How to come up with Solvable Lattice Models?

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* Sorry for the clickbait!
* It's work in progress.
Outline:
1. Spherical Lattice Models
2. Spherical Whittaker functions
3. How to come up with spherical models?
4. How to come up with colored models and my results.

Main takeaway: You can (without guessing)
1. start with functions from representation theory of $p$-adic groups,
2. decompose them into pieces geometrically,
3. describe pieces combinatorically,
4. realize combinatorical data as lattice models.

I've done it for spherical and Iwahori Whittaker $C$-wts of $GL_n$ and got spherical and colored lattice models.
Spherical Lattice Model

Consider model \( \mathcal{G}_\mu \) given as follows:

The top boundary is determined by strictly dominant partition \( \mu \). Above \( \mu = (3,1,0) \).

Only the following 6 vertices are allowed:

\[
\begin{align*}
& a_1 & a_2 & b_1 & b_2 & c_1 & c_2 \\
& 1 & 2i & -q^{-1} & 2i & 1 & 2i (1 - q^{-1})
\end{align*}
\]
Thm (Brubaker, Bump, Friedberg, ’03)
Let \(p = (n-1, n-2, \ldots, 1, 0)\), and write \(\mu = \lambda + p\). Then the partition function \(Z\mu\) of \(\mathcal{G}\mu\) is given by

\[
Z\mu(q, z) = \prod_{i < j} (z_i - q^{1/2} z_j) S_{\lambda}(z),
\]

\[
= z^p \prod_{i < j} (1 - q^{1/2} z_j) S_{\lambda}(z),
\]

where \(S_{\lambda}\) is the Schur polynomial.

Example \(\lambda = (2, 0), p = (1, 0), \mu = \lambda + p = (3, 0)\)
\[ 2\mu(z) = 2^3 + z^3 z_2 (1 - q^{-1}) + z_1 z_2^3 (1 - q^{-1}) + (-q^{-1}) z_2^3 =
\]
\[ = (z_1 - q^{-1} z_2) (z_1^2 + z_1 z_2 + z_2^2) =
\]
\[ = \prod_{i<j} (z_i - q^{-1} z_j) S_\lambda(z) = 2^p \prod_{i<j} (1 - q^{-1} \frac{z_i}{z_j}) S_\lambda(z). \]
Spherical Whittaker function

Let $G = GL_n(F)$, where $F$ local (non-arch) field (e.g., $\mathbb{Q}_p$). $q$-coad of reg field

$\mathcal{O}_F$-ring field of integers.

$\mathcal{O}_F \subseteq \mathcal{O}$, uniformizer.

$\mathcal{T} = \begin{pmatrix} \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \end{pmatrix}$, $W = \frac{NG(T)}{T} \cong S_n$ ("permutation matrices")

$K = GL_n(\mathcal{O}_F) = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F & \cdots & \mathcal{O}_F \\ \vdots & \ddots & \vdots & \vdots \\ \mathcal{O}_F & \mathcal{O}_F & \cdots & \mathcal{O}_F \end{pmatrix}$

$\Phi^+ = \{ (i,j) \in [n]^2 \mid i < j \}$ — positive roots,

let $N = |\Phi^+| = \frac{n(n-1)}{2}.$

Also, for partition $\lambda$ we denote $\mathcal{W}^\lambda = \begin{pmatrix} \mathcal{W}_{\lambda_1} & \cdots & \mathcal{W}_{\lambda_k} \\ \vdots & \ddots & \vdots \\ \mathcal{W}_{\lambda_n} \end{pmatrix}.$
Let \( \psi: \mathcal{U}^* \rightarrow \mathbb{C}^* \) be given by

\[
\psi \left( \begin{pmatrix}
\frac{1}{x_1} & 1 \\
x_2 & 1 \\
\vdots & \vdots \\
x_{n-1} & 1
\end{pmatrix} \right) = \psi(x_1) \ldots \psi(x_{n-1}),
\]

where \( \psi_0: F \rightarrow \mathbb{C}^* \) is a character which is trivial on \( \Theta_F \) but not on \( \rho_1^{-1} \Theta_F \).

**Blackbox** The spherical Whittaker function \( W^2: G \rightarrow \mathbb{C} \) is given by the following integral.

\[
W^2(g) = \int_{\mathcal{U}^*} f^2(ug) \psi(u) du,
\]

where \( z \in \mathbb{C}^r \) and \( f^2 \) is given by

\[
f^2(g) = f(g) = f(u \overline{w^\mu k}) = z^\mu, \quad (z \text{ is fixed})
\]

where we use the Iwasawa decomposition \( G = \mathcal{U} \mathcal{K} \).
Thm (Casselman-Shalika, ’80) The values of the spherical Whittaker function are given by

\[ W^2(g) = \begin{cases} 0, & \text{if } g \in \mathfrak{U} \bar{\omega} \mathfrak{K} \text{ with } \lambda \text{ not dominant} \\ \Pi \left(1 - q^{-1/2} \frac{z_i}{z_j}\right) S_{\lambda}(z), & \text{otherwise}. \end{cases} \]

In other words, the partition function $\varphi_\lambda$ gives the values of spherical Whittaker function.

\[ Z_{\mu}(z) = z^\nu \cdot W^2(g), \quad g \in \mathfrak{U} \bar{\omega} \mathfrak{K}, \quad \lambda \in \Lambda^+. \]

The usual proof of the connection is ad hoc. You show that both sides satisfy the same functional equations imposed by the intertwining operators for $W^2(g)$, and Yang-Baxter equations for $Z_{\mu}(z)$. But how to come up with $\varphi_\lambda$?
How to come up with spherical lattice model \( \mathfrak{g}_x \)?

Let's start with 

\[
W^2(g) = \int f(ug)\psi(u)du.
\]

Recall that 

\[
f(g) = f(u, \bar{w}^{\mu}w) = 2^\mu.
\]

It follows that 

\[
W^2(g) \text{ is determined on } \bar{w}.
\]

So we get 

\[
\int f(u, \bar{w}^\mu)\psi(u)du.
\]

After change of variables 

\[
u \mapsto \bar{w}^\mu \bar{u} \bar{w}^{-1},
\]

we get (up to constant) 

\[
\int f(u)\psi_x(u)du, \text{ where}
\]

\[
\psi_x(u) = \psi(\bar{w}^\mu \bar{u} \bar{w}^{-1}).
\]

We study 

\[
\psi_x(z) = \int f(u)\psi_x(u)du.
\]
**Theorem (McNamara, 09)** We can write
\[ U^{-} = \bigcup_{m \in \mathbb{N}^n} C_m, \quad \text{and} \quad f|_{C_m} = 2^{\text{wt}(m)}. \]
(there is actually a lot more.)

Using the theorem, we get
\[ \psi_\lambda(z) = \int f(u) \psi_\lambda(u) du = \sum_{m \in \mathbb{N}^n} 2^{\text{wt}(m)} \cdot \int \delta_{C_m} \psi_\lambda(u) du. \]

Denote \( G_\lambda(m) = \int \delta_{C_m} \psi_\lambda(u) du = \prod_{\text{dep}^+} G_\lambda(k,m) \)

For almost all \( m \), we have \( G_\lambda(m) = 0 \).

More precisely, there is finite set \( L_u(\lambda + p) \) of \( N \)-tuples such that \( G(m) = 0 \) unless \( m \in L_u(\lambda + p) \).

**Punchline:** States of spherical model correspond to \( L_u(\lambda + p) \) and their weights to \( 2^{\text{wt}(m)} G_\lambda(m) \). States have "geometric meaning."
Let's concentrate on bijection between states. 
$L(u; \lambda+p) \leftrightarrow G(u; \lambda+p)$, Gel'fand–Tsetlin patterns.

**Example:**

\[ \begin{align*}
&GT(3, 1, 0): \\
&\{} 310, 100 \{} 310, 010 \{} 310, 001 \{}
\end{align*} \]

\[ \begin{align*}
&3210, 3201 \{} 3210, 2301 \{} 3210, 2103 \{}
\end{align*} \]

\[ \begin{align*}
&3210, 2301 \{} 3210, 2103 \{} 3210, 2103 \{}
\end{align*} \]

\[ \begin{align*}
&3210, 2301 \{} 3210, 2103 \{} 3210, 2103 \{}
\end{align*} \]

**Bijection:**

\[ \begin{align*}
&310 \leftrightarrow 3210 \leftrightarrow 3210 \leftrightarrow 3210 \\
&20 \leftrightarrow 210 \leftrightarrow 210 \leftrightarrow 210 \\
&1 \leftrightarrow 3 \leftrightarrow 3 \leftrightarrow 3 \\
\end{align*} \]
States in $\mathcal{E}_{\lambda+p}$ $\leftrightarrow$ GT-patterns $\leftrightarrow$ Lusztig data $\lambda+p$

weight-preserving bijections.

**Remark** Calculations in terms of Lusztig data can be made uniformly for any split reductive group (e.g., $Sp(2n)$ or $O(n)$).

**Summary**

1. Start with spherical Whittaker function.
2. $\int = \sum_{m} \int_{U^m} C_m$ by Mehta-Narayana's decomposition.
3. $\int \neq 0$ for finite set $\mathcal{L}(\lambda+p) \leftrightarrow \text{GT}(\lambda+p)$.
4. $\mathcal{L}(\lambda+p) \leftrightarrow \text{GT}(\lambda+p) \leftrightarrow$ states in $\mathcal{E}_{\lambda+p}$. 


How to come up with colored lattice models?

[Show the table with patterns]

Colored lattice model \( E_{\mu, 3, 3'} \)

Top row is \( \mu \), input permutation is \( 3' \), and output permutation is \( 2 \). Models were introduced in 2019 by BBBG and BW.
Blackbox 2

Iwahori-Whittaker functions $W_\theta^\natural : G \to \mathbb{C}$ for $\theta \in W \times S_n$ are given by

$$W_\theta^\natural(g) = \int_{U^-} f_\theta^\natural(ug \psi(u)) du,$$

where $f_\theta^\natural(g) = f_\theta^\natural(u \cdot \omega \cdot \sigma' \cdot \kappa) = \begin{cases} 0, & \text{if } \sigma' \neq \sigma \\ e^v & \text{if } \sigma' = \sigma. \end{cases}$

Then McNamara's decomposition

$$U^- = \bigcup_m C_m$$

is not enough

because $f_\theta^\natural$ is not constant on $C_m$

We need to explore $C_m \cap B \mathcal{S} J(\sigma' \sigma^{-1})$,

the intersections of $C_m$ with double Iwahori cosets.

$$U^- = \bigcup_m C_m$$ and $$U^- = \bigcup_w B w J$$
**Thm (N, ’20)** \( C_{N_1} \cap B_{G_1} J_1(g_1)^{-1} = \bigcup_{n} S_{m,n} \).

**Punchline** Each \( S_{m,n} \) corresponds to a state in colored lattice model with fixed input and output colors.

**Thm (N, ’20)** There are weight preserving bijections between States of colored lattice model ↔ Colored GT patterns ↔ Colored Lusztig data.

**Thm (N, ’20)** We can express Iwahori Whittaker functions in terms of colored data above.
Questions:

A. Other groups besides $GL_n$, $Sp(2n)$, $O(n)$?

B. Metaplectic covers? In progress.

C. Other decompositions of $U$?

There are two "good" ones.

\[
\begin{pmatrix}
GL_{n-1} & 1 \\
-1 & 1
\end{pmatrix}
\quad \quad
\begin{pmatrix}
\dot{\cdot} \\
\dot{\cdot} & GL_{n-1}
\end{pmatrix}
\]