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*The work of science is to substitute facts for appearances  
and demonstrations for impressions.—RUSKIN*

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egation is in conflict with rating principles. Delegation to contract reserves may also lead to contingent claims being overlooked or their recognition deferred since the shortfall of future revenues in meeting contract reserve expenditures can be nebulous under contracts where the insurer has the right of nonrenewal and/or the right to adjust future premiums.

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## MEASURING THE INTEREST RATE RISK

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### ABSTRACT

This paper develops the theory of the measurement of interest rate risks from its foundations, beginning with the question of which asset values (market or book) are economically relevant and therefore at risk. Upon this foundation, the paper builds a flexible and general theory of the measurement of interest rate risk that includes the familiar Macaulay-Redington theory as one special case. The theory is applied using a simple model of interest rates to allow separate measurement of the risks associated with permanent and transient changes in interest rates.

### I. INTRODUCTION

Few subjects have generated more controversy or been the topic of more meetings among actuaries than the subject of the interest rate risk. Whatever an actuary's field of practice, he faces the thorny problem of judging or, if possible, measuring the risk of changing interest rates to the financial security plans he advises. That problem has become more important in recent years with the increasing volatility of interest rates and, for U.S. actuaries, the introduction of interest rate futures contracts, the huge and unprecedented federal budget deficits, and the accompanying uncertainties about likely future movements in interest rates.

It is fundamental that the interest rate risk is an equity risk—losses are incurred only when assets fall more (or rise less) than liabilities. The Macaulay-Redington theory of immunization, which has become fairly well known among North American actuaries over the last decade, reflects an awareness of that fact. That theory, which is reviewed in section III, measures the vulnerability of a portfolio of assets in relation to a set of liabilities using a construct called a "duration." The durations,  $D_A$  of assets and  $D_L$  of liabilities, are defined so that a one percent increase in the interest rate will result in approximately a  $D_A$  percent decrease in the value of the assets and a  $D_L$  percent decrease in the value of the liabilities. According to this theory, when the values of the assets and liabilities are equal, the difference,  $D_A - D_L$ , measures the loss that would occur from a 1 percent increase in the interest rate, computed as a fraction of total assets. A positive difference indicates that asset values will fall faster than liabilities when the interest

rate rises and rise faster than liabilities when the interest rate falls. Later, I will review the computation of the duration measure as well as related measures. The important point is that they all purport to provide an index of the vulnerability of a portfolio of assets to interest rate changes in relation to a set of liabilities.

Some of the controversy over immunization theory concerns the question of just what "the interest rate" means. Some argue that it means the market rate of interest, though that cannot be precisely right, since there is not a single rate that applies to all investments in fixed-income securities, independent of the term to maturity, the call provisions, the issuer's solvency, and so on. Others argue that "the interest rate" means the rate that the actuary reasonably forecasts to apply on average to investments that will be made in the future. Such an evaluation, of course, is inherently subjective, and it is subject to the criticism that there is no good reason to forecast a single rate for all years in the future. Vanderhoof and other advocates of immunization theory have argued the reverse proposition that immunization theory, because it identifies the assured return from a particular investment strategy relative to a particular set of liabilities, should guide the interest assumption made by actuaries in valuing liabilities.

The proper resolution of these disputes depends on the way interest rates are determined in the bond markets. If long-term interest rates fluctuated randomly around some "normal" level, so that very high rates in any one year were simply an aberration and rates could be counted on to fall in the very near future, then a loss in asset value exceeding the decrease in liabilities caused by high current market rates would be transitory and therefore no cause for concern. Indeed, very high current rates would represent an unusually favorable opportunity to invest in long-term bonds to lock-in the current attractive returns. That seems to be the interest rate model that those who advocate basing asset valuations on actuarial assumptions have in mind. However, if high long-term interest rates are caused by a change in the environment that is likely to persist for awhile, so that currently owned assets may eventually have to be sold in a high interest rate (low price) environment, then the loss in market value of assets from high interest rates is a real cause for concern. Thus, the right interest rate to use for studying the value and vulnerability of an asset depends very much on one's theory of how bond markets work.

To illuminate the relationship between prevailing market interest rates and "true" asset values, we must begin by returning to the fundamentals of the theory of present value. This is done in section II, using the standard eco-

nomic treatment of the theory, which differs in some small but crucial details from the way the theory has usually been treated by actuaries. The theory is developed for investments where the only risk is that prevailing interest rates will change. The main conclusion is that if the cost of bond trading is negligibly small and market prices for bonds are "internally consistent" (or "arbitrage free") then every cash-flow stream has a single objective value at every point in time. That value is the stream's present value computed at the year-by-year interest rates implicit in current bond prices.

Apart from the assumption that bonds are liquid assets and that bond prices are internally consistent, the fundamentals reviewed in section II do not require any additional assumptions about the details of how the bond markets operate or how interest rates vary over time. In that general context, there is no way to measure interest rate risk in terms of a few simple indexes like the duration index, without making further assumptions about the shapes of possible "yield curves." Therefore, in section III, I consider how a theory of the shape of yield curves can lead to one or more indexes of interest rate risk. It is in that context that I develop the general theory of measurement of the interest rate risk.

Unfortunately, there is no empirically proven theory of the term structure of interest rates at this time. For the practical actuary, the best course for now seems to be to use a rough-and-ready model of interest rates. The Macaulay-Redington theory is based on such a model in which the term structure of interest rates has an unvarying shape, so that a 1 percent increase in the short-term rate is always mirrored by a 1 percent increase in the yields to maturity of bonds of all durations. The fact, however, is that the shape of yield curves varies over time and that short-term interest rates are very much more volatile than long-term rates. As a result, the duration measure overstates the sensitivity of long-term bond values to changes in the interest rate environment and understates the sensitivity for short-term bonds.

In section IV, we suggest a refinement of immunization theory based on a simple two-parameter theory of yield curves, which allows short-term rates and long-term rates to move separately but requires intermediate-term rates to be the interpolated value between them. The result is a pair of measures, reflecting the sensitivity of the portfolio of assets and liabilities to changes in short- and long-term rates separately. These measures are offered as temporary expedients. Research into the actual term structure of interest rates is proceeding rapidly (see the recent work by Cox, Ingersoll and Ross [1]), and the results of that research offer the promise of a more reliable set of measures in the foreseeable future.

## II. THE ECONOMIC FUNDAMENTALS OF PRESENT VALUE

There is a small but crucial difference between the foundations of the theory of present value as traditionally developed in the actuarial mathematics of compound interest and the foundations developed by economists. In traditional actuarial theory, there is a "bank" that stands willing to accept deposits or lend money at fixed interest rate  $i$ . If the interest rate varies in a predictable way over time, so that the rate in year  $t$  is  $i_t$ , there is no problem: the value of any certain (that is, nonrandom) stream of cash flows  $(F_1, \dots, F_n)$  is simply its present value, computed at the year-by-year varying interest rates. The real problem begins when the interest rate offered by the bank is volatile and unpredictable; one does not know then what rate to use in discounting flows in future years.

The problem is not as bad as one might think because long-term bonds exist which allow one to lock in an interest rate over an extended period. An approach that some actuaries have advocated for evaluating cash-flow streams is to discount the flows using rates of interest that are a blend of the rate forecasted to be available on future investments and the rate locked-in by existing assets. The appeal of this procedure is that it reduces the amount of subjectivity in the interest rate forecasts, at least for years in the near future, because the rates obtained depend largely on the existing portfolio of assets. Still, unlike the economic theory that follows, this theory admits a substantial role for subjective element.

The economic theory of present value is a branch of price theory. As a matter of terminology, let us say that if some item, say a toaster, can be bought or sold in the marketplace at a given price, say \$15, then its "economic value" is \$15, regardless of whether the owner likes toast. No one will be willing to pay more than \$15 for a toaster when he can buy one for that amount in the marketplace. Anyone would be happy to buy a toaster for \$14 if he can resell it for \$15. In an idealized, "frictionless" market where toasters can easily be bought and sold at a fixed price, a toaster has the same value to everyone, in the sense that anyone would be happy to buy a toaster for any amount less than \$15 and nobody would be willing to pay even a penny more. It is in this sense that economic values are objective.

To apply this theory to the bond market, we must suppose that the costs incurred in buying and selling bonds are a negligible factor in the determination of value. The theory applies if bonds are sufficiently "liquid." The implications of an objective theory of value are far reaching. Suppose, for example, that we wish to evaluate some particular investment, represented by the sequence of cash flows  $F = (F_1, F_2, F_3)$  over a period of three years,

and that there are bonds available in the market with coupons  $C_1$ ,  $C_2$ , and  $C_3$  that mature in one, two, and three years, respectively, for a maturity value of 1. A portfolio of these bonds is represented by a vector  $x = (x_1, x_2, x_3)$ , where  $x_n$  is the number of bonds with maturity  $n$  in the future that the investor owns. The cash flows  $y = (y_1, y_2, y_3)$  generated by the portfolio  $x$  can be computed by multiplying the matrix  $B$  of bond returns by the vector  $x$  describing the portfolio:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 + C_1 & C_2 & C_3 \\ 0 & 1 + C_2 & C_3 \\ 0 & 0 & 1 + C_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (1)$$

or, more compactly,  $y = Bx$ .<sup>1</sup>

Thus, a portfolio  $x$  exactly matches the cash flow  $F$  if and only if it is a solution to:  $F = Bx$ . Notice, however, that the matrix  $B$  is upper triangular (that is, all entries below the diagonal are zeros) with nonzero entries on the diagonal. Such a matrix is always invertible, and therefore there is a unique solution  $x = B^{-1}F$  to equation (1).<sup>2</sup> Suppose the prices of the three bonds are  $p_1$ ,  $p_2$ , and  $p_3$ , respectively, and let  $p$  be the vector of the three prices:  $p = (p_1, p_2, p_3)$ . Then the net outlay required to purchase the portfolio  $x$  is  $\sum p_j x_j$  or, in vector notation (treating  $p$  as a row vector and  $x$  as a column vector),  $px$ .<sup>3</sup> Since  $x = B^{-1}F$ , the purchase price can be expressed as  $pB^{-1}F$ .

According to economic price theory, as long as the investor is free to buy or sell bonds at the prices  $p$ ,  $pB^{-1}F$  is the unique objective value of the cash-flow stream  $F$ . This is so even though the interest rates that will be available on future investments may be unknown. If an investor had to pay some amount  $P > pB^{-1}F$  to acquire the stream  $F$ , then by buying the portfolio  $x = B^{-1}F$  he could acquire the same cash-flow stream at a lower price, so he should decline to make the purchase. If, instead, the investor could buy the stream  $F$  for some amount  $P' < pB^{-1}F$ , by selling the portfolio  $x$  (with safe proceeds  $pB^{-1}F$ ) and purchasing the stream  $F$ , he would leave

<sup>1</sup>I use column vectors ( $n \times 1$  matrices) to represent both cash-flow streams and portfolios. Later, prices are represented by row vectors ( $1 \times n$  matrices).

<sup>2</sup>In general, some components of the solution  $x$  may be negative. In that case, the investment strategy needed to match  $F$  involves selling some kinds of bonds. This corresponds to the need in the actuarial version of the theory to borrow from the "bank" to justify the present-value evaluation of some cash-flow streams.

<sup>3</sup>The purist may object that  $px$  is a  $1 \times 1$  matrix, rather than a real number, so that the expression should be the trace of the expression I have written. To avoid unnecessary notation, I shall not distinguish between numbers and  $1 \times 1$  matrices.

future cash flows unchanged while enjoying immediate net cash receipts of  $pB^{-1}F - P'$ . Since these are positive, it is obviously profitable to make the purchase. Any investor, regardless of his preferences or the other assets in the portfolio, should be willing to pay  $P$  to acquire the cash-flow stream  $F$  if and only if  $P < pB^{-1}F$ . This is the meaning of our claim that  $P$  is the objective value of the cash-flow stream  $F$ , a value that depends neither on the investor's preferences nor on the content of the portfolio.

The foregoing analysis is easily generalized to numbers of periods other than three. Given a set of  $n$  bonds with varying maturities including one of each maturity ranging from one year to  $n$  years, one can construct an upper triangular matrix  $B$  of bond cash flows in the manner illustrated. The matrix will have nonzero entries on its diagonal. Then, for any cash-flow stream  $F$  that expires in  $n$  periods or less, there is a unique portfolio  $x$  that precisely matches the cash flow  $F$ , that is, a unique solution to  $F = Bx$ . That portfolio can be purchased for the price  $px = pB^{-1}F$ , which is therefore the economic value of the stream  $F$ .

The given matrix form is useful for simplifying the economic argument, but for computations it may be more convenient to express the economic value of  $F$  in present-value terms.

**PROPOSITION.** *The economic value of the cash-flow stream  $F$  is  $pB^{-1}F$ . This value is the present value of the cash-flow stream  $F$  computed using the period-by-period interest rates  $i_1, \dots, i_n$  implicit in the bond prices, that is, the rates that set the present value of the cash flows on each bond equal to its price.*

*Proof.* Let  $PV(F)$  denote the present value of any cash-flow stream  $F$  using the interest rates described in the proposition. As is well known, and easily verified, the present value function  $PV$  is linear, that is, for any two cash-flow streams  $F$  and  $F'$  and any constants  $\alpha$  and  $\beta$ ,

$$PV(\alpha F + \beta F') = \alpha PV(F) + \beta PV(F').$$

Regarding  $pB^{-1}F$  as a function of  $F$ , it, too, is linear. By construction, these two linear functions agree in their evaluation of the bond cash flows, and those are a basis for the  $n$ -dimensional vector space. Hence, the two linear functions must be identical. Q.E.D.

Thus, when bonds are a liquid investment in the sense that they can be bought and sold with negligible transactions costs, any riskless cash-flow stream has an objective value, which is the present value of the stream at the year-by-year interest rates implicit in the bond prices. This conclusion means that the proper interest rate to use in discounting cash flows for the

analysis of an investment opportunity depends neither on the other investments held by the firm, nor on the nature of the investor's liabilities, nor on any other similar factor. There is no room for subjectivity in the choice of interest rates in the world we have described.<sup>4</sup> When liquid bond market investments are available, an investment outside the bond market can be worthwhile only if it is less expensive than the bond market investment with the identical cash flows. Moreover, an outside investment is always worthwhile if it can be financed by selling bonds from the portfolio in such a way as to exactly match the investment cash flows while leaving a positive amount of extra cash on the table today.

I wish to emphasize at this point that the foregoing analysis does not mean that there is an objectively best strategy for investing in the bond market. It asserts that one should always accept "bargains" (bonds offered for less than their market price) and sell bonds which are overpriced relative to other bonds. If an investor expects long-term rates to rise soon, it is generally wise to sell long-term bonds. Conversely, when the investor expects the rates to fall, then to be consistent with his expectation, the investor will buy long-term bonds. These are standard conclusions that are unaffected by the prescriptions I have been offering.

At this point, what happens if there are many bonds traded in the marketplace each of different maturity? Is it not possible that there is another set of bonds with cash-flow matrix  $B'$  and price  $p'$  such that  $p'B'$  is different from  $pB$ , so that there are two different "economic values" for a cash-flow stream  $F$ ? This question is akin to the question that one might ask in the "banker" model of present value, if there were two bankers offering different interest rates. If the world were like that, it would be possible to borrow large sums from the bank charging the low rate to use for making deposits in the bank offering the high rate, netting large and certain profits to the investor. Such a situation could not persist, since a surge of investors exploiting the opportunity would soon force one of the banks to change its policy or fall into bankruptcy.

Similarly, an inconsistency in the prices in the bond market offers what is known as an arbitrage opportunity. Investors or firms could sell (or issue) the bonds that were relatively overpriced to finance the purchase of underpriced bonds, earning a certain profit. Investors exploiting an arbitrage opportunity would soon force brokerage houses to change their price quotations, to balance supply and demand, or to deplete the inventories of the offending

<sup>4</sup>However, the world I have described omits all tax considerations, which are important in the United States and Canada at the present time. I leave the analysis of tax consequences to others.

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brokerage house so that the arbitrage opportunity disappears. Arbitrage opportunities—the flaw in the market that permits one to earn huge sums at no risk with a tiny initial investment—are the dream of every new investor. Practical investors soon learn that such opportunities, if they exist at all, are rare and fleeting, just as economic theory predicts. We can safely build a theory of present value on the hypothesis that arbitrage opportunities do not exist or, at least, are of negligible importance for investment analysis. It was this no-arbitrage hypothesis that I alluded to in the Introduction as the assumption of “internal consistency.”

To understand the major conclusions of financial economic theories and the perspectives they lend, one must grasp fully the importance and extent of the no-arbitrage hypothesis. When one finds two bonds with nearly identical coupons and maturities offering quite different yields, the hypothesis directs us to find an explanation in terms of differences in call provisions, convertibility provisions, default risk, tax treatment, or some similar factor. To isolate the interest rate on a riskless investment in the United States, one must use U.S. Treasury bonds, which are fully call-protected and suffer virtually no default risk.

In the outlined theory, there is no single measure that summarizes the sensitivity of a portfolio of assets to changes in the market interest rates, either alone or in relationship to a set of liabilities. For example, suppose that a certain liability can be represented by a series of cash flows  $L = (L_1, \dots, L_n)$ , and that there is an associated set of assets of equal present value whose flows are represented by  $A = (A_1, \dots, A_n)$ . How sensitive is the difference in the present values of  $A$  and  $L$  to changes in the prevailing level of market interest rates? The question is difficult even to pose in the present context for the notion of a prevailing level of rates is ill-defined. There are, after all,  $n$  interest rates  $i = (i_1, \dots, i_n)$ —one for each year, and the sensitivity of values to each of them is different. There is little value to reporting  $n$  separate measures, though this is precisely what is required for a complete evaluation of risk in an environment where each interest rate is free to change independently of all others. Fortunately for those who seek simple measures, the interest rates  $(i_1, \dots, i_n)$  have some tendency to change together. That is where immunization theory comes in.

### III. YIELD CURVES AND THE INTEREST RATE RISK

With no theory of how interest rates move, the problem of measuring the vulnerability of a portfolio of assets relative to a set of liabilities has no practical solution because there is a different sensitivity of the portfolio to

the interest rate at each different maturity. However, the interest rates for each different maturity do not fluctuate completely independently. Long-term rates tend to move up and down together, and there is even some linkage between long- and short-term rates. The function  $i_t$  that specifies the yield available on bonds maturing in each year  $t$  is often presented in graphical form, and called the “yield curve.” By making assumptions about the possible shapes of the yield curve, one can simplify the problem of measuring interest rate risk. The accuracy of the measures depends in part on the accuracy of the theory of yield curves used and in part on the kind of portfolios to which the measures are applied.

The simplest theory of this form is the Macaulay-Redington (M-R) immunization theory, which normally assumes that  $i_t = i$ , that is, that the rate of interest is the same for all durations. The mathematical analysis works out most neatly when one works with the continuously compounded rate (or force) of interest  $\delta$ , and a formulation of this kind also makes possible a significant extension of the M-R theory. The key assumption of the theory is that the continuously compounded rate of interest applicable to time  $t$  is  $\delta(t) = \delta + \Delta(t)$ . The parameter  $\delta$  determines the overall level of the yield curve while the function  $\Delta(t)$  fixes its shape—the variant reported by Vaunderhoof specifies  $\Delta(t) = 0$  for all  $t$ . If the level of interest rates changes gradually over time so that investment managers can rebalance portfolios as conditions change, then the relevant risk to measure is the risk to the portfolio of small changes in  $\delta$ . Letting  $A$  represent the cash-flow stream associated with the assets and  $L$  the stream associated with the liabilities, we assume that the two streams initially have the same present values. Also, we assume that the two streams expire after  $n$  years. Then,

$$\sum_{t=1}^n A_t \exp\left(-\int_0^t \delta(s) ds\right) = \sum_{t=1}^n L_t \exp\left(-\int_0^t \delta(s) ds\right).$$

Letting  $D(t) = \int_0^t \Delta(s) ds$ , we may rewrite this as:

$$PV(A; \delta) = \sum A_t e^{-\delta t - D(t)} = \sum L_t e^{-\delta t - D(t)} = PV(L; \delta). \quad (2)$$

We regard these present values as functions of the overall level of interest rates  $\delta$ . Then the duration  $D_A$  of the assets is defined to be minus the derivative of  $PV(A; \delta)$  divided by  $PV(A; \delta)$ ; this measures the percentage decrease in  $PV(A; \delta)$  per unit of increase in the level of interest rates. Thus,

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Letting primes denote differentiation with respect to  $\delta$ , one defines

$$D_A = \frac{-dPV(A; \delta)/d\delta}{PV(A; \delta)} = \frac{\sum t A_t e^{-\delta t}}{\sum A_t e^{-\delta t}} \quad (3)$$

A similar expression describes  $D_L$ , the duration of the liabilities. The "duration" of assets is so named because it represents a present-value weighted average of the time until the cash flow occurs. For example, for an asset which generates flows only in year five, the duration is simply five.

The duration of assets or liabilities is not significant by itself. The difference in duration between assets and liabilities is important since that measures the change in surplus per 1 percent change in the overall rate of interest measured as a fraction of total asset values. To express the same thing in symbols,

$$D_A - D_L = \frac{PV'(A; \delta) - PV'(L; \delta)}{PV(A; \delta)}$$

The advantage of measuring durations in percentage terms is that the duration of a portfolio of assets is the average of the durations of the assets in the portfolio. Therefore, if one wants to raise the portfolio duration from  $D_A$  to  $D'_A$ , one can do so by purchasing bonds with durations exceeding  $D'_A$  using new funds or the proceeds from selling bonds with shorter durations.

Vanderhoof's account of immunization theory (with  $\Delta(r) = 0$ ) also gives a role to the second derivative of  $PV(A; \delta)$  and  $PV(L; \delta)$  with respect to  $\delta$ , arguing that if the first derivatives are equal and the second derivative of  $PV(A; \delta)$  exceeds that of  $PV(L; \delta)$ , then any small change in the overall level of interest rates from  $\delta$  to  $\delta'$  will result in a profit:  $PV(A; \delta') > PV(L; \delta')$ . This conclusion, though mathematically correct, is not a proper or useful interpretation of immunization theory. Indeed, one can say more generally that the no-arbitrage hypothesis implies that there is no place in immunization theory for measures based on second derivatives of the present-value function. What has actually been shown by this mathematical argument based on second derivatives is that if the flat yield curve theory were correct and if interest rates change frequently and by small amounts, then there must be an arbitrage opportunity. To see this, let  $x_A$  and  $x_L$  be portfolios of assets that duplicate the cash-flow streams  $A$  and  $L$ , respectively, with  $PV(A; \delta) = PV(L; \delta)$ ,  $PV'(A; \delta) = PV'(L; \delta)$ , and  $PV''(A; \delta) > PV''(L; \delta)$ . Then buying the portfolio  $x_A$  and selling  $x_L$  generates a cash-flow stream  $A - L$

with present value zero and  $PV'(A-L; \delta) = 0$ , but  $PV''(A-L; \delta) > 0$ , that is, a portfolio that costs nothing but yields a certain profit whenever  $\delta$  changes, which it certainly will do. Such a portfolio represents an arbitrage opportunity. If the no-arbitrage hypothesis is correct, then there must be something fundamentally wrong with the hypothesis that the yield curve is always flat. Of course, actual yield curves are not flat and do not maintain the rigid shape prescribed by our assumptions as they vary over time. The interest rate model underlying the theory is only a very rough approximation. It may be suitable for use in generating approximate measures of asset vulnerability but it should not be used for fine-tuning investment choices.

At this point, we can also address the question about whether interest rates fluctuate randomly around a "normal" level. The idea that bond prices (and therefore interest rates) fluctuate in that way is the basis for the argument that market rates should not be used for actuarial valuations. The main point is that randomly fluctuating long-term interest rates on a day-to-day or week-to-week basis always generate effective arbitrage opportunities. If the random fluctuations theory correctly described the workings of bond markets, one could do a statistical study to estimate the normal level and find out whether rates are currently higher or lower than that level, and then buy or sell accordingly in the market for bonds or interest rate futures, financing the purchases by sales of short-term assets. That would lead to positive expected profits for each day or week. Then, since the fluctuations are presumed to be independent over time, the result of a consistent strategy of trading in bonds (or interest rate futures contracts) would be a certain improvement in the yield of the portfolio in the long run (by the Strong Law of Large Numbers of probability theory), which constitutes an arbitrage opportunity. Thus, the no-arbitrage hypothesis—a hypothesis that is strongly supported by both economic theory and the experiences of investors—implies that the random fluctuations view of long-term interest rates must be incorrect.

Some readers may think the previous paragraph too harsh a critique of the random fluctuations view. After all, one might argue, nobody believes that the fluctuations in interest rates around their normal level are statistically independent on a day-to-day, week-to-week, or even a month-to-month basis. But to concede that point is to give up the argument. If changes in interest rates are persistent over long periods and if assets and liabilities are not perfectly matched, then one may have to trade at near current market prices, so there is no merit in the view that fluctuations in asset prices should be mostly ignored as temporary and irrelevant phenomena.

As noted above, the flat yield curves originally used for developing the

duration measure are not consistent with the no-arbitrage hypothesis, that is they imply that there exist arbitrage opportunities.<sup>5</sup> Economic theories of the yield curve (such as the Cox-Ingersoll-Ross theory) that do not allow arbitrage opportunities are often much more mathematically complex than the simple theories already analyzed and have not yet been subjected to the kind of empirical scrutiny needed to lend some degree of confidence to them. Moreover, there is really no prospect of ever specifying a perfectly correct theory of interest rates upon which to base a theory of immunization, so the best course for practical people is to base immunization measures on simple, approximate theories of interest rates. These theories should be used only to construct vulnerability indexes based on first derivatives of the present-value function, in order to avoid the kind of misleading recommendations that necessarily result from the second derivative measures, which are effectively based on trying to identify by mechanical means that rare and elusive animal—the arbitrage opportunity.

To build a general theory to measure interest rate risk, let  $I = (I_1, \dots, I_k)$  denote a series of  $k$  economic indexes that summarize both the current interest rate environment and whatever other factors affect expectations about future cash flows. Let  $A_t(I)$  and  $L_t(I)$  denote the expected cash flows from a certain asset portfolio and set of liabilities, respectively, in year  $t$ . These generally depend on  $I$  since, for example, the cash surrenders and policy loans on individual ordinary life insurance are sensitive to prevailing interest rates, and the payments on health insurance plans are sensitive to the inflation index for health care costs. Let  $\delta(r; I)$  denote the continuously compounded rate of interest implicit in current bond market prices for time  $t$  when the index values are  $I$ . We assume that all functions of  $I$  are continuously differentiable. We wish to measure the vulnerability of the cash-flow stream to a change in index  $I_j$ , with other indexes held constant. Let  $PV(A; I)$  represent the present value of the cash-flow stream  $A$  and  $PV_L(A; I)$  the partial derivative with respect to  $I_j$  (with other index values being held constant). Then the ratio  $V_j(A) = -PV_L(A; I)/PV(A; I)$  is the decrease in the asset value per unit increase in the index  $I_j$ , as a percentage of the total asset value. A similar calculation can be done to compute a vulnerability index  $V_j(L; I)$  for liabilities.

Since the indexes  $V_j(A; I)$  and  $V_j(L; I)$  are basically first derivatives of

<sup>5</sup>In fact, one can show that this model is consistent with the requirement that there is no possibility of arbitrage if and only if  $\delta(I) = \alpha + \beta I$  for some constants  $\alpha$  and  $\beta$ . The magnitude of  $\beta$  depends on the volatility of interest rates. In particular,  $\beta$  can be zero only if interest rates do not fluctuate at all. A reasonable specification would set  $\beta > 0$ .

the present-value function, it may appear that they give a good idea of vulnerability only for very small changes in interest rates. One can extend things a bit using the argument given earlier in connection with the discussion of Vanderhoof's theory. (NOTE: The conclusions reached in the remainder of this paragraph and in the next paragraph are incorrect. See the author's review for corrections to this analysis.) In particular, if  $A$  and  $L$  have equal present values and equal vulnerabilities when evaluated at market interest rates, then the no-arbitrage hypothesis implies that  $PV_L(A; I) = PV_L(L; I)$ . This fact suggests two important conclusions: First, equating indexes based on first derivatives immunizes against small changes in interest rates even if they are large enough to alter the first derivative term (provided the first derivative itself is approximately linear in the relevant region). Second, maintaining a strategy of approximate immunization over time does not require an excessive amount of trading, since an immunized portfolio remains (approximately) immunized as  $I$  fluctuates in a neighborhood of its original value.

As we have seen, if one rebalances the asset portfolio over time to maintain (approximate) equality of the asset and liability vulnerability indexes, then the change in the  $PV(A - L; I)$  over any short period is of "second order."<sup>6</sup> The crucial mathematical fact now is that if the changes are second-order over all short periods of time, then their sum over longer periods is also negligible. Therefore, immunizing effectively against the losses suffered from small changes in  $I$  over all short periods also solves the larger problem of immunizing against the large movements in  $I$  that may take place over longer periods.

In general, the risk indexes  $I$  that we have been discussing abstractly need not be interest rate indexes; they could equally well consist of an oil price index, a health care costs index, etc., and such indexes may well be useful for evaluating the risks associated with certain specialized kinds of insurance or with portfolios containing common stock, real estate, etc. To be useful, the number of indexes in the theory must be small. For bond portfolios, it is probably necessary to use at least two indexes of interest rates, since yield curves vary over time in both their slope and their level. In the next section, we introduce a particularly simple and familiar two parameter theory of the yield curve that allows the curve to vary in both slope and level, and we derive the resulting risk measures.

<sup>6</sup>This means that the change over any period of time, during which the portfolio was not rebalanced, is on the order of the square of the length of time involved. If the relevant period is one month,  $1/12$  of a year, then the squared value is  $1/144$ , which is a negligibly small number for applications of this kind.



## IV. A SIMPLE TWO-FACTOR THEORY OF IMMUNIZATION

A two-parameter model of the term structure of interest rates makes it possible to separate the effects of changes in long-term rates from those of short- and medium-term rates. By so doing, it reduces the bias that is inherent in the Macaulay-Redington theory due to its false assumption that long- and short-term rates are equally volatile and closely tied. For a useful two index model of interest rates, I propose a variant of a model that is familiar to many actuaries—a model that has been widely used in pension valuations, premium rate calculations, GAAP reserve calculations for life companies, and other applications. Specifically, I use a two-point graded interest-rate model, with the grading occurring up to some predetermined time  $T$ . More precisely, the continuously compounded rate (force) of interest at any time  $t > T$  is the long-term rate  $\delta(t) = I_1$  and for  $t \leq T$  the rate is linearly interpolated between the long rate  $I_1$  and the short rate  $I_1 + I_2$ :  $\delta(t) = I_1 + (1-t/T) I_2$ . Letting  $M(t) = \min(t, T)$ , we have:<sup>7</sup>

$$\delta(t) = I_1 + (1-M(t)) I_2. \quad (4)$$

I use a long-term rate plus a transient component (rather than a long rate and a short rate) as interest indexes to make the first vulnerability measure take the same form as the traditional duration measure. It is equally valid and perhaps more readily interpretable to construct vulnerability indexes using the long- and short-rate indexes—this is primarily a matter of style and taste.

Before proceeding, I offer a brief warning: the theory specified is inconsistent with the no-arbitrage hypothesis. I shall use it only to develop first-derivative based measures of vulnerability, in the hope that these measures may approximate the ones that could be obtained from the "correct theory."

Given our specification, the present value  $PV(A;T)$  is given by:

$$\begin{aligned} PV(A;T) &= \sum_{t=1}^n A_t \exp\left[-\int_0^t \delta(s) ds\right] \\ &= \sum_{t=1}^n A_t \exp\{-I_1 t - I_2 T M(t) [1 - M(t)/2]\}. \end{aligned} \quad (5)$$

<sup>7</sup>Of course, it is also possible to specify:

$$\delta(t) = I_1 + (1-m(t)) I_2 + \Delta(t)$$

and to allow  $M(\cdot)$  to be nonlinear.

The vulnerability measure for the first index is then given by  $D_{A1} = -PV_1(A;T)/PV(A;T)$ , where

$$-PV_1(A;T) = \sum_{t=1}^n t A_t \exp\left[-\int_0^t \delta(s) ds\right]. \quad (6)$$

The identity of form between this measure and the duration measure masks an important distinction; this measure reflects the sensitivity of asset values to permanent changes in the interest rate environment, that is, changes that are reflected in yields-to-maturity in bonds of all maturities. Such changes tend to be smaller than changes in any overall interest rate index which give substantial weight to short-term interest rates.

The second measure records the vulnerability of an asset portfolio to transient changes in the interest rate environment, that is, changes whose effects are not reflected in the rates that apply to years from year  $T$  on. Once again, to measure vulnerability in the convenient percentage of assets format, we use the form:  $D_{A2} = -PV_2(A;T)/PV(A;T)$ . Using the second expression in equation (5), one may compute the derivative as:

$$-PV_2(A;T) = \sum_{t=1}^n A_t T M(t) [1 - M(t)/2] \exp\left(-\int_0^t \delta(s) ds\right). \quad (7)$$

Just as the first index is a present-value weighted average of the times  $t$  to payment of the cash flows, this second index is a present-value weighted average of a particular function of time, namely,  $T M(t) [1 - M(t)/2]$ . This index is easy to compute for both assets and liabilities, and it reflects the relatively large sensitivity of the value of medium-term cash flows to transient fluctuations in interest rates.

## V. CONCLUSION

The theory of immunization is not a theory that is properly studied in isolation without reference to theories of financial markets, present values, and the like. The controversies that rage over matters such as valuation interest rate assumptions and the nature and measurement of the interest rate risk are founded in differences in the underlying theory of bond markets that the debaters carry in their minds. The main issues can be resolved only by exposing and examining the underlying theories, reaching a consensus based on the best available evidence, and building an immunization theory and a set of measures of risk on those sound foundations.

I have tried to carry out just such a program in this paper. In section II,

the economic theory of valuation is laid out, founded on the observation, which is almost universally affirmed by financial economists and business people, that the bond markets are not teeming with arbitrage opportunities. The conclusion is that when the subject of interest is real economic values of the kind that should guide pricing and investment decisions, cash-flow streams should be valued using the rates of interest implicit in bond prices. In particular, assets should be evaluated at market values and liabilities should be evaluated by a present-value calculation using the corresponding market-determined interest rates. These are the real values of the assets and liabilities whose vulnerability to fluctuating interest rates requires measurement.<sup>9</sup>

Having identified the values to be immunized, I observed that without some knowledge of the structure of the yield curves, one cannot have a sound theory of immunization. I summarized the Macaulay-Redington theory and set it in the framework of a general theory, which can be specialized to take account of whatever may be learned in future studies of the yield curve. Finally, I offered a simple, practicable enhancement of the Macaulay-Redington theory which is similar to it in form but which allows separate measurement of the risks associated with transient and permanent changes in the interest rate environment. In general, permanent changes of any given magnitude have a larger effect on asset values than transient changes of the same magnitude, but transient fluctuations (in short-term rates) tend to be more frequent and larger in magnitude. The importance of the enhancement depends on the relative volatility of long- and short-term rates and the nature of the assets and liabilities under study.

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<sup>9</sup>In a regulated environment, some nominal values may also be important, simply because the regulations say they are. Management may then care about the sensitivity of these asset values to changing interest rates. However, except when threats to solvency are immediate, it is the real values of assets, that is, the market values, that should be of principal concern.

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