Schubert polynomials and the inhomogeneous TASEP on a ring

Donghyun Kim

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(joint work with Lauren Williams)
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The ASEP has many applications in a broad range including protein synthesis, traffic flow, formation shocks, surface growth, and sequence alignments.
The inhomogenous TASEP definition

Consider a lattice with $n$ sites arranged in a ring. Let $St(n)$ denote the $n!$ labelings of the lattice by distinct numbers $1, 2, \ldots, n$, where each number $i$ is called a \textit{particle of weight} $i$. 

Eg.) $1 \ 3 \ 2 \ 5 \ 4$ 

$\begin{align*} r_{1,3} &= x_1 - y_3 \ 2 \ 4 \ 5 \ 1 \ 3 \end{align*}$
Consider a lattice with \( n \) sites arranged in a ring. Let \( St(n) \) denote the \( n! \) labelings of the lattice by distinct numbers \( 1, 2, \ldots, n \), where each number \( i \) is called a particle of weight \( i \).

The inhomogeneous TASEP on a ring of size \( n \) is a Markov chain with state space \( St(n) \) where at each time \( t \) a swap of two adjacent particles may occur: a particle of weight \( i \) on the left swaps its position with a particle of weight \( j \) on the right with transition rate \( r_{i,j} \) given by:

\[
    r_{i,j} = \begin{cases} 
        x_i - y_{n+1-j} & \text{if } i < j \\
        0 & \text{otherwise.}
    \end{cases}
\]
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- Eg.)

\[
\begin{array}{c}
1 \\
3 \\
4 \\
5 \\
2 \\
\end{array} \quad r_{1,3} = x_1 - y_3 \\
\begin{array}{c}
3 \\
4 \\
5 \\
2 \\
1 \\
\end{array}
\]
Cantini proved that the inhomogenous TASEP is a solvable lattice model. ("Inhomogenous Multispecies TASEP on a ring with spectral parameters", 2016)
The inhomogeneous TASEP $n = 3$

Figure: The transition diagram for the inhomogeneous TASEP for $n = 3$
Renormalized steady state probabilities

- The steady state probabilities for \( n = 3 \) inhomogeneous TASEP
  States 123, 231, 312: \[ \frac{x_1 - y_1}{6x_1 + 3x_2 - 6y_1 - 3y_2} \]
  States 132, 321, 213: \[ \frac{x_1 + x_2 - y_1 - y_2}{6x_1 + 3x_2 - 6y_1 - 3y_2} \]
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- we multiply all steady state probabilities by the same constant, obtaining “renormalized” steady state probabilities $\psi_w$, so that

$$\psi_{123...n} = \prod_{i<j} (x_i - y_{n+1-j})^{j-i-1}.$$
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- \( n = 3 \)

  \[
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- Observation: $\psi_w$ is a positive polynomial in $x_i$’s and -$y_i$’s!
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Schubert polynomials? $x_1 - y_1 = \mathcal{S}_{(2,1)}$, $x_1 + x_2 - y_1 - y_2 = \mathcal{S}_{(1,3,2)}$
Definition: double Schubert polynomials

For the longest permutation $\sigma_0 \in S_n$

$$\mathcal{G}_{\sigma_0}(x; y) = \prod_{i+j \leq n} (x_i - y_j)$$

for generic $\sigma \in S_n$

$$\mathcal{G}_{\sigma}(x; y) = \partial_{\sigma^{-1}\sigma_0} \mathcal{G}_{\sigma_0}(x; y)$$

where $\partial_{\sigma} = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_l}$ ($s_{i_1} s_{i_2} \cdots s_{i_l}$ is a reduced decomposition of $\sigma$)

$$\left(\partial_{i} P\right)(x_1, \ldots, x_n) = \frac{P(\ldots, x_i, x_{i+1}, \ldots) - P(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}$$
<table>
<thead>
<tr>
<th>State $w$</th>
<th>Probability $\psi_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1234</td>
<td>$(x_1 - y_1)^2(x_1 - y_2)(x_2 - y_1)$</td>
</tr>
<tr>
<td>1324</td>
<td>$(x_1 - y_1)S_{1432}$</td>
</tr>
<tr>
<td>1342</td>
<td>$(x_1 - y_1)(x_2 - y_1)S_{1423}$</td>
</tr>
<tr>
<td>1423</td>
<td>$(x_1 - y_1)(x_1 - y_2)(x_2 - y_1)S_{1243}$</td>
</tr>
<tr>
<td>1243</td>
<td>$(x_1 - y_2)(x_1 - y_1)S_{1342}$</td>
</tr>
<tr>
<td>1432</td>
<td>$S_{1423}S_{1342}$</td>
</tr>
</tbody>
</table>
Table $n = 5, y_i = 0$

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<tr>
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<tr>
<td>12345</td>
<td>$x^{(6,3,1)}$</td>
</tr>
<tr>
<td>12354</td>
<td>$x^{(5,2,0)}S_{13452}$</td>
</tr>
<tr>
<td>12435</td>
<td>$x^{(4,1,0)}S_{14532}$</td>
</tr>
<tr>
<td>12453</td>
<td>$x^{(4,1,1)}S_{14523}$</td>
</tr>
<tr>
<td>12534</td>
<td>$x^{(5,2,1)}S_{12453}$</td>
</tr>
<tr>
<td>12543</td>
<td>$x^{(3,0,0)}S_{14523}S_{13452}$</td>
</tr>
<tr>
<td>13245</td>
<td>$x^{(3,1,1)}S_{15423}$</td>
</tr>
<tr>
<td>13254</td>
<td>$x^{(2,0,0)}S_{15423}S_{13452}$</td>
</tr>
<tr>
<td>13425</td>
<td>$x^{(3,2,1)}S_{15243}$</td>
</tr>
<tr>
<td>13452</td>
<td>$x^{(3,3,1)}S_{15234}$</td>
</tr>
<tr>
<td>13524</td>
<td>$x^{(2,1,0)}(S_{164325} + S_{25431})$</td>
</tr>
<tr>
<td>13542</td>
<td>$x^{(2,2,0)}S_{15234}S_{13452}$</td>
</tr>
<tr>
<td>14235</td>
<td>$x^{(4,2,0)}S_{13542}$</td>
</tr>
<tr>
<td>14253</td>
<td>$x^{(4,2,1)}S_{12543}$</td>
</tr>
<tr>
<td>14325</td>
<td>$x^{(1,0,0)}(S_{1753246} + S_{265314} + S_{2743156} + S_{356214} + S_{364215} + S_{365124})$</td>
</tr>
<tr>
<td>14352</td>
<td>$x^{(1,1,0)}S_{15234}S_{14532}$</td>
</tr>
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</tr>
<tr>
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</tr>
<tr>
<td>15342</td>
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Conjecture 1

The steady state probability $\psi_w$ is a positive polynomial in $x_i$’s and $-y_i$’s

Lam and Williams in 2010 studied this model for $y_i = 0$ and made the above conjectures in that setting.

For $y_i = 0$, Conjecture 1 has been proved by Arita and Mallick in 2012 by giving a monomial expansion formula in terms of multiline queues as conjectured by Ayyer and Linusson.

Later, Cantini in 2016 generalized the model by putting $y$-parameters and established a solvability. Then he made the above conjectures.
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The steady state probability $\psi_w$ is a positive polynomial in $x_i$'s and $-y_i$'s.

Conjecture 2

The steady state probability $\psi_w$ is a positive sum of double Schubert polynomials.
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(Eg)

```
1
2 1 2 2
3 2 1 2 2
```

Type: (2, 2, 1, 4, 4, 4, 2, 3)
A vacancy in $Q$ is called $i$ — covered if it is traversed by a $i$-bully path, but not traversed by $i'$-bully path for $i' < i$. 

Let $z_{r,i}$ be the number of $i$-covered vacancies on row $r$. We define a weight of a multiline queue $Q$ as follows:

$$\text{weight}(Q) = \prod_{1 \leq i < r \leq L} (x_{r,i})^{z_{r,i}}.$$ 

(Eq)
Multiline queues

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(Eg)

$$weight(Q) = \left( \frac{x_2}{x_1} \right) \left( \frac{x_3}{x_1} \right)^2.$$
Theorem (Arita, Mallick)

The steady state probability $\psi_w$ is proportional to a weighted sum over multiline queues of type $w$

$$\psi_w \propto \sum_{Q: \text{type } w \text{ multiline queue}} \text{weight}(Q).$$
Multiline queues

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Limits

- Cannot deal with the model with general $y_i$'s. It is an open problem to define a weight putting $y_i$'s.
- Hard to explain the appearance of Schubert polynomials except for a special case.
Special case (Inverse of Grassmannian permutations)

**Theorem (K, Williams)**

If $w$ is an inverse of Grassmannian permutation that starts with 1, then there exists a (weight-preserving) bijection between multiline queues of type $w$ and certain flagged semistandard Young tableaux.

- **Consequences**

  The steady state probability $\psi_w$ is proportional to a certain flagged Schur function. Flagged Schur functions are Schubert polynomials for certain vexillary (2143-avoiding) permutations.

- **Definition: pattern avoidance**

  We say that a permutation $\pi$ avoids a pattern $\sigma$, or $\sigma$-avoiding, if $\pi$ does not contain a subsequence in a same relative order with $\sigma$. Eg. $(1, 4, 2, 6, 3, 5, 7)$ is not 2143-avoiding $\rightarrow (1, 4, 2, 6, 3, 5, 7)$
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Bijection
Cantini introduced a family $\psi_w(z_1, \cdots, z_n)$ that recovers the steady-state probability $\psi_w$ by taking the leading coefficient in $z$-variables (specializing to $z = \infty$). We call $\psi_w(z_1, \cdots, z_n)$ a $z$-deformed steady-state probability.
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- Let $V$ be a vector space with formal basis $e_1, \cdots, e_n$ where $e_i$ represents the particle type $i$.
- Consider $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ ($V_i = V$), and represent each state of the inhomogeneous TASEP as a basis of this space.

\[
\begin{array}{c}
1 \\
4 \\
3 \\
5 \\
2 \\
\end{array} \quad e_1 \otimes e_3 \otimes e_2 \otimes e_5 \otimes e_4
\]
Integrability

Let $W_n$ be the subspace of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ spanned by basis elements that represent possible states of the inhomogeneous TASEP.
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- Let $M$ be the Markov matrix of the inhomogenous TASEP, then $M = M_1 + M_2 + \cdots + M_n$ is the sum of local terms $M_i$. The matrix $M_i$ acts on $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ by acting locally on $V_i \otimes V_{i+1}$ parts.
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Suppose we have an operator $R_i(a, b)$ for formal variables $a$ and $b$ acting locally on $V_i \otimes V_{i+1}$ such that

$$R_i(a, a) = 1, \quad \frac{dR_i(a, b)}{da} \bigg|_{a=b=\infty} \propto M_i$$

and a vector $\psi(z_1, \cdots, z_n)$ that satisfies

$$R_i(z_i, z_{i+1})\psi(z_1, \cdots, z_n) = s_i \psi(z_1, \cdots, z_n)$$

where $s_i$ acts by exchanging $z_i$ and $z_{i+1}$. We call the above equation, exchange equation.
Integrability

Claim

$\psi(z_1, \cdots, z_n)|_{z=\infty}$ is proportional to the steady state probability of the inhomogeneous TASEP.
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Claim

\[ \psi(z_1, \cdots, z_n) \big|_{z=\infty} \] is proportional to the steady state probability of the inhomogeneous TASEP.

Proof.

Differentiating the exchange equation with \( z_i \) and plugging in \( z = \infty \) gives

\[ \mathcal{M}_i \psi(z_1, \cdots, z_n) \big|_{z=\infty} = \partial_{i+1} \psi(z_1, \cdots, z_n) \big|_{z=\infty} - \partial_i \psi(z_1, \cdots, z_n) \big|_{z=\infty}. \]

Summing over \( i = 1 \) to \( n \) completes the proof.
Cantini found such operator $R_i(a, b)$ satisfying the additional two equations (unitary relation, braid Yang-Baxter equation)

$$R_i(a, b)R_i(b, a) = 1$$
$$R_i(b, c)R_{i+1}(a, c)R_i(a, b) = R_{i+1}(a, b)R_i(a, c)R_{i+1}(b, c).$$
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- The vector $\psi(z_1, \cdots, z_n)$ is the common eigenvector of scattering matrices

$$S_i = \mathcal{R}R_{i-2}(z_i, z_{i-1}) \cdots R_{i+1}(z_i, z_{i+2})R_i(z_i, z_{i+1})$$

where $\mathcal{R}$ acts by rotation

$$\mathcal{R}(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (v_n \otimes v_1 \otimes \cdots \otimes v_{n-1})$$
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$$\mathcal{R}(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (v_n \otimes v_1 \otimes \cdots \otimes v_{n-1})$$

$S_i$ and $S_j$ commute by braid Yang-Baxter equation
The exchange equation has a unique polynomial solution up to multiplication by a symmetric function of \( z \).

Expanding the exchange equation component by component gives

\[
\psi_{s_{i_{w}}} (z_1, \ldots, z_n) = \pi_{i_{w}} (w_i, w_{i+1}; n) \psi_{w} (z_1, \ldots, z_n)
\]

if \( w_i > w_{i+1} \),

where \( \pi_{i_{w}} (\beta, \alpha; n) \) is the isobaric divided difference operator defined by

\[
\pi_{i_{w}} (\beta, \alpha; n) G(z) = \left( z_i - y_{n+1} - \beta \right) \left( z_{i+1} - x_{\alpha} \right) - \left( z_i - z_{i+1} \right) G(z) - s_{i_{w}} G(z)
\]
Integrability

**Theorem, Cantini**

The exchange equation has a unique polynomial solution up to multiplication by a symmetric function of $z$.

The initial condition is given by

$$
\psi_{(1,2,\ldots,n)}(z_1, \ldots, z_n)
= \prod_{1 \leq i < j \leq n} (x_i - y_{n+1-j})^{j-i-1} \prod_{i=1}^{n} \left( \prod_{j=1}^{i-1} (z_i - x_j) \prod_{j=i+1}^{n} (z_i - y_{n+1-j}) \right)
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Theorem, Cantini

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Expanding the exchange equation component by component gives

$$\psi_{s_i w}(z_1, \cdots, z_n) = \pi_i(w_i, w_{i+1}; n) \psi_w(z_1, \cdots, z_n) \quad \text{if } w_i > w_{i+1},$$

where $\pi_i(\beta, \alpha; n)$ is the isobaric divided difference operator defined by

$$\pi_i(\beta, \alpha; n) G(z) = \frac{(z_i - y_{n+1-\beta})(z_{i+1} - x_\alpha)}{x_\alpha - y_{n+1-\beta}} \frac{G(z) - s_i G(z)}{z_i - z_{i+1}}.$$
We can compute every component $\psi_w(z_1, \cdots, z_n)$ of $\psi(z_1, \cdots, z_n)$ from the initial condition by applying sequences of isobarbic divided difference operators.

\[\psi(1, 2, 3)(z_1, z_2, z_3) = (x_1 - y_1)(z_1 - y_2)(z_1 - y_3)(z_2 - x_1)(z_2 - y_1)(z_3 - x_1)(z_3 - x_2)\]

\[\psi(3, 2, 1)(z_1, z_2, z_3) = \pi_3^3(3, 1; 3)\psi(1, 2, 3)(z_1, z_2, z_3)\]

Taking the leading coefficients in $z$-variables (specializing to $z_1 = \infty$) gives

\[\psi(1, 2, 3) = x_1 - y_1\]

and

\[\psi(3, 2, 1) = (x_1 + x_2 - y_1 - y_2)\]
We can compute every component $\psi_w(z_1, \cdots, z_n)$ of $\psi(z_1, \cdots, z_n)$ from the initial condition by applying sequences of isobarbic divided difference operators.

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\[ \psi_{(1,2,3)}(z_1, z_2, z_3) = (x_1 - y_1)(z_1 - y_2)(z_1 - y_1)(z_2 - x_1)(z_2 - y_1)(z_3 - x_1)(z_3 - x_2) \]

\[ \psi_{(3,2,1)}(z_1, z_2, z_3) = \pi_3(3, 1; 3)\psi_{(1,2,3)}(z_1, z_2, z_3) \]
\[ = (z_1 - x_1)(z_2 - x_1)(z_2 - y_1)(z_3 - y_1) \times \]
\[ ((x_1 + x_2 - y_1 - y_2)z_3z_1 + (x_1x_2 - y_1y_2)(z_3 + z_1) - x_1x_2y_1 - x_1x_2y_2 + x_1y_1y_2) \]
$n = 3$ example

We can compute every component $\psi_w(z_1, \cdots, z_n)$ of $\psi(z_1, \cdots, z_n)$ from the initial condition by applying sequences of isobarbic divided difference operators.

$$\psi_{(1,2,3)}(z_1, z_2, z_3) = (x_1-y_1)(z_1-y_2)(z_1-y_1)(z_2-x_1)(z_2-y_1)(z_3-x_1)(z_3-x_2)$$

$$\psi_{(3,2,1)}(z_1, z_2, z_3) = \pi_3(3, 1; 3) \psi_{(1,2,3)}(z_1, z_2, z_3)$$

$$= (z_1 - x_1)(z_2 - x_1)(z_2 - y_1)(z_3 - y_1) \times ((x_1 + x_2 - y_1 - y_2)z_3z_1 + (x_1x_2 - y_1y_2)(z_3 + z_1) - x_1x_2y_1 - x_1x_2y_2 + x_1y_1y_2)$$

Taking the leading coefficients in $z$-variables (specializing to $z = \infty$) gives $\psi_{(1,2,3)} = x_1 - y_1$ and $\psi_{(3,2,1)} = (x_1 + x_2 - y_1 - y_2)$. 

Donghyun Kim (UC Berkeley) - 21 / 30
Definition (K, Williams)

We say that $w \in S_n$ is a evil-avoiding, if: $w_1 = 1; w$ avoids the patterns 2413, 3214, 4132, and 4213. We say $w \in St(n, k)$ if $w$ is evil-avoiding and $w^{-1}$ has exactly $k$ descents.
Main results

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**Theorem (K, Williams)**

For $w \in St(n, k)$, the steady-state probability $\psi_w$ is given as a trivial factor times product of $k$ double Schubert polynomials.

Eg) $w = (1, 2, 5, 4, 3)$, $w^{-1} = (1, 2, 5, 4, 3)$. $w \in St(5, 2)$.

$$\psi_w = (x_1 - y_1)(x_1 - y_2)(x_1 - y_3) \mathcal{G}_{(1,4,5,2,3)}(x; y) \mathcal{G}_{(1,3,4,5,2)}(x; y)$$
The number of evil-avoiding permutations in $S_n$ is $\frac{(2+\sqrt{2})^{n-1}+(2-\sqrt{2})^{n-1}}{2}$. Previously the formula (explaining the appearance of (double) Schubert polynomials) for $n$ out of $n!$ states were known by Cantini.
The number of evil-avoiding permutations in $S_n$ is $\frac{(2+\sqrt{2})^{n-1}+(2-\sqrt{2})^{n-1}}{2}$. Previously the formula (explaining the appearance of (double) Schubert polynomials) for $n$ out of $n!$ states were known by Cantini.

- Cantini gave an explicit formula for $\psi_w(z_1, \cdots, z_n)$ for the permutation $w$ of the form $w(n, h) := (1, h + 1, h + 2, \ldots, n, h, h - 1, \ldots, 2)$
- We will present an explicit formula for $\psi_w(z_1, \cdots, z_n)$ for evil-avoiding permutations.
The number of evil-avoiding permutations in $S_n$ is $\frac{(2+\sqrt{2})^{n-1}+(2-\sqrt{2})^{n-1}}{2}$. Previously the formula (explaining the appearance of (double) Schubert polynomials) for $n$ out of $n!$ states were known by Cantini.

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- We will present an explicit formula for $\psi_w(z_1, \cdots, z_n)$ for evil-avoiding permutations.
- We introduce $z$-Schubert polynomials $\zeta^n_{\lambda}(z; x; y)$ to do that.
Lemma, Double Schubert polynomials

**Code** $c(w) = (a_1, \cdots, a_n)$ of a permutation $w \in S_n$ is an integer vector such that $a_i$ is the number of $w_i > w_j$ for $j > i$.

Example $c((5, 1, 3, 4, 2)) = (4, 0, 1, 1, 0)$ and $c((1, 3, 4, 2)) = (0, 1, 1, 0)$

**Proposition (K, Williams)**

Let $w$ and $w'$ be permutations such that $c(w) = (c_1, \ldots, c_n)$ and $c(w') = (M, c_1, \ldots, c_n)$. If $M$ is "sufficiently big" then we have the equation

$$S_{w'}(x; y) = S_{w}(x_1; y) \prod_{k=1}^{M} (x_1 - y_k).$$

$x_1$ means we shift indices of $x$-variables by 1 ($x_1 \rightarrow x_2, x_2 \rightarrow x_3, \ldots$)

Example

$$S_{(5,1,3,4,2)} = S_{(1,3,4,2)}(x_1) \prod_{i=1}^{4} (x_1 - y_i)$$
z-Schubert polynomials $\mathcal{S}_\lambda^n(z; x; y)$

- We say that a partition $\lambda$ is *valid* for $n$ if $\lambda$ is properly contained in a $\text{length}(\lambda) \times (n - \text{length}(\lambda))$ rectangle.
z-Schubert polynomials $\mathcal{G}_\lambda^n(z; x; y)$

- We say that a partition $\lambda$ is valid for $n$ if $\lambda$ is properly contained in a $\text{length}(\lambda) \times (n - \text{length}(\lambda))$ rectangle.
- $z$-Schubert polynomial $\mathcal{G}_\lambda^n(z; x; y)$ is defined for $\lambda$ that is valid for $n$
We say that a partition \( \lambda \) is **valid** for \( n \) if \( \lambda \) is properly contained in a \( \text{length}(\lambda) \times (n - \text{length}(\lambda)) \) rectangle.

\( z \)-Schubert polynomial \( \mathcal{G}_\lambda^n(z; x; y) \) is defined for \( \lambda \) that is valid for \( n \).

\( \mathcal{G}_\lambda^n(z; x; y) \) is defined recursively from \( \mathcal{G}_{\lambda'}^{n-1}(z; x; y) \) where \( \lambda' \) is the partition obtained by deleting the first part of \( \lambda \).

\[
\mathcal{G}_\lambda^n(z; x; y) = \partial_{n-\lambda_1-\text{mul}(\lambda)} \cdots \partial_1 \left( \mathcal{G}_{\lambda'}^{n-1}(\sigma^{\lambda_1-\lambda_2+1}z; x_1; y) \right)
\]

\[
\times \prod_{l=1}^{n-\text{mul}(\lambda)} (x_1 - y_l) \prod_{i=1}^{(\lambda_1-\lambda_2+1)n-\lambda_1-\text{mul}(\lambda)+1} \prod_{m=2} (z_i - x_m).
\]
We say that a partition $\lambda$ is valid for $n$ if $\lambda$ is properly contained in a $\text{length}(\lambda) \times (n - \text{length}(\lambda))$ rectangle.

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$\mathcal{S}_\lambda^n(z; x; y)$ is defined recursively from $\mathcal{S}_{\lambda'}^{n-1}(z; x; y)$ where $\lambda'$ is the partition obtained by deleting the first part of $\lambda$.

$$
\mathcal{S}_\lambda^n(z; x; y) = \partial_{n-\lambda_1-\text{mul}(\lambda)} \cdots \partial_1 \left( \mathcal{S}_{\lambda'}^{n-1}(\sigma^{\lambda_1-\lambda_2+1} z; x_1; y) \right)
\times \prod_{l=1}^{n-\text{mul}(\lambda)} (x_1 - y_l) \prod_{i=1}^{(\lambda_1-\lambda_2+1)} \prod_{m=2}^{n-\lambda_1-\text{mul}(\lambda)+1} (z_i - x_m) .
$$

$\sigma^a(z)$ means we shift indices of $z$ variables by $a$. 

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Leading coefficient of $S^n_\lambda(z; x; y)$ (specializing to $z = \infty$) is a double Schubert polynomial. And there is a simple rule to get a code of the permutation indexing double Schubert polynomial.

(Eg.)

\[
S^4_{(1,1)}(z; x; y) \rightarrow S_{(1,3,4,2)}
\]

\[
S^5_{(2,1,1)}(z; x; y) \rightarrow S_{(1,3,5,4,2)}
\]
**z-Schubert polynomials** $S^n_\lambda(z; x; y)$

- Leading coefficient of $S^n_\lambda(z; x; y)$ (specializing to $z = \infty$) is a double Schubert polynomial. And there is a simple rule to get a code of the permutation indexing double Schubert polynomial. (Eg.)

$$S^4_{(1,1)}(z; x; y) \rightarrow S_{(1,3,4,2)}$$

$$S^5_{(2,1,1)}(z; x; y) \rightarrow S_{(1,3,5,4,2)}$$

- Recursive construction of $z$-Schubert polynomials is consistent with the equations of Schubert polynomials.

$$S_{(1,3,5,4,2)} = \partial_2 \partial_1(S_{(5,1,3,4,2)}) = \partial_2 \partial_1(S_{(1,3,4,2)}(x_\hat{i}) \prod_{i=1}^{4}(x_1 - y_i))$$

$$S^5_{(2,1,1)}(z; x; y) = \partial_2 \partial_1(S^4_{(1,1)}(z; x_\hat{i}; y) \prod_{i=1}^{4}(x_1 - y_i) \prod_{i=1}^{2} \prod_{m=2}^{3}(z_i - x_m))$$
For $w \in St(n, k)$, let $c(w^{-1}) = (c_1, \ldots, c_n)$; and denote descents positions by $a_1, \ldots, a_k$. For $1 \leq i \leq k$, define

$$\lambda^i = \left(n - a_i\right)^{a_i} - \left(0, \ldots, 0, c_{a_i - 1 + 1}, c_{a_i - 1 + 2}, \ldots, c_{a_i}\right)_{a_{i-1}}.$$
For \( w \in St(n, k) \), let \( c(w^{-1}) = (c_1, \ldots, c_n) \); and denote descents positions by \( a_1, \ldots, a_k \). For \( 1 \leq i \leq k \), define
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\]

(Eg) \( w = (1, 2, 5, 4, 3) \in St(5, 2) \) with \( c(w^{-1}) = (0, 0, 2, 1, 0) \). So \( \lambda_1 = (2, 2) \) and \( \lambda_2 = (1, 1, 1) \).
Main results

For $w \in St(n, k)$, let $c(w^{-1}) = (c_1, \ldots, c_n)$; and denote descents positions by $a_1, \ldots, a_k$. For $1 \leq i \leq k$, define

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Theorem (K, Williams)

For $w \in St(n, k)$ we have

$$\psi_w(z_1, \ldots, z_n) = \text{"Trivial Factor" } \times \prod_{i=1}^{k} G_{\lambda_i}^{n}(\sigma^{t_i}(z); x; y)$$
Main results

For $w \in St(n, k)$, let $c(w^{-1}) = (c_1, \ldots, c_n)$; and denote descents positions by $a_1, \ldots, a_k$. For $1 \leq i \leq k$, define

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(Eg) $w = (1, 2, 5, 4, 3) \in St(5, 2)$ with $c(w^{-1}) = (0, 0, 2, 1, 0)$. So $\lambda_1 = (2, 2)$ and $\lambda_2 = (1, 1, 1)$

**Theorem (K, Williams)**

For $w \in St(n, k)$ we have

$$\psi_w(z_1, \cdots, z_n) = "\text{Trivial Factor}" \times \prod_{i=1}^{k} \mathfrak{S}_{\lambda_i}^n(\sigma^{t_i}(z); x; y)$$

→ As a corollary, we have that $\psi_w$ for $w \in St(n, k)$ is given as a trivial factor times product of $k$ Schubert polynomials.
Main Results

Proof idea

- We first prove the theorem for $St(n, 1)$. (Inverse of Grassmannian permutations that start with 1)
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- We first prove the theorem for $St(n, 1)$. (Inverse of Grassmannian permutations that start with 1)

- Use factorization theorem by Cantini
  $\rightarrow$ Suppose that $w$ splits as $w^{(1)}w^{(2)} \cdots w^{(k)}$ such that every component in $w^{(i)}$ is bigger than $w^{(j)}$ if $i > j$.
  Then $\psi_w(z) = \bar{\psi}_{w^{(1)}}(z)\bar{\psi}_{w^{(2)}}(z) \cdots \bar{\psi}_{w^{(k)}}(z)$
Future directions

- Give a combinatorial formula for $S^n_\lambda(z; x; y)$. Leading coefficient is a double Schubert polynomial for a vexillary permutation $(2143$-avoiding) $\rightarrow$ flagged semistandard young tableau
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- Put a parameter $\beta$ to the inhomogeneous TASEP so that Schubert polynomials lift to $\beta$-Grothendieck polynomials
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- Put a parameter $\beta$ to the inhomogeneous TASEP so that Schubert polynomials lift to $\beta$-Grothendieck polynomials

- Define $z$-Schubert polynomials in a more general setting. And understand $\psi_w(z)$ for $w$ other than evil-avoiding permutations (as a sum of $z$-Schubert polynomials). I have checked that this is possible for some states whose probability is a sum of two Schubert polynomials.
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Geometric interpretation?
Thanks for your attention!

Extended abstract is available (https://arxiv.org/abs/2102.00560)

