trivial! ($= B\Gamma$, classifying space of $\Gamma$)
For this talk, $\Gamma = T = (\mathbb{C}^\times)^n$, maximal torus of $G_t := \text{GL}_n(\mathbb{C})$.

First let $n=1$. Then we can take

$$ET = \left\{ (z_i)_{i>0} \mid z_i \in \mathbb{C}, \text{ finitely many} \right\}$$

so

$$ET/\Gamma = \mathbb{C}P^\infty \quad \& \quad H_T(pt) = \mathbb{Z}[y]$$

For general $n$,

$$H_T(pt) = \mathbb{Z}[y_1, \ldots, y_n].$$
Consider

\[ \Pi : X \longrightarrow \text{pt} \]

This induces

\[ \Pi^* : H^n_\pi(\text{pt}) \longrightarrow H^n_\pi(X) \]

(so \( H^n(X) \) is a \( H^n_\pi(\text{pt}) \)-module)

& (by Poincaré duality)

\[ \Pi_* : H^n_\pi(X) \longrightarrow H^n_\pi(\text{pt}) \]

which we use to define the Poincaré/intersection pairing:

\[ \langle a, b \rangle = \Pi_* (a \cup_{\text{prod}} b) \]
Equivariant $K$-theory

Similar, but now:

- $K^0(X)$ consists of equivariant vector bundles $E \to X$ (or, equivalently if $X$ smooth, equivariant sheaves)

- $K^0(pt) \cong \text{Rep}(\Gamma)$

  \[ \text{so } K^0(pt) = \mathbb{Z} \left[ e^{\pm t_1}, \ldots, e^{\pm tr} \right] \]

  \[ \text{(characters } \leftrightarrow \text{ basis of } \text{Lie}(\Gamma)) \]
Advantage of equivariant-ness:

Often, all the information about $H^*_\alpha(x)$ is captured by the cohomology of the fixed point locus $X^\alpha$. 
Example: Frozen pipes model

(exposition heavily influenced by Zinn-Justin,
"Lectures on Geometry, Quantum Integrability, &
Symmetric functions")

( can extend to )

we are here (+ connective K-theory)

focus today
(simplest case)

Figure 1. Three “orthogonal” directions to generalize classical Schubert calculus

Let

\[ G = \text{GL}_n \]
\[ B = (\bigotimes) \]
\[ T = (\mathbb{C}^*)^n \]
\[ X = G/B \text{ flag variety} \]

Then

\[ X = \bigsqcup_{w \in S_n} C_w \quad \text{(Schubert cell / Bruhat decomp)} \]
\[ X_w := \overline{C_w} = \bigsqcup_{v \leq w} C_w \quad \text{is a Schubert variety} \]
The Schubert classes \( S_w := [X_w] \) form an additive basis of \( \mathcal{H}^*_T(X) \) "equivariant parameters"

\[
\mathcal{H}^*_T(X) \cong \frac{\langle C[x_1, \ldots, x_n, y_1, \ldots, y_n] \rangle}{\langle f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n) \mid \text{f \in H}_G(pt) \rangle}
\]

In this case, \([X_w] \rightarrow \text{Schubert polynomial} \)

\[
(K\text{-theory: Grothendieck poly})
\]

\[
(\text{connective K}\text{-theory: } \mathbb{Q}\text{-Grothendieck poly})
\]

In the equivariant theory, also have basis of twisted Schubert classes for every \( v \in S_n \):

\[
S_w^{(v)} := [\nu X_w \nu^{-1}]
\]
T- fixed points: coordinate flags

\[ F_w = \frac{wB}{B} \quad (\leftarrow 0 \subset \langle e_{\omega(1)} \rangle \subset \langle e_{\omega(1)}, e_{\omega(2)} \rangle \subset \ldots \subset \mathbb{C}^n) \]

Localization

\[ \tilde{H}_T^*(X) := \text{localize } H_T^*(X) \]
\[ \text{at } H_T(pt) \]

Thm The classes \( l_\omega := [F_w] \) form a basis of \( \tilde{H}_T^*(X) \) as a vector space over \( \tilde{H}_T(pt) \).

How to actually decompose in this basis?
Thm \[ [X] = \sum_{w \in S_n} \frac{[F_w]}{\text{weights of } T_{F_w}X \text{ acting on } T_{F_w}X} \text{ (tangent space)} \]

Now, how are we going to get a lattice model out of this??

Fix an \( n \)-dim'el vector ("ket") of colors \( \vec{c} = | \begin{array}{cccc} 0 & 0 & 0 & 0 \end{array} \rangle \in \mathbb{V}^n \)

Then
\[
\begin{bmatrix}
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\end{array}
\end{bmatrix}
\xleftrightarrow{w \vec{c}}
\rightarrow
S_w
\]

\[
\begin{bmatrix}
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\end{array}
\end{bmatrix}
\xleftrightarrow{w \vec{c}}
\rightarrow
\text{Poincaré dual to } S_w (= S_w^{(\omega_0)})
\]
& the Boltzmann weights will come from the change of basis matrix, \( R_v \), between \( \{ S_w \} \) & \( \{ S_w^{(v)} \} \):

\[
S_w = \sum_{u \in S_n} (R_v)_{uw} S_w^{(v)}
\]

**Calculating** \( R_{si} \), \( S_i = (i, i+1) \):

\[\text{Let } P_i = \{ \begin{pmatrix} \star & * \\ i+1 & \star \end{pmatrix} \} \]

Consider \( P_i \times_B X_w \)

\[ \begin{array}{c}
\text{two fixed pts, } [1] \leftrightarrow [\square] \\
& \text{& } [s; i] \leftrightarrow [\square]
\end{array} \]
Then by the localization formula:

\[
[1^P] = \frac{[1]}{y_{i+1} - y_i} + \frac{[s_i]}{y_i - y_{i+1}}
\]

*divided difference operator!

So since

\[
f_* g^* [1] = [X_w]
\]

\[
f_* g^* [s_i] = [s_i \cdot X_w]
\]

\[
f_* g^* [1^P] = \begin{cases} [P_i \cdot X_w] & \text{if dim } P_i \cdot X_w = \text{dim } X_w + 1 \\ 0 & \text{else} \end{cases}
\]

we have

\[
[X_w] = [s_i \cdot X_w] + (y_{i+1} - y_i) [P_i \cdot X_w]_{\text{dim } P_i \cdot X_w = \text{dim } X_w + 1} + \sum_{i \neq j} \sum_{j \neq i} \sum_{s_i \neq s_j} S_{s_i(s_j)} W_{C_i < C_{i+1}}
\]

\[
= S_{s_i(w)}
\]
So $R_{si}$ acts nontrivially only on the $i, i+1$ vector entries. If are the colors appearing in these spots, this nontrivial piece of $R_{si}$ has the form:

$$
\begin{pmatrix}
1 & & & \\
0 & 1 & & \\
& 1 & y_{i+1} - y_i & \\
& & & 1
\end{pmatrix}
$$

$\leftarrow$ Frozen pipes Boltzmann weights, e.g. (with $\beta = 0$, $y_j \rightarrow -y_j$)
& the row/column R-matrix weights.

\[ \begin{align*}
\text{column} & : y_{i+1} - y_i \\
\text{row} & : x_{i+1} - x_i
\end{align*} \]

So... one way to analyze this lattice model is by applying operators to Schubert classes.
Another way: algebraic Bethe ansatz.

Think of one row of a lattice as an operator.

- "A" operator
- "B" operator
- "C" operator
- "D" operator
Yang–Baxter eqn $\Rightarrow$ commutation relations between these.

The operators + their relations generate a "Yang–Baxter algebra"

($\sim$ degeneration of Yangian / quantum gp; special case of Maulik Okounkov construction)

Bethe ansatz: Look @

$$B_1(x_1) \cdots B_n(x_n) |\phi\rangle$$

$$= \begin{array}{c}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{array}$$

$y_1$ $y_n$

$\text{top boundary not specified yet}$
Use commutations to find conditions on $x_1, \ldots, x_n$ so that this is an eigenvector of the transfer matrix $T(u) = \sum_{i=1}^{n} D_{ii}(u)$

\[ \exists \]

need $\exists x_1, \ldots, x_n$ to be a permutation of $\exists y_1, \ldots, y_n$

Further,

\[ y_1 \quad y_n \]

\[ \begin{array}{c}
  y_{w(1)} \\
  y_{w(2)} \\
  \vdots \\
  y_{w(n)}
\end{array} \]

\[ \leftrightarrow \quad l_w \quad \text{fixed pt class} \]

& the partition function is the expansion of $l_w$ into $S_v$'s: $\sum_{v \in S_n} \langle l_w, S_v \rangle S_v$
So picking a top boundary \( \vec{c} \) implies calculating the coefficient

\[
\langle l_w, S_\nu \rangle \in H^*_{\nu}(pt) \\
\cong \mathbb{Z}[y_1, \ldots, y_n]
\]

aka the double Schubert polynomial evaluated at \( \{ x_i \mapsto y_{w(i)} \} \).

**Cartoons:**

\[
\begin{array}{cccc}
\text{id} & y_1 & y_2 & \cdots & y_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
y_{w(1)} & \ldots & \ldots & \ldots & y_{w(n)} \\
l_w & \phi & \phi & \phi \\
\end{array}
\]

\[
\begin{array}{cccc}
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi \\
\langle l_w, S_\nu \rangle & \phi & \phi & \phi \\
\end{array}
\]
Example: Colored & uncolored Whittaker function lattice models

In lattice model world, we will be here:

\[ K_T(T^*X) \text{ (or really, } K_T(pt) \cong R(T)) \]

**Figure 1.** Three “orthogonal” directions to generalize classical Schubert calculus
Quick review of Whittaker functions

Take $G = \text{GL}_n(F)$, $F$ non-archimedean field, $\mathfrak{o}$ ring of integers of $F$, $\mathfrak{p} = \langle \omega \rangle$ maximal ideal of $\mathfrak{o}$, $\mathfrak{o}/\mathfrak{p} = \mathbb{F}_q$ the residue field, $\{((\mathfrak{o}:\mathfrak{p})\}, = J$ Iwahori subgroup.

$$y_z : \left( \begin{array}{cccc} \omega^{\lambda_1} & & & \\ & \ddots & & \\ & & \omega^{\lambda_n} \\ & & & \omega^{\lambda_n} \end{array} \right) \mapsto z^\lambda = \prod_{i=1}^n z_i^{\lambda_i}$$

unramified character of torus $T(F)$

The principal series representation $(\pi, \mathcal{I}(z))$ is:

$$\mathcal{I}(z) := \text{Ind}_{B(F)}^{G(F)} (\phi^{\frac{1}{2}} J_z)$$

with $\pi$ the right regular action of $G$. 
The space $I(z)^T$ of Iwahori-fixed vectors generates $I(z)$ as a $G$-module, has size $|S_n|$, & has two well-known bases:

1. Standard basis $\{ \overline{\chi}_w \}_{w \in S_n}$ of characteristic functions on the orbits of $G = \bigsqcup_w BwT$

2. Casselman's basis $\{ f_w \}$ defined as dual to intertwining operators $A_w : I(z) \to I(\omega z)$:

$$A_w (f_v)(1) = S_{\omega, v}$$
Aluffi, Mihalcea, Schürmann, & Su define an isomorphism \( \psi \) between

\[
\widetilde{K}_T(G/B)[q] \otimes_{K_T(pt)} \mathbb{C} \mathbb{J}_\mathbb{Z}
\]

\&

\( I(\mathbb{Z}) \mathbb{J} \)

such that

\[
\psi(MC^\vee(y(w)^0) \otimes 1) = \Omega_{\omega} \mathbb{Z}
\]

\&

\[
\psi(\omega) = f_{\omega}
\]
Brubaker, Bump, Buciumas, & Gustafsson (2020) define a colored lattice model that computes the Whittaker functions

\[ \tilde{Z}^\lambda \cdot \phi_{\omega, \lambda} (z) := \tilde{Z}^\lambda \cdot \sum_{\omega} \left( \prod_{i=1}^{\omega} \omega_i \right) \Phi_{\omega}^{z^{-1}} \]

Whittaker functional \( \lambda + \varphi \)

\( \omega \tilde{C} \)
& the sum
\[
\sum_{\mathcal{w}} \phi_{\mathcal{w}, \lambda}(z)
\]

computes the spherical Whittaker function $\leftrightarrow$ uncolored “Tokuyama” lattice model.

Like with the Frozen Pipes model, we have two ways of analyzing the Tokuyama model, one of which $\leftrightarrow$ Schubert-like basis & the other $\leftrightarrow$ fixed point basis.
Two ways of calculating:

1. As sum of colored Iwahori partition functions
   "Macro YBE" proof

\[ \sum_n \sum_{\pi} \chi_{\pi} \]

Calculate using Demazure-Whittaker operators via train argument in geometry:

\[ \sum_n \langle L_{\lambda+p}, (-q)^{l(w_0)} \text{MC}^\vee(y(w)^\circ) \rangle \]

= \langle L_{\lambda+p}, \text{MC}^\vee(y(1)) \rangle

2. Using algebraic Bethe ansatz inspired method
   "Micro YBE" proof

get Weyl character-like formula

\[ \prod_{i<j} \frac{z_i - q^{-1}z_j}{z_i - z_j} \sum_w (-1)^{l(w)} z_w^{\lambda+p} \]

in geometry:

\[ \sum (-q)^{l(w_0)} \prod_{\omega > 0} \frac{1 - q^{-1}e^\xi}{1 - e^\xi} \langle L_{\lambda+p}, b_{\omega} \rangle \]

\[ \sum (-q)^{l(w_0)} \prod_{\omega < 0} \langle L_{\lambda+p}, b_{\omega} \rangle \]

\[ \text{Casselman basis vector } f_w \]
proof of these geometric interpretations: very careful formal geometric manipulations & convention matching 😊

• Can extract from this a (modified version of) the Langlands–Gindikin–Karpelevich formula expanding \( \Phi^- \) into the bw's.

( & its geometric version, expanding \( \widetilde{MC}(Y(1)) \) into fixed pts.)

• Also have these cartoons now:

\[
\begin{array}{c}
\text{Owahou} \\
\lambda + \varrho
\end{array}
\]

\[
\begin{array}{c}
\langle \lambda + \varrho, \widetilde{MC}(Y(\omega)^0) \rangle
\end{array}
\]

\[
\begin{array}{c}
\text{Spherical} \\
\lambda + \varrho
\end{array}
\]

\[
\begin{array}{c}
\langle \lambda + \varrho, \widetilde{MC}(Y(1)) \rangle
\end{array}
\]
A picture to give the idea of the algebraic Bethe ansatz - like method in the simplest case:

Take each state of the model: & commute columns separately until all A's & D's are below B's:

\[ \begin{align*}
\text{factor from commutation} & \quad \text{weight } z_1 = z \\
B \leftrightarrow D & \\
\end{align*} \]

\[ \begin{align*}
\text{factor from commutation} & \quad \text{weight } z_2 = z \omega (\chi + p) \\
B \leftrightarrow A & \\
\end{align*} \]
Future directions?

Initially, I hoped to find a lattice proof of the Bump-Naruse-Nakasuji conjecture describing the expansion

$$
\Phi_n = \sum_w M_{u,w} f_w.
$$

$$
\left( \leftrightarrow \quad MC'(\gamma(\omega)^c) = \sum_w M_{u,w} b_w \right)
$$

It is potentially extractable from a formula of Borodin & Wheeler obtained via the nested Bethe ansatz:
non-symmetric Hall-Littlewood poly specializes to [factor of (-1)^s] \times (1 + 1)

sum over fixed pts?

\[
\Pi \left( \prod_{i=2}^{n} \Sigma_{\sigma_{i}} \left( \prod_{j=1}^{\sigma_{i}(n)} x_{\sigma_{i}(j)} \right) \right)
\]

would need to collapse this inner sum

\[
\sum_{\sigma_{1} \in S_{1}} \prod_{i=2}^{n} \sum_{\sigma_{i} \in S_{i}} \left( \prod_{j=1}^{\sigma_{i}(n)} x_{\sigma_{i}(j)} \right)
\]

\[
\left( x_{\sigma_{1}(1)} - x_{\sigma_{1}(n)} \right) \left( x_{\sigma_{2}(1)} - x_{\sigma_{2}(n)} \right) \cdots \left( x_{\sigma_{n}(1)} - x_{\sigma_{n}(n)} \right)
\]

Ch. 7: Bowdoin & Wheeler 2018
Other ideas...? 😊