Wreath Products of Distributive Forest Algebras

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Introduction

A much-studied problem in the theory of automata is: 
*Can a given regular language $L$ be defined by a formula of some logic?*
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*Give an effective characterization of the precise expressive power of the logic.*
Introduction

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- In other words, to *Give an effective characterization of the precise expressive power of the logic.*
- For automata over **words**, we have effective tests for definability in many temporal and predicate logics.
A much-studied problem in the theory of automata is: *Can a given regular language \( L \) be defined by a formula of some logic?*

In other words, to *Give an effective characterization of the precise expressive power of the logic.*

For automata over **words**, we have effective tests for definability in many temporal and predicate logics.

For automata over **trees**, situation is quite different: Effectively deciding expressibility in \( \text{CTL}, \text{CTL}^*, \text{FO}[\preceq], \text{PDL}, \ldots \) remains open.
Logics on Finite Trees

For all of these logics, decidability remains an open problem. Bojańczyk, et. al., (2012) proved characterizations in terms of iterated wreath products of certain forest algebras.
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For all of these logics,

- Decidability remains an open problem
- Bojańczyk, et. al., (2012) proved characterizations in terms of iterated wreath products of certain forest algebras.
1. For any regular word language $L$, the language of ‘Forests where at least one path is in $L$’ is in PDL.
1. Let $L_1, \ldots, L_n$ PDL languages that partition the set of forests.

Given a forest $f$, add to each node a label $L_i$ if the forest below it is in $L_i$.

Set $f \in L$ iff the resulting forest has a path in $L$. (follows Bojańczyk, et. al., 2012)
Propositional Dynamic Logic

1. Let $L_1, \ldots, L_n$ PDL languages that partition the set of forests.

Let $L$ a regular word language over $\Sigma \times \{a_1, \ldots, a_n\}$. 

(follows Bojańczyk, et. al., 2012)
Propositional Dynamic Logic

1. Let $L_1, \ldots, L_n$ PDL languages that partition the set of forests.

Let $L$ a regular word language over $\Sigma \times \{a_1, \ldots, a_n\}$. Then the following language $\mathcal{L}$ is in PDL:

Given a forest $f$, add to each node a label $L_i$ if the forest below it is in $L_i$.

Set $f \in \mathcal{L}$ iff the resulting forest has a path in $L$.

(follows Bojańczyk, et. al., (2012))
Open Question
Given a regular forest language, can we decide whether it is definable in PDL?
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Given a regular forest language, can we decide whether it is definable in PDL?

Following Bojańczyk, et. al., (2012) and Straubing (2013), we attack this problem using algebraic characterizations in terms of **wreath products** of **forest algebras**.
Background: Forest Algebras

Definition (Bojańczyk and Walukiewicz, 2008)
A tuple \((H, V)\) is called a forest algebra if:

1. \(H\) and \(V\) are monoids
2. There is a faithful action \(V \times H \rightarrow H\)
3. There is a map \(I. : H \rightarrow V\) such that \(I_h h' = h +_H h'\).
4. For each \(h \in H\), there is \(v \in V\) such that \(h = v \cdot_V 0_H\).
Example (Free Forest Algebra)

- \( H = \) set of finite forests over some finite alphabet \( \Sigma \)

\[
\{ \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array},
\begin{array}{c}
\text{a} \\
\text{b}
\end{array},
\begin{array}{c}
\text{a} \\
\text{b}
\end{array},
\begin{array}{c}
\text{a} \\
\text{b}
\end{array},
\begin{array}{c}
\text{a} \\
\text{b}
\end{array},
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array}
\}
\]
Example (Free Forest Algebra)

- $H =$ set of finite forests over some finite alphabet $\Sigma$

\[
\begin{array}{l}
  a, \\
  b
\end{array}
\]

- $V =$ set of such forests where one leaf is a variable

\[
\begin{array}{l}
  X, \\
  X, \\
  X
\end{array}
\]
Example (Free Forest Algebra)

H is a monoid:

\[
\begin{align*}
H &= \{a, b, c\} \\
H &= \{a, a, a\} \\
H &= \{a, a, a\}
\end{align*}
\]

\[
\begin{align*}
b + c &= b \\
c &= c
\end{align*}
\]

V is a monoid:

\[
\begin{align*}
V &= \{a, X\} \\
V &= \{a, a, a\} \\
V &= \{a, a, a\}
\end{align*}
\]

\[
\begin{align*}
X \cdot X &= a \\
X &= a
\end{align*}
\]

Action:

\[
\begin{align*}
\text{Action} &= \{a, a, a\} \\
\text{Action} &= \{a, a, a\} \\
\text{Action} &= \{a, a, a\}
\end{align*}
\]

\[
\begin{align*}
X \cdot a &= a \\
X &= a
\end{align*}
\]
Forest Algebras and Languages

- A forest language is a set of forests – that is, a subset of $H_\Sigma$.
- A forest language $L \subset H_\Sigma$ is recognized by $(H, V)$ iff there is a morphism
  \[ \phi : (H_\Sigma, V_\Sigma) \to (H, V) \]
  such that
  \[ L = \phi^{-1}(\phi(L)) \]

**Proposition**

Regular forest languages are exactly those recognized by finite forest algebras.
Wreath Product

Definition (Bojańczyk, et. al., (2012))

Let \((H_1, V_1), (H_2, V_2)\) be forest algebras.

\[
(H_1, V_1) \wr (H_2, V_2) := (H_1 \times H_2, V_1^{H_2} \times V_2)
\]
Wreath Product

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\]

Action:

\[
(f, v)(h_1, h_2) := (f(h_2)h_1, vh_2)
\]
Wreath Product

Definition (Bojańczyk, et. al., (2012))
Let \((H_1, V_1), (H_2, V_2)\) be forest algebras.

\[[H_1, V_1] \wr [H_2, V_2] := (H_1 \times H_2, V_1^{H_2} \times V_2)\]

Action:
\[(f, v)(h_1, h_2) := (f(h_2)h_1, vh_2)\]

Multiplication:
\[(f, v)(f', v') := (f'', vv')\]
with \(f''(h) := (f(v'h)) \cdot (f'(h))\).
Distributive Forest Algebras

Definition (Bojańczyk, et. al., (2012))

A forest algebra \((H, V)\) is called distributive if any morphism into it respects the identities:

\[
\begin{align*}
A \sqcap B & \equiv v A \sqcap v B, \\
A \sqcup B & \equiv B \sqcup A, \\
A \sqcap A & \equiv A
\end{align*}
\]

Distributive forest algebras can only distinguish forests with different sets of paths.
Characterisation of PDL

Theorem (Bojańczyk, et. al., (2012))

For a regular forest language $\mathcal{L} \subset H_{\Sigma}$, the following are equivalent:

1. $\mathcal{L}$ is definable in PDL
2. There are finite distributive forest algebras $(H_1, V_1), \ldots, (H_n, V_n)$ such that $(H_1, V_1) \wr \ldots \wr (H_n, V_n)$ recognizes $\mathcal{L}$
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- ‘There is a path in $L$’ is recognized by a distributive algebra
- Wreath product simulates composition operation
Characterisation of PDL

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1. $L$ is definable in PDL
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- Wreath product simulates composition operation

Similar characterizations for other classes:

- $CTL$: only one specific distributive algebra is allowed
- $CTL^*$: only aperiodic distributive algebras are allowed
- ...
Iterated Distributive Laws

Definition (cf. Straubing (2013))

For each $k \geq 1$, define a congruence $\sim_k$ on $\Sigma^\Delta = (H_\Sigma, V_\Sigma)$:

1. Base case: $\sim_1$ is the congruence generated by:

\[
\begin{align*}
\begin{array}{c}
A \\
\end{array}
\quad & \sim_1 \\
\begin{array}{c}
B
\end{array}
\end{align*}
\]

2. Inductive case: $\sim_{k+1}$ is the congruence generated by:

\[
\begin{align*}
A \sim_k B & \Rightarrow aA \sim_1 aB
\end{align*}
\]

$k$-fold wreath products of distr. algebras respect $\sim_{k+1}$. 
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Distributive forest algebras respect $\sim_1$. 

```
A
\sim_1
B
```
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   Distributive forest algebras respect $\sim_1$.

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   ![Diagram showing $\sim_1$] 

   Distributive forest algebras respect $\sim_1$.

2. Inductive case: $\sim_{k+1}$ is the congruence generated by:

   ![Diagram showing $\sim_{k+1}$]

   $k + 1$-fold wreath products of distr. algebras respect $\sim_{k+1}$.
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Definition (cf. Straubing (2013))

For each \( k \geq 1 \), define a congruence \( \sim_k \) on \( \Sigma^\Delta = (H_\Sigma, V_\Sigma) \):

1. Base case: \( \sim_1 \) is the congruence generated by:

```
      a
     / \  \(\sim_1\)
   A   B   A   B
```

Distributive forest algebras respect \( \sim_1 \).

2. Inductive case: \( \sim_{k+1} \) is the congruence generated by:

```
      a
     / \  \(\sim_{k+1}\)
   A   B   A
     \   \(\sim_k\)
      a   B
```

\( k + 1 \)-fold wreath products of distr. algebras respect \( \sim_{k+1} \).

A converse to this statement would yield a characterization of PDL!
Iterated Distributive Laws

Definition

$(H, V)$ is \textit{k-distributive} if any morphism into it respects $\sim_k$. 
Iterated Distributive Laws

Definition

$(H, V)$ is $k$-distributive if any morphism into it respects $\sim_k$.

Proposition

- Given $(H, V)$, it is decidable whether there is $k$ such that $(H, V)$ is $k$-distributive.
- Any PDL language is recognized by a $k$-distributive forest algebra, for some $k$.
Iterated Distributive Laws

Definition

\((H, V)\) is \(k\)-distributive if any morphism into it respects \(\sim_k\).

Proposition

- Given \((H, V)\), it is decidable whether there is \(k\) such that \((H, V)\) is \(k\)-distributive.
- Any PDL language is recognized by a \(k\)-distributive forest algebra, for some \(k\).

Open Question

Conversely, are all languages recognized by finite \(k\)-distributive algebras in PDL?
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- Any PDL language is recognized by a $k$-distributive forest algebra, for some $k$.

Open Question

Conversely, are all languages recognized by finite $k$-distributive algebras in PDL?

An affirmative answer would settle decidability of PDL!
Our Main Result

We solve this for $k = 2$:

**Theorem**

1. Let $(H, V)$ be finite and 2-distributive. Then every language recognized by $(H, V)$ is definable in PDL. Further, a product of 4 distributive algebras is enough.

2. It is decidable whether a regular language is recognized by some 2-distributive forest algebra.
Proof of the Main Result

Theorem

Let \((H, V)\) be finite and 2-distributive. Then every language recognized by \((H, V)\) is definable in PDL.

Further, a product of 4 distributive algebras is enough.
Proof of the Main Result

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Let \((H, V)\) be finite and 2-distributive. Then every language recognized by \((H, V)\) is definable in PDL. Further, a product of 4 distributive algebras is enough.

- Given a 2-distributive forest algebra \((H, V)\),
- we seek distributive algebras \((H_1, V_1), \ldots, (H_4, V_4)\) such that

\[
(H, V) \preceq (H_1, V_1) \smile (H_2, V_2) \smile (H_3, V_3) \smile (H_4, V_4)
\]
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\]

We solve two sub-problems related to the left and right factors:

1. To obtain a right factor, we study the problem of separating forest languages by looking at paths only
2. We then compute a ‘minimal’ left factor and show that it is distributive.
Proof of the Main Result

Theorem

Let \((H, V)\) be finite and 2-distributive. Then every language recognized by \((H, V)\) is definable in PDL. Further, a product of 4 distributive algebras is enough.

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\[(H, V) \prec (H_1, V_1) \wr (H_2, V_2) \wr (H_3, V_3) \wr (H_4, V_4)\]

We solve two sub-problems related to the left and right factors:

1. To obtain a right factor, we study the problem of separating forest languages by looking at paths only

2. We then compute a ‘minimal’ left factor and show that it is distributive.

Approach is parallel to much work on logic over words
Separation Lemma

- For a forest $f$, let $\pi(f)$ be the set of paths in the forest.

Lemma (Separation Lemma)

Let $\mathcal{L}_1, \mathcal{L}_2 \subseteq H_\Sigma$ be regular forest languages such that

$$\pi(\mathcal{L}_1) \cap \pi(\mathcal{L}_2) = \emptyset$$
Separation Lemma

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Lemma (Separation Lemma)
Let $\mathcal{L}_1, \mathcal{L}_2 \subseteq H_\Sigma$ be regular forest languages such that

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Then there is a PDL language $X$ such that

$$\mathcal{L}_1 \subseteq X \subseteq (H_\Sigma - \mathcal{L}_2)$$

Further, $X$ can be written as the wreath product of three distributive algebras.
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Further, $X$ can be written as the wreath product of three distributive algebras.

Similar results are often used to prove decidability for logics over words.
Separation Lemma: Example

$\mathcal{L}_1$: ‘Each path ends in siblings labeled $b$ and $c$’

$\mathcal{L}_2$: ‘Each path ends in siblings labeled $b$ and $a[c]$’
Separation Lemma: Example

\( \mathcal{L}_1 \): ‘Each path ends in siblings labeled b and c’

\( \mathcal{L}_2 \): ‘Each path ends in siblings labeled b and a[c]’

- Elements of \( \mathcal{L}_1, \mathcal{L}_2 \) can be told apart just from looking at the sets of paths.
- No finite distributive forest algebra can separate them.
- Nonetheless, PDL separates them.
Proof of the Main Result

1. Use separation lemma to construct right factor

2. Remaining problem is then to find a distributive left-hand factor \((H_1, V_1)\) such that

\[(H, V) \prec (H_1, V_1) \bowtie (H_2, V_2) \bowtie (H_3, V_3) \bowtie (H_4, V_4) \quad (1)\]
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\[
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\]

3. Construct the ‘minimal’ left-hand factor \( (H_1, V_1) \)
   - In the case of groups, the solution is the kernel group
   - For monoids, the analog is a category (Tilson, 1987)
   - For forest algebras, result is forest category (Straubing, 2018)
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4. Can show that this factor can be replaced by a distributive forest algebra
Decidability of 2-Distributivity

Theorem

*It is decidable whether a regular language is recognized by some 2-distributive forest algebra.*

\[
\begin{align*}
A \sim_1 B & \implies
\begin{array}{c}
A \\
\end{array}
\begin{array}{c}
B
\end{array}
\sim
\begin{array}{c}
A
\end{array}
\begin{array}{c}
B
\end{array}
\end{align*}
\]
Decidability of 2-Distributivity

**Theorem**

*It is decidable whether a regular language is recognized by some 2-distributive forest algebra.*

1. Transform regular forest languages into a normal form by mapping forests to representatives that only depends on the set of paths.
Decidability of 2-Distributivity

Theorem

It is decidable whether a regular language is recognized by some 2-distributive forest algebra.

1. Transform regular forest languages into a normal form

2. This transformation preserves regularity of languages

3. Find pairs $A, B$ by checking intersection of images under this transformation
Discussion

- Decidability remains open for prominent tree logics
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- 2-distributive finite forest algebras describe a *decidable subset* of PDL
- Generalizing this to $k > 2$ would settle decidability of PDL

- Builds on algebraic characterization that is similar to related logics for which decidability is open
- Our results may shed light on this larger family of open problems
- Our proof method relates to results from the study of word languages
- Shows how classical theory generalizes to trees and forests
Discussion

- Decidability remains open for prominent tree logics
- **2-distributive** finite forest algebras describe a **decidable subset** of PDL
- Generalizing this to $k > 2$ would settle decidability of PDL
- Builds on algebraic characterization that is **similar to related logics** for which decidability is open
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