Projection Pricing

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Abstract. A significant problem in modern finance theory is how to price assets whose payoffs are outside the span of marketed assets. In practice, prices of assets are often assigned by using the Capital Asset Pricing Model. If the market portfolio is efficient, the price obtained this way is equal to the price of an asset whose payoff, viewed as a vector in a Hilbert space of random variables, is projected orthogonally onto the space of marketed assets. This paper looks at the pricing problem from this projection viewpoint. It is shown that the results of the CAPM formula are duplicated by a formula based on the Minimum Norm Portfolio, and this pricing formula is valid even in cases when there is no efficient portfolio of risky assets. The relation of the pricing to other aspects of projection are also developed. In particular a new pricing formula, called the Correlation Pricing Formula, is developed that yields the same price as the CAPM but is likely to be more accurate and more convenient than the CAPM in some cases.
Key Words. Asset pricing, projection theorem, correlation pricing, Hilbert space.

1 Introduction

The pricing formula derived from mean–variance portfolio theory (Ref. 1) is one of the principal foundations of modern finance theory. Starting with a set of assets with known return characteristics, this pricing formula expresses the expected return of any one of those assets in terms of the expected excess return (above the risk-free return) of an optimal portfolio and in terms of the beta of the asset with respect to the optimal portfolio. Mathematically, the pricing formula is a tautology; it is essentially a restatement of the necessary conditions for the optimal portfolio.

The pricing formula takes on vast power, however, when it is assumed (by an equilibrium argument) that the optimal portfolio is the market portfolio. Then, expected returns of individual assets can be determined from a knowledge only of their betas with respect to the market portfolio. This is the essence of the Capital Asset Pricing Model (CAPM).

In practice the CAPM formula is applied to any asset, including assets not necessarily in the family of marketed assets. Application of the CAPM formula in this way extends the range of mean–variance theory, but it raises the question of how to interpret the price that is produced.

One way to characterize this price is in terms of approximation. If the new asset is approximated as closely as possible (in the sense of minimum expected squared error) by a linear combination of marketed assets, the new asset is assigned a price equal to that of the corresponding approximating linear combination of market assets.

This can be stated more elegantly in terms of orthogonal projection in Hilbert space. The random payoffs of an original set of n priced assets can be regarded as n vectors in a subspace of a Hilbert space of random variables. To assign a price to a payoff vector outside of this subspace, this payoff is first projected orthogonally onto the subspace; then the price is assigned to be consistent with prices in the subspace.

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Orthogonal projection not only provides a nice interpretation of the CAPM-style pricing formula, it opens up the possibility of applying the powerful results of Hilbert space projection to other aspects of the pricing problem. This paper explores the relation between Hilbert space projection methods and pricing (both to interpret known results and to develop new ones).

One important result, based on a duality associated with projection, is that the basic CAPM-style pricing formula can be stated in terms of a minimum norm vector rather than in terms of an optimal portfolio; see Ref. 2. We shall show that an advantage of this result is that the minimum norm vector always exists, whereas the optimal portfolio may not. Indeed, the minimum norm method provides a pricing formula even when the efficient frontier contains no (nontrivial) marketed asset.

The Hilbert space view also leads to an alternative pricing formula in which the optimal portfolio (or market portfolio) is replaced by a marketed asset most correlated with the asset being priced. This *Correlation Pricing Formula* is a rigorous implementation of the common idea of finding a comparable marketed asset when pricing a new one. It is likely that this method will be preferred in many pricing situations.

In the final section, the projection approach is extended to the case where portfolios are evaluated according to a utility function.

2 Projection Theorems

The classic projection theorem generalizes the highly intuitive fact that in 3-dimensional space the shortest distance between a point and a plane is achieved by a line perpendicular to the plane (Ref. 3). The generalized theorem is stated in an arbitrary Hilbert space, and in that setting it characterizes the solution to approximation and efficiency problems in numerous applied fields. There are, in addition, several extensions and modifications of the classic theorem which also have strong intuitive geometric interpretations and numerous applications. This section states some of these important results in a general Hilbert space setting. Subsequent sections apply these results to issues of asset pricing.

A (real) *pre-Hilbert space* H is a vector space with a real-valued *inner product* defined for any two elements. The inner product of vectors x and y in H is denoted $(x|y)$. The inner product is linear in each argument and $(x,x) > 0$ unless $x = \theta$, the zero element of H. The *norm* of an element $x \in H$ is $\|x\| = \sqrt{(x|x)}$. A pre-Hilbert space is a *Hilbert space* if it is complete[2] under the norm. We say x is *orthogonal* to y if $(x|y) = 0$; in this case we write $x \perp y$. One important lemma is the Cauchy–Schwarz inequality: $|(x,y)| \leq \|x\| \cdot \|y\|$ for any x and y in H, with equality if and only if $x = \alpha y$ for some α or $y = \beta x$ for some β.

A subspace M of a pre-Hilbert space is a set $M \subset H$ that is itself a linear space. Many of the results stated here apply only to closed subspaces. However, in our applications, the relevant subspaces are finite dimensional—and all finite-dimensional subspaces are closed.

The classic projection theorem is in many respects the most important theorem of Hilbert space.

[2]A normed space is complete if every Cauchy sequence converges (with respect to the norm) to a limit in the space.

It is the theorem that generalizes the concept that the shortest distance from a point to a plane is attained by the line perpendicular to the plane. See Ref. 3 for proof.

Theorem 2.1 (The Classic Projection Theorem) *Let H be a Hilbert space and M a closed nonempty subspace of H. Let $x \in H$. Then there is an $m_0 \in M$ such that $\|x - m_0\| \leq \|x - m\|$ for all $m \in M$. Furthermore, $x - m_0 \perp m$ for all $m \in M$. Conversely, if $m_0 \in H$ is a vector such that $x - m_0 \perp m$ for all $m \in M$, then $\|x - m_0\| \leq \|x - m\|$ for all $m \in M$.*

In practice, computation of the projection may be difficult. It is often advisable to carry out the computation in a series of steps, each of which is a simpler projection problem. The abstract way to do this is to represent the subspace as the sum of smaller subspaces. For example, we may write a subspace S as $S = M + N$ where M and N are smaller subspaces. The overall projection decomposes if M and N are mutually orthogonal; that is, if $(m|n) = 0$ for all $m \in M$ and $m \in N$. We shall use this principle to derive the Correlation Pricing Formula. We state the general principle here.

Theorem 2.2 (Projection Splitting) *Let H be a Hilbert space and let M and N be nonempty subspaces which are closed and mutually orthogonal. Then the orthogonal projection of x onto the subspace $M + N$ is $m_0 + n_0$ where m_0 and n_0 are the projections of x onto M and N, respectively.*

Proof. We find $(x - m_0 - n_0|m) = (x - m_0|m) - (n_0|m) = 0$ for all $m \in M$. Likewise $(x - m_0 - n_0|n) = 0$ for all $n \in N$. Hence $x - m_0 - n_0$ is orthogonal to all elements in $M + N$. By the projection theorem $m_0 + n_0$ is the the projection of x onto $M + N$. ■

There is another way to carry out the projection operation in steps. Suppose M and N are subspaces with $N \subset M$, and we wish to find the projection of x onto N. We can first project x onto M and then project that result onto N.

Theorem 2.3 (Nested Projection) *Let M and N be subspaces of a Hilbert space with $N \subset M$. Let $x \in H$. Let x_M and x_N be the projections of x onto M and N, respectively. Then $x_N = (x_M)_N$, the projection of x_M onto N.*

Proof. We have $x - (x_M)_N = x - x_M + x_M - (x_M)_N$. The first two terms on the right form a vector that is orthogonal to N because it is orthogonal to M and $N \subset M$. The second two terms form a vector orthogonal to N because it is the difference between x_M and its projection onto N. ■

The classic projection theorem has a dual, which is expressed in terms of a maximization problem. Geometrically, again consider 3-dimensional space. Given a horizontal plane that contains the origin, and given a vector x lying outside the plane, we seek the vector of unit length in the plane that has maximum cosine with x. It should be clear that the appropriate vector in the plane is "directly below" x. That is, it is pointing in the direction of the projection of x onto the plane. This result, when generalized, is the content of the dual projection theorem. In the dual theorem given here, the cosine of the angle between two vectors is replaced by the inner product of those vectors. As long as the norms are fixed, this is a valid generalization.

Theorem 2.4 (Dual Projection Theorem) *Let H be a Hilbert space and M a closed nonempty subspace of H. Let $x \in H$. Then there is an $m' \in M$ with $\|m'\| = 1$ such that $(x|m') \geq (x|m)$ for all $m \in M$ with $\|m\| = 1$. Furthermore, if the projection m_0 of x onto M is nonzero, then m' is unique and is a positive multiple of m_0.*

Proof. If x is orthogonal to every $m \in M$ then we may take m' to be any vector in M with $\|m'\| = 1$. Otherwise, let m_0 be the projection of x onto M. Since $m_0 \neq \theta$ we may set $m' = m_0/\|m_0\|$. Then for any $m \in M$ with $\|m\| \leq 1$ we have $(x|m) = (x - m_0 + m_0|m) = (m_0|m) \leq \|m_0\| \cdot \|m\| \leq \|m_0\|$ by the Cauchy–Schwarz inequality. On the other hand, $(x|m') = (x - m_0 + m_0|m_0)/\|m_0\| = \|m_0\|$. Comparing the last two sentences proves that m' has the required property.

Now suppose $m_0 \neq \theta$ and suppose $m'' \in M$ with $\|m''\| = 1$ is another vector with the desired property. Then $(x|m'') = (x|m')$. Hence $0 = (x|m''-m') = (x-m_0+m_0|m''-m') = (m_0|m''-m') = (m'|m''-m')\|m_0\|$. Thus $(m'|m'') = (m'|m')$. This means $1 = (m'|m') = (m'|m'') \leq \|m'\| \cdot \|m''\| = 1$. Equality must hold throughout, and the condition for equality in the Cauchy–Schwarz inequality requires $m'' = m'$. ■

The next theorem is a powerful result for pricing theory. Stated in algebraic terms the result may be surprising; but stated geometrically the result is again quite intuitive. The theorem combines duality with the projection theorem.

Geometrically, suppose that in 3 dimensions there is a given plane containing the origin. (Think of it as a horizontal plane.) There is also a given line in that plane not passing through the origin. We want to construct an additional plane that contains the given line and is such that the projection of any point in the new plane onto the first plane falls on the given line. It should be clear that the new plane rises perpendicularly from the first. (It must be a vertical plane.) This new plane has the property that the minimum distance from this plane to the origin is achieved by a vector in the first plane (that is, it is a horizontal vector).

Here is an interpretation of the theorem that will be used in the context of asset pricing (and is closely related to the CAPM formula). Suppose that on M, a closed subspace of H, there is a linear functional f (defined only on M). It can be thought of as a way to assign a price to any vector in M. The price of $m \in M$ is $f(m)$. We would like to assign prices to vectors outside of M as well. One way to assign a price to a vector x is to project x onto M giving, say m_x, and then assign the price $f(m_x)$ to x. This is a two-step process: project, and then apply f. The next theorem states that there is a shortcut method. There is a vector $g \in M$ such that the price of any vector $x \in H$ is $(g|x)$. Furthermore the vector g is proportional to the vector in M of minimum norm having price 1.

Theorem 2.5 (Orthogonal Extension Theorem) *Let H be a Hilbert space and let M be a closed nonempty subspace of H. Let f be a linear functional defined on M. Then there is a unique vector $g \in M$ such that for any $x \in H$ there holds $(g|x) = f(m_x)$ where m_x is the projection of x onto M. Furthermore, if $f \neq \theta$, the vector g is the (unique) vector of minimum norm in M with $f(g) = \|f\|^2 \equiv \{\sup f(m) : \|m\| = 1,\ m \in M\}$.*

Proof. According to the Riesz–Fréchet theorem (Ref. 3), there is a unique vector $g \in M$ such that $f(m) = (g|m)$ for all $m \in M$. Suppose $x \in H$. By the projection theorem $x = m_x + n_x$ where

$n_x \perp m$ for all $m \in M$. Therefore, $(g|x) = (g|m_x) + (g|n_n) = (g|m_x) = f(m_x)$. This shows that a g exists which duplicates f on M and gives $f(x) = f(m_x)$ in H. Clearly $\|f\| = \|g\|$.

Assume now $f \neq \theta$. Then $g \neq \theta$. Suppose that h is a vector in M such that $f(h) = \|f\|^2$. Then $(g|h) = \|g\|^2$. By the Cauchy–Schwarz inequality, $(h|g) \leq \|h\|\|g\|$, with equality only if $h = \alpha g$ for some $\alpha \geq 0$. If equality does not hold we have $\|g\|^2 < \|h\|\|g\|$; and thus $\|g\| < \|h\|$. If equality does hold then $h = \alpha g$ and $f(h) = (h|g) = \alpha(g|g)$. But the requirement $f(h) = \|g\|^2$ implies $\alpha = 1$; so in this case $h = g$. ■

3 Basic Pricing Concepts

In order to use the concepts of the previous section we must set up the appropriate Hilbert space. Vectors in this space are random variables.

Corresponding to a random variable y, we denote the expected value of y by $\mathrm{E}(y)$ or by $\overline{y}$. The variance of y is $\sigma^2 = \mathrm{E}[(y - \overline{y})^2]$. The standard deviation of y is $\sigma = \sqrt{\sigma^2}$. We consider only those random variables that are bounded in the sense that $\mathrm{E}[y^2] < \infty$.

The Hilbert space we use has elements (vectors) which are bounded random variables.[3] The inner product of two such elements y_1, y_2 is $(y_1|y_2) = \mathrm{E}[y_1 y_2]$. The norm of an element y in H is thus $\|y\| = \sqrt{\mathrm{E}[y^2]}$. Throughout this paper, there is no loss in generality in assuming that H is finite-dimensional: that is, a finite number of independent vectors (random variables) generate all the others by linear combination.

We associate assets with random variables. Assets are purchased at time 0 and sold at time 1. The amount for which an asset is sold is its *payoff*, which is a random variable. Assume that, initially, there is available a set of n assets with payoffs $y_1, y_2, \ldots, y_n$ respectively. These payoffs are each members of H.

Associated with each asset is a price, which is paid at time 0. The n prices corresponding to the n assets are $p_1, p_2, \ldots, p_n$, respectively.

We assume that any linear combination of the given n payoff vectors defines an asset with a price determined by the linear combination. That is, the asset $w_1 y_1 + w_2 y_2 + \cdots + w_n y_n$ has price $w_1 p_1 + w_2 p_2 + \cdots + w_n p_n$. This linear pricing rule can be inferred from an arbitrage argument if the n original assets can be arbitrarily divided and bought or sold without transactions costs. The n original assets and their linear combinations define a subspace M in H.

The basic problem we consider is how to price payoffs y that are not in M. It is well known that there is not a unique solution to this problem. We investigate one simple (and standard) method.

Standard Projection

We price an arbitrary payoff $x \in H$ by projecting x onto the subspace M generated by $y_1, y_2, \ldots, y_n$. The price of the projection is then found by the linear pricing rule in M, and this price is assigned to x.

[3] We assume throughout that there is a fixed underlying probability space and probability measure. All of our random variables are measurable functions on this space.

To carry out this process, we seek the vector m_x of the form $m_x = w_1y_1 + w_2y_2 + \cdots + w_ny_n$ that minimizes $\|x - m_x\|$, where the weights $w_1, w_2, \ldots w_n$ are real numbers.

The solution is given immediately by noting that according to the projection theorem, the error $x - m_x$ must be orthogonal to each of the vectors $y_1, y_2, \ldots, y_n$. In equation form, we have $(x - m_x|y_i) = 0$ for all $i = 1, 2, \ldots, n$.

These equations can be expressed algebraically in terms of the (symmetric) matrix

$$\mathbf{Y} = \begin{bmatrix} (y_1|y_1) & (y_1|y_2) & \cdots & (y_1|y_n) \\ (y_2|y_1) & (y_2|y_2) & \cdots & (y_2|y_n) \\ \cdots & \cdots & \cdots & \cdots \\ (y_n|y_1) & (y_n|y_2) & \cdots & (y_n|y_n) \end{bmatrix}$$

and the vectors

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} (y_1|x) \\ (y_2|x) \\ \vdots \\ (y_n|x) \end{bmatrix}.$$

Using this notation, the orthogonality conditions can be written as $\mathrm{E}(x - \mathbf{y}^T\mathbf{w}|y_i) = 0$ for $i = 1, 2, \cdots n$, where $\mathbf{y}$ is considered a column vector of the random variables y_i, $i = 1, 2, \cdots, n$. With a further obvious extension of notation we may combine all n orthogonality conditions and write them together as

$$(x - \mathbf{y}^T\mathbf{w}|\mathbf{y}) = \mathbf{0}.$$

This becomes

$$\mathbf{Y}\mathbf{w} = \mathbf{b}.$$

If the matrix $\mathbf{Y}$ is nonsingular, we may write the explicit solution

$$\mathbf{w} = \mathbf{Y}^{-1}\mathbf{b}. \tag{1}$$

Hence the projection of x onto M is

$$m_x = \mathbf{b}^T\mathbf{Y}^{-1}\mathbf{y}. \tag{2}$$

Let $\mathbf{p}$ denote the column vector of prices associated with the y_i's. The vector m_x has the price $p_x = w_1p_1 + w_2p_2 + \cdots + w_np_n = \mathbf{p}^T\mathbf{w}$. In view of (1) we have

$$p_x = \mathbf{p}^T\mathbf{w} = \mathbf{p}^T\mathbf{Y}^{-1}\mathbf{b}. \tag{3}$$

This p_x is the price that is assigned to x. Using the definition of $\mathbf{b}$ this can be written as

$$p_x = \mathbf{p}^T\mathrm{E}[\mathbf{Y}^{-1}\mathbf{y}\,x]. \tag{4}$$

Orthogonal Extension

The orthogonal extension theorem can now be employed—which within the framework we have developed is merely a rearrangement of the solution. However, this rearrangement provides an important interpretation of the solution.

We may rewrite (3) as

$$p_x = \mathbf{g}^T\mathbf{b} \tag{5}$$

or, equivalently,

$$p_x = \mathrm{E}[\mathbf{g}^T\mathbf{y}\, x] = (\mathbf{g}^T\mathbf{y}|x) = (g|x) \tag{6}$$

where $\mathbf{g} = \mathbf{Y}^{-1}\mathbf{p}$. Hence

$$g = \mathbf{g}^T\mathbf{y} = \mathbf{p}^T\mathbf{Y}^{-1}\mathbf{y}. \tag{7}$$

This shows that p_x can be expressed as an inner product with a fixed vector g independent of x. We have extended the pricing functional defined only on the subspace M to the entire space H, and the extension is consistent with the result that would be obtained by orthogonal projection of x onto M.

The advantage of this version is that, once $\mathbf{g}$ is computed and the random variable $g = \mathbf{g}^T\mathbf{y}$ formed, equation (6) can be used to find the price of any x; it is not necessary to first compute the projection of x onto the subspace M generated by the y_i's.

We use the second part of the orthogonal extension theorem to develop an alternative interpretation of g and an alternative method of computation. According to that theorem the proper g is, first of all, in M; which means that $g = \mathbf{g}^T\mathbf{y}$ for some $\mathbf{g}$. Second, this g has minimum norm with respect to all vectors in M that have price equal to $\|f\|^2$ where f is the original pricing function on M. (We do not need to know $\|f\|$ yet.)

Hence the minimum norm problem is

$$\min_{\mathbf{g}} \mathrm{E}[\mathbf{g}^T\mathbf{y}\mathbf{y}^T\mathbf{g}]$$
$$\text{subject to } \mathbf{g}^T\mathbf{p} = \|f\|^2.$$

Since $\mathbf{Y} = \mathrm{E}[\mathbf{y}\mathbf{y}^T]$ the above problem is

$$\min_{\mathbf{g}} \mathrm{E}[\mathbf{g}^T\mathbf{Y}\mathbf{g}]$$
$$\text{subject to } \mathbf{g}^T\mathbf{p} = \|f\|^2.$$

Introducing a Lagrange multiplier 2λ for the constraint, the appropriate first-order necessary conditions are

$$\mathbf{Y}\mathbf{g} - \lambda\mathbf{p} = \mathbf{0}.$$

Hence

$$\mathbf{g} = \lambda\mathbf{Y}^{-1}\mathbf{p}$$

We know from the previous development that $\lambda = 1$ or it can be found by calculating $\|f\|^2$ directly.[4]

[4] $\|f\|^2 = \max\ (\mathbf{w}^T\mathbf{p})^2/\mathrm{E}[(\mathbf{w}^{\mathrm{T}}\mathbf{y})^2] = \max\ (\mathbf{w}^T\mathbf{p})^2/(\mathbf{w}^{\mathrm{T}}\mathbf{Y}\mathbf{w}) = \mathbf{p}^{\mathrm{T}}\mathbf{Y}^{-1}\mathbf{p}$.

In general, to avoid calculating $\|f\|^2$ we may find $g' \in M$ minimizing the norm subject to the price of g' being 1. The pricing vector g is then a scalar multiple of this g'. The scale factor can be found by matching the known price of any asset in M.

Covariance Form

It is conventional to express the results of a pricing problem in terms of the means and covariance matrix of the underlying assets. To convert the pricing formula above to this form, we define the n-dimensional vector $\overline{\mathbf{y}}$ as the vector of expected values of the y_i's and we define the covariance matrix $\mathbf{V} = \mathrm{E}[(\mathbf{y}-\overline{\mathbf{y}})(\mathbf{y}-\overline{\mathbf{y}})^T]$, and throughout this paper we assume that $\mathbf{V}$ is nonsingular.

We will make use of the following standard lemma (which can be verified by cross multiplication).

Lemma 3.1 (Sherman–Morrison Formula) Let $\mathbf{A}$ be a nonsingular symmetric $n \times n$ matrix and let $\mathbf{a}$ be an n-dimensional column vector. If $\mathbf{A} + \mathbf{a}\mathbf{a}^T$ is nonsingular, then

$$[\mathbf{A} + \mathbf{a}\mathbf{a}^T]^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{a}\,\mathbf{a}^T\mathbf{A}^{-1}}{1 + \mathbf{a}^T\mathbf{A}^{-1}\mathbf{a}}.$$

Applying the Sherman–Morrison formula to the matrix $\mathbf{Y} = \mathbf{V} + \overline{\mathbf{y}}\,\overline{\mathbf{y}}^T$, we may rewrite (7) in the explicit form

$$g = \mathbf{g}^T\mathbf{y} = \mathbf{p}^T\mathbf{Y}^{-1}\mathbf{y} = \mathbf{p}^T\{\mathbf{V}^{-1} - \frac{\mathbf{V}^{-1}\overline{\mathbf{y}}\,\overline{\mathbf{y}}^T\mathbf{V}^{-1}}{1+\overline{\mathbf{y}}^T\mathbf{V}^{-1}\overline{\mathbf{y}}}\}\mathbf{y} \tag{8}$$

$$= \frac{\mathbf{p}^T\mathbf{V}^{-1}\overline{\mathbf{y}}}{1+\overline{\mathbf{y}}^T\mathbf{V}^{-1}\overline{\mathbf{y}}} - \{\mathbf{p}^T\mathbf{V}^{-1} - \frac{\mathbf{p}^T\mathbf{V}^{-1}\overline{\mathbf{y}}\,\overline{\mathbf{y}}^T\mathbf{V}^{-1}}{1+\overline{\mathbf{y}}^T\mathbf{V}^{-1}\overline{\mathbf{y}}}\}(\mathbf{y}-\overline{\mathbf{y}}) \tag{9}$$

This is the general pricing equation. We can greatly simplify it by determining the price it implies for a risk-free asset. This will define an implied risk-free return which we denote by R_0. In particular, the implied price of the asset with payoff identically 1 is defined to be $1/R_0$.

Using g from (9) as a pricing vector, the price of the asset with payoff identically equal to 1 is found from $(g|1) = \mathrm{E}(g) = 1/R_0$ to be[5]

$$\frac{1}{R_0} = \frac{\mathbf{p}^T\mathbf{V}^{-1}\overline{\mathbf{y}}}{1+\overline{\mathbf{y}}^T\mathbf{V}^{-1}\overline{\mathbf{y}}}. \tag{10}$$

[5]Another important return value is R_{mv} which is the expected return of the minimum-variance point. We easily find

$$R_{rm} = \frac{\overline{\mathbf{y}}^T\mathbf{V}^{-1}\mathbf{p}}{\mathbf{p}^T\mathbf{V}^{-1}\mathbf{p}}.$$

We then have

$$\frac{R_0}{R_{mv}} = \frac{(1+\overline{\mathbf{y}}^T\mathbf{V}^{-1}\overline{\mathbf{y}})(\mathbf{p}^T\mathbf{V}^{-1}\mathbf{p})}{(\overline{\mathbf{y}}\mathbf{V}^{-1}\mathbf{p})^2} > \frac{(\overline{\mathbf{y}}^T\mathbf{V}^{-1}\overline{\mathbf{y}})(\mathbf{p}^T\mathbf{V}^{-1}\mathbf{p})}{(\overline{\mathbf{y}}\mathbf{V}^{-1}\mathbf{p})^2} \geq 1$$

by the Cauchy–Schwarz inequality. Hence $R_0 > R_{mv}$.

Substituting this into (8) we have

$$\mathbf{g}^T = \mathbf{p}^T\mathbf{V}^{-1} - \frac{1}{R_0}\overline{\mathbf{y}}^T\mathbf{V}^{-1} \tag{11}$$

We may now state the following theorem.

Theorem 3.1 (Minimum Norm Pricing) *Let x be a payoff. Then the price of x as determined by its projection onto the space of priced assets is*

$$p_x = \frac{1}{R_0}[\overline{x} - \mathrm{cov}(\mathbf{z}^T\mathbf{V}^{-1}\mathbf{y}, x)] \tag{12}$$

where

$$\mathbf{z} = \overline{\mathbf{y}} - R_0\mathbf{p}.$$

Example 3.1 Suppose there are two assets. Their returns have expected values $R_1 = 1.4,\ R_2 = .8$, and standard deviations $\sigma_1 = \sigma_2 = .20$, respectively. The two assets are uncorrelated. In terms of the previous notation we have

$$\mathbf{V} = \begin{bmatrix} .04 & 0 \\ 0 & .04 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \overline{\mathbf{y}} = \begin{bmatrix} 1.4 \\ .8 \end{bmatrix}.$$

One way to find the pricing formula related to these assets is to directly minimize the norm of a portfolio subject to the price being 1. We have assumed that the original assets are normalized so that their prices are 1; and hence their payoffs are equal to their returns. We let w be the amount of asset 1 and $1 - w$ be the amount of asset 2 in a portfolio with payoff y. The price constraint is then satisfied for all w. The norm of y is $[\mathrm{E}(y)^2 + \mathrm{var}(y)]^{1/2}$; hence the problem becomes:

$$\min\ [1.4w + .8(1-w)]^2 + .04w^2 + .04(1-w)^2 \tag{13}$$

which has solution $w = -1$, $1 - w = 2$. The price of any asset x will accordingly be

$$p_x = \alpha\mathrm{E}[(-y_1 + 2y_2)x)]$$

where α is a scale factor. We can find α by applying the formula to y_1 which gives

$$p_1 = \alpha\mathrm{E}[-y_1^2 + 2y_1y_2] = \alpha[-1.4^2 - .04 + 2(1.4)(.8)] = 0.24\alpha.$$

Since $p_1 = 1$ we find $\alpha = 1/0.24$. The pricing vector g is thus $g = (-y_1 + 2y_2)/0.24$.

As an application of the formula, we can compute the implied risk-free return R_0, determined by projecting the 1 payoff onto the space of the two risky assets. We have

$$\frac{1}{R_0} = \mathrm{E}[g] = (-1.4 + 1.6)/.24 = 1/1.2,$$

which means $R_0 = 1.20$.

The situation can be illustrated in a familiar Expected Return–Standard Deviation diagram as shown in Fig. 1. The returns of the two assets are shown as heavy dots. The curved line represents payoffs of linear combinations of the two assets. The line drawn from the origin to the curve is the shortest distance to the curve, and is the minimum norm vector. It is formed by taking a weight of -1 for asset 1 and 2 for asset 2 in accordance with the solution to (13). When properly scaled this vector serves as the general pricing vector g. The dashed line in the figure is the line tangent to the curve at the minimum norm point. The vertical intercept of this line is $\|g\|^2/\mathrm{E}(g) = \alpha\|g\|^2/(\mathrm{E}(g)\alpha) = R_0\alpha\|g\|^2 = R_0$, the last equality following from the fact that the price of g is 1. Therefore, $R_0 = 1.2$ is equal to the vertical intercept of the dashed line. (The minimum-variance expected return is $R_{mv} = 1.1$ in this case.) The vector g is found by scaling the minimum norm vector so that its vertical height is 1.2, which in this case means scaling by a factor of $1.2/.2 = 6$.

Extension of Standard Form

We can find the a general formula for the scale factor α to apply to the minimum-norm vector which converts the pricing formula to familiar form. We have

$$p_x = \alpha\mathrm{E}(y_0\, x) \tag{14}$$

where y_0 is the minimum-norm portfolio. We can apply this formula to the payoff $y_0 - \overline{y}_0$ which has price $p_0 - \overline{y}_0/R_0$. Thus

$$p_0 - \overline{y}_0/R_0 = \alpha\mathrm{E}[y_0(y_0 - \overline{y}_0)] = \alpha\sigma_0^2.$$

Solving for α and substituting in (14) we find $p_x = \alpha\mathrm{E}[y_0\, x] = \alpha\mathrm{E}[y_0(\overline{x} + x - \overline{x})]$ and hence, we may state the following result.

Theorem 3.2 (Standard Form for Minimum Norm Pricing) *Let x be a payoff. Then the price of x as determined by its projection on the space of priced assets is*

$$p_x = \frac{1}{R_0}[\overline{x} - \mathrm{cov}(y_0, x)(\overline{y}_0 - p_0 R_0)/\sigma_0^2]$$

where y_0 is a minimum-norm payoff, R_0 is the implied risk-free return, and σ_0^2 and p_0 are the variance and price of y_0.

This looks similar to the CAPM formula in pricing form with R_0 used as the risk-free return, but with the minimum norm payoff used in place of the efficient portfolio.

4 Inclusion of a Given Risk-Free Asset

Suppose that in addition to the n risky assets, there is an asset (the $(n+1)$-st asset) which is risk free, (as first considered in Ref. 4). This asset is priced such that its return is R_f. Inclusion of such an asset does not change the general principle of minimum norm pricing, and in fact does not much influence the formula. It does, however, allow a direct comparison between the minimum norm formula and the standard CAPM-style pricing formula, where the price is expressed in terms of a unique efficient portfolio of risky assets. As we shall see, the minimum norm solution is the more general of the two approaches, since it always exists while the efficient portfolio of risky assets may not.

Minimum Norm Formulation

We may directly formulate the problem of finding the minimum norm vector with price equal to 1. We denote by $\mathbf{w}$ the vector of weights for the risky assets. Then $1 - \mathbf{p}^T\mathbf{w}$ is the weight of the risk-free asset. The problem is

$$\min\ [\mathbf{w}^T\overline{\mathbf{y}} + (1 - \mathbf{w}^T\mathbf{p})R_f]^2 + \mathbf{w}^T\mathbf{V}\mathbf{w}$$

This leads to

$$\mathbf{w} = -R_f[\mathbf{V} + \mathbf{z}\mathbf{z}^T]^{-1}\mathbf{z} \tag{15}$$

where

$$\mathbf{z} = \overline{\mathbf{y}} - R_f\mathbf{p}.$$

Using the Sherman–Morrison formula this becomes

$$\mathbf{w} = -R_f\left[\mathbf{V}^{-1} - \frac{\mathbf{V}^{-1}\mathbf{z}\mathbf{z}^T\mathbf{V}^{-1}}{1 + \mathbf{z}^T\mathbf{V}^{-1}\mathbf{v}}\right]\mathbf{z} = -\frac{\mathbf{V}^{-1}\mathbf{z}}{1 + \mathbf{z}^T\mathbf{V}^{-1}\mathbf{z}}R_f \tag{16}$$

Hence the pricing vector g is of the form

$$\begin{aligned} g &= \alpha\left[\mathbf{w}^T\mathbf{y} + (1 - \mathbf{w}^T\mathbf{p})R_f\right] \\ &= \alpha\left[-\frac{\mathbf{z}^T\mathbf{V}^{-1}(\mathbf{y} - R_f\mathbf{p})}{1 + \mathbf{z}^T\mathbf{V}^{-1}\mathbf{z}} + 1\right]R_f \end{aligned}$$

We may find α by applying the pricing vector to the risk-free asset with payoff 1. We have

$$\begin{aligned} \frac{1}{R_f} &= \mathrm{E}(g) \\ &= \alpha\left[-\frac{\mathbf{z}^T\mathbf{V}^{-1}\mathbf{z}}{1 + \mathbf{z}^T\mathbf{V}^{-1}\mathbf{z}} + 1\right]R_f \\ &= \alpha\frac{R_f}{1 + \mathbf{z}^T\mathbf{V}^{-1}\mathbf{z}} \end{aligned}$$

Thus

$$\alpha = (1 + \mathbf{z}^T\mathbf{V}^{-1}\mathbf{z})/R_f^2 \tag{17}$$

and we have

$$
\begin{aligned}
g &= \frac{1+\mathbf{z}^T\mathbf{V}^{-1}\mathbf{z}}{R_f^2}\left[-\frac{\mathbf{z}^T\mathbf{V}^{-1}(\mathbf{y}-\overline{\mathbf{y}}+\overline{\mathbf{y}}-R_f\mathbf{p})}{1+\mathbf{z}^T\mathbf{V}^{-1}\mathbf{z}}+1\right]R_f \\
&= \frac{1+\mathbf{z}^T\mathbf{V}^{-1}\mathbf{z}}{R_f^2}\left[-\frac{\mathbf{z}^T\mathbf{V}^{-1}(\mathbf{y}-\overline{\mathbf{y}}+\mathbf{z})}{1+\mathbf{z}^T\mathbf{V}^{-1}\mathbf{z}}+1\right]R_f \\
&= \frac{1}{R_f}[1-\mathbf{z}^T\mathbf{V}^{-1}(\mathbf{y}-\overline{\mathbf{y}})]
\end{aligned}
$$

Theorem 4.1 (Minimum Norm Pricing with Risk-Free Asset) *When there is a risk-free asset with return R_f, the price of an arbitrary asset with payoff x is*

$$
p_x = \frac{1}{R_f}\left[\overline{x} - \mathrm{cov}(\mathbf{z}^T\mathbf{V}^{-1}\mathbf{y},\ x)\right] \tag{18}
$$

where $\mathbf{z} = \overline{\mathbf{y}} - R_f\mathbf{p}$. Equivalently,

$$
p_x = \frac{1}{R_f}[\overline{x} - \mathrm{cov}(y_0, x)(\overline{y}_0 - p_0 R_f)/\sigma_0^2], \tag{19}
$$

where y_0 is a minimum-norm payoff and p_0 and σ_0^2 are the price and variance of that payoff.

The second part of this result is derived from the first in the same way that Theorem 3.2 is derived. Note that in general a minimum-norm payoff may be modified by adding or subtracting a multiple of the risk-free asset. Likewise, y_0 may be scaled arbitrarily.

Optimal Portfolio Formulation

A standard way to derive the asset pricing formula when there is a risk-free asset is to find the portfolio of risky assets that maximizes the price of risk: $(\overline{y}_M - p_M R_f)/\sigma_M$ (where $\overline{y}_M$, p_M, and σ_M are the expected value, the price, and standard deviation of the optimal portfolio), leading to a CAPM-type formula. (See Refs. 5, 6, 7.)

Mathematically, the payoff of the optimal portfolio is of the form $\mathbf{w}^T\mathbf{y}$ where $\mathbf{w}^T\mathbf{p} > 0$. The $\mathbf{w}$ that maximizes the price of risk solves

$$
\text{maximize } \frac{\mathbf{w}^T(\overline{\mathbf{y}} - R_f\mathbf{p})}{\sqrt{\mathbf{w}^T\mathbf{V}^{-1}\mathbf{w}}} \tag{20}
$$

Under appropriate conditions (to be discussed later), this will have a solution $\mathbf{w}$ and a corresponding optimal y_M of the form

$$
y_M = \gamma\mathbf{z}^T\mathbf{V}^{-1}\mathbf{y} \tag{21}
$$

where $\gamma > 0$.

We may eliminate γ by noting that the price of y_M is $p_M = \gamma\mathbf{z}^T\mathbf{V}^{-1}\mathbf{p}$. Thus

$$
y_M = \frac{\mathbf{z}^T\mathbf{V}^{-1}\mathbf{y}}{\mathbf{z}^T\mathbf{V}^{-1}\mathbf{p}} p_M. \tag{22}
$$

Then we note that

$$\mathbf{z}^T\mathbf{V}^{-1}\mathbf{y} = \left[\frac{\mathbf{z}^T\mathbf{V}^{-1}\mathbf{y}}{\mathbf{z}^T\mathbf{V}^{-1}\mathbf{p}}p_M\right]\left[\frac{\mathbf{z}^T\mathbf{V}^{-1}\mathbf{z}}{\mathbf{z}^T\mathbf{V}^{-1}\mathbf{p}}p_M\right]\left[\frac{\mathbf{z}^T\mathbf{V}^{-1}\mathbf{z}}{(\mathbf{z}^T\mathbf{V}^{-1}\mathbf{p})^2}p_M^2\right]^{-1}.$$

This can be written[6]

$$\mathbf{z}^T\mathbf{V}^{-1}\mathbf{y} = y_M(\overline{y}_M - p_M R_f)/\sigma_M^2$$

and hence the the minimum norm formula (18) can be written as

$$p_x = \frac{1}{R_f}\left[\overline{x} - \frac{\mathrm{cov}(y_M, x)(\overline{y}_M - p_M R_f)}{\sigma_M^2}\right] \tag{23}$$

which is the standard CAPM-style formula in pricing form.

A weakness of this formulation is that the required portfolio y_M may not exist. This is true even though the mathematical manipulations used to obtain it lead to the same solution as the minimum norm solution. The problem is with the maximization of the price of risk ratio. This ratio may not have a maximum even though the necessary conditions have a solution. Indeed there may be no portfolio of risky assets on the upper portion of the feasible region. This is the case if the risk-free return R_f is higher than some threshold value. The price of risk ratio will then have a minimum, rather than a maximum, and in that case we must take $\gamma < 0$ in (21). The minimum norm solution still exists of course, and gives the price corresponding to orthogonal projection. Since the formal mathematics for the expression of $\mathbf{z}^T\mathbf{V}^{-1}\mathbf{y}$ in terms of y_M is identical to the case where this is a maximum, we see that the pricing formula can be written exactly like (23) where y_M is the portfolio that minimizes the price of risk ratio if a minimum exists.

The threshold value of R_f, defining the upper limit of risk-free returns that lead to maximum y_M's, corresponds to the case where $\mathbf{w}^T\mathbf{z} = 0$ and a risk-free return equal to the return at the minimum-variance point, namely

$$R_{mv} = \frac{\overline{\mathbf{y}}^T\mathbf{V}^{-1}\mathbf{p}}{\mathbf{p}^T\mathbf{V}^{-1}\mathbf{p}} \tag{24}$$

For $R_f > R_{mv}$ there is no portfolio of risky assets that achieves a maximum price of risk ratio, as pointed out, for example, in Ref. 8. For $R_f = R_{mv}$ there is neither a maximum nor a minimum.

Example 4.1 Suppose we have the same two risky assets as in example 3.1 and we adjoin a risk-free asset with return R_f. According to (24) (or by inspection) the threshold value for this example is $R_{mv} = (1.4 + .8)/2 = 1.1$.

If $R_f < 1.1$, there is a portfolio that maximizes the price of risk. For example, for $R_f = 1$ we find easily from (22), with $p_M = 1$, that $y_M = (.4y_1 - .2y_2)/.2 = 2y_1 - y_2$. Hence $\overline{y}_M = 2.8 - .8 = 2.0$ and $\sigma_M^2 = .04(4+1) = .20$, which gives $\sigma_M = .447$. This y_M can be used as the optimal portfolio in the CAPM-style pricing formula.

[6] Note that

$$\frac{\mathbf{z}^T\mathbf{V}^{-1}\mathbf{z}}{\mathbf{z}^T\mathbf{V}^{-1}\mathbf{p}}p_M = \frac{\mathbf{z}^T\mathbf{V}^{-1}(\overline{\mathbf{y}} - R_f\mathbf{p})}{\mathbf{z}^T\mathbf{V}^{-1}\mathbf{p}}p_M = y_M - p_M R_f.$$

We can also find the corresponding minimum-norm vector for $R_f = 1$. From (16) we have $w_1 = -(1.4-1.0)/(.04+(.16+.04)) = -10/6 = -1.66$; and $w_2 = -(.8-1.0)/(.04+(.16+.04)) = 5/6 = .833$. The amount in the risk-free asset must therefore be $1+(10/6)-(5/6) = 11/6 = 1.833$. The portfolio with these weights has expected return $\overline{R} = (-10 \cdot 1.4 + 5 \cdot .8 + 11)/6 = 1/6 = 1.666$ and standard deviation $\sigma = \sqrt{(10^2+5^2).04/6^2} = .372$. Note that the ratio of weights w_1/w_2 is -2, which is exactly the same as the ratio of weights in y_M

If $R_f > 1.1$ there is no portfolio that maximizes the price of risk, but there is a portfolio that minimizes the price of risk. For example, suppose $R_f = 1.3$. Then from (22) the minimizing portfolio is $y_M = -(.1y_1 - .5y_2/(.4) = -.25y_1 + 1.25y_2$. This has $\overline{y}_M = .65$, $\sigma_M = .255$ which is on the lower part of the minimum-variance frontier of risky assets.

For $R_f = 1.3$ there is also a minimum-norm portfolio (of $y_1,\ y_2,$ and R_f). It is found from (16) to have weights $w_1 = -1.3(1.4-1.3)/(.04+(.01+.25)) = -.433$ and $w_2 = -1.3(.8-1.3)/(.04+(.01+.25)) = 2.166$. Hence the weight of the risk-free asset is $1+.433-2.166 = -.733$. This yields $\overline{R} = .1733$ and $\sigma = .2\sqrt{65^2+13^2}/30 = .442$. When normalized using the α of (17) this vector serves as a pricing vector. Again, the result is the same as using y_M for it can be seen that the ratio of weights is w_1/w_2 is $-1/5$ in both cases.

What about the critical case of $R_f = 1.1$?. There is neither a maximum nor a minimum extremum of the price of risk. However, there is still a minimum-norm solution. We find $z_1 = .3$ and $z_2 = -.3$. Using (16) we find $w_1 = -.3 \times 1.1/(.04(1+.09 \times 2/.04)) = -1.5$. Likewise, $w_2 = 1.5$. Notice that in this case the price of $w_1y_1 + w_2y_2$ is zero. (This is why the standard method breaks down.) The weight of R_f in the minimum-norm portfolio is 1. The portfolio has $\overline{R} = -1.5 \times 1.4 + 1.5 \times .8 + 1.1 = .20$ and $\sigma = 1.5 \times .2 \times \sqrt{2} = .424$. This is a point on the line tangent to the lower part of the minimum-variance curve of risky assets.

All three cases are shown in Fig. 2. Part (a) of the figure represents the typical situation when the risk-free return is lower than the threshold (minimum-variance) value. The second shows the situation when the risk-free return is higher than the threshold value. And the third shows the situation when the risk-free return is exactly equal to the threshold value, in which case there is no solution to extremizing the price of risk.

5 Correlation Pricing

We now employ the remaining three concepts of projection presented in Section 2; namely, the duality theorem, the splitting theorem, and the nested projection theorem. These lead to a new pricing formula, equivalent to the CAPM-style formula, but which is advantageous in many situations.

Suppose, as before, we are give $n+1$ assets, n of which are risky and the last of which is risk free. For convenience, we assume that the $n+1$ assets are linearly independent. These $n+1$ assets span a subspace S. The first n assets define the subspace $M \subset S$ of risky assets. Consider the new set of payoffs $y'_i = y_i - \overline{y}_i$ for $i = 1, 2, \ldots n$ and R_f. These payoffs span the same subspace S as the original payoffs. Let us denote the subspace spanned by the first n of these transformed payoffs by M' and the (one-dimensional) subspace spanned by the risk-free payoff by N. The risk-free payoff is orthogonal to each of the other payoffs, since $\mathrm{E}[y'_i R_f] = 0$, for each $i = 1, 2, \ldots, n$. Hence M' and

N are mutually orthogonal. According to the orthogonal splitting theorem, we may determine the projection of an arbitrary payoff x onto the subspace S by projecting separately onto M' and N and then adding the results.

To carry out the projection onto M' we use the duality theorem. According to that theorem, the projection onto M' is proportional to the payoff $y'_{m'}$ of norm 1 in M' that maximizes the inner product $\mathrm{E}(y'_{m'}x)$. Since any vector in M' has expected value equal to 0, it follows that $\mathrm{E}(y'x) = \mathrm{cov}(y', x)$ for any $y' \in M'$. Hence $y'_{m'}$ maximizes $\mathrm{cov}(y', x)$ subject to $\sigma^2(y') = 1,\ y' \in M'$.

Actually, any $y' \in M'$ that maximizes the correlation coefficient

$$\rho(y', x) = \mathrm{cov}(y', x)/[\sigma(y')\sigma(x)]$$

will work, because these are scalar multiples of each other. The projection of x onto M' is then $x_{M'} = \beta_{x,m'}y'_{m'}$ where $\beta_{x,m'} = [\mathrm{cov}(y'_{m'}, x)/\sigma^2(y'_{m'})]$. (That $x_{M'}$ is the projection can be verified by noting that $x - x_{M'}$ is orthogonal to M'.

The projection of x onto N is easily computed to be $\overline{x}$ and hence the total projection of x on S is

$$x_S = \overline{x} + \beta_{x,m'}y'_{m'}.$$

The vector $y'_{m'}$ is a unique linear combination of the y'_i's, and so we define y_m as the corresponding linear combination of the y_i's. Furthermore, if $y'_{m'}$ maximizes $\mathrm{cov}(y', x)/\sigma(y')$ over M', then the corresponding $y_m = y'_{m'} + \overline{y}_m$ maximizes $\mathrm{cov}(y, x)/\sigma(y)$ over the subspace M because adding a constant to any y does not change covariance or variance. We may therefore write the projection of x onto S as

$$x_S = \overline{x} + \beta_{x,m}[y_m - \overline{y}_m].$$

Since the price of the payoff y_i is p_i, the price of the payoff $y_i - \overline{y}_i$ is $p_i - \overline{y}_i/R_f$. It follows that the price of $y_m - \overline{y}_m$ is $p_m - \overline{y}_m/R_f$ and the price of the projection x_S is

$$p_x = \frac{1}{R_f}\left[\overline{x} + \beta_{x,m}(y_m)(R_f p_m - \overline{y}_m)\right]. \tag{25}$$

This leads to the following result.

Theorem 5.1 (Correlation Pricing Theorem) *The projection price of a payoff x is*

$$p_x = \frac{\overline{x} - \beta_{x,m}[\overline{y}_m - p_m R_f]}{R_f} \tag{26}$$

where y_m is the payoff of a priced asset that is most correlated with x, p_m is the price of y_m, and

$$\beta_{x,m} = \mathrm{cov}(x, y_m)/\sigma^2(y_m). \tag{27}$$

Note that the scale factor in y_m is arbitrary, since the expression for price is homogeneous of degree zero in y_m. Likewise, it is clear that the addition of a constant payoff to y_m does not effect the formula, and therefore the theorem is written without the restriction that y_m belong to M;

any payoff in S which achieves the maximum correlation can be used. (We may think of y_m in the theorem as denoting a "most correlated" asset.)

Formula (26) is similar in structure to the familiar CAPM formula in pricing form, with the correlated payoff y_m replacing the market (efficient) payoff. A major difference between the two formulas, however, is that y_m depends on x, while the efficient portfolio does not.

An advantage of the correlation pricing formula is that a most correlated asset frequently may be computed more reliably than the efficient portfolio computed either directly from data or indirectly by using the market portfolio and assuming that the market is mean–variance efficient. In many cases a most correlated asset may be close at hand. For example, if the assets are securities of various corporations, an asset most correlated with the payoff of a new corporation may be the security of a similar corporation in the same industry. Indeed, it is common practice to value companies through a comparative analysis that looks to similar companies. Correlation pricing is a formalization and generalization of this common pricing technique.

Let us examine some special cases of the formula. First, suppose $x = y_i$ for some $i = 1, 2, \ldots, n$. In that case a risky asset most correlated with x is y_i itself. Substituting $y_m = y_i$, the pricing formula gives $p_x = (\overline{y}_i - [\overline{y}_i - p_i R_f])/R_f = p_i$, as it should.

Second, suppose that given x we can find a y_m that is most correlated with x but we do not know the price of y_m. We can use the standard CAPM to find this latter price. Let y_M be the return of the efficient portfolio of risky assets (or, more generally, the risky part of the scaled minimum norm payoff). Then according to the CAPM-style formula,

$$p_m = \frac{1}{R_f}[\overline{y}_m - \beta_{m,M}(\overline{y}_M - R_f)].$$

Substituting this in (26), we find

$$p_x = \frac{\overline{x} - \beta_{x,m}[\overline{y}_m - [\overline{y}_m - \beta_{m,M}(\overline{y}_M - R_f)]}{R_f} = \frac{\overline{x} - \beta_{x,m}\beta_{m,M}(\overline{y}_M - R_f)}{R_f}.$$

According to the nested projection theorem, the projection of x onto y_M can be found by first projecting onto M and then projecting the result onto y_M. From this it can be deduced that $\beta_{x,m}\beta_{m,M} = \beta_{x,M}$, and the pricing formula reduces to

$$p_x = \frac{1}{R_f}[\overline{x} - \beta_{x,M}(\overline{y}_M - R_f)],$$

which is the CAPM formula for the price of x. (Alternatively, since the CAPM holds, it must be the case that $\beta_{x,m}\beta_{m,M} = \beta_{x,M}$.)

6 Zero-Level Pricing

Suppose an investor is an expected utility maximizer with utility function U for wealth, where U is continuously differentiable, increasing, and concave. We assume that the investor has cash endowment e which is invested optimally in the $n+1$ assets with payoffs $y_1, y_2, \ldots, y_n, R_f$ which span the subspace S. For convenience we let $y_{n+1} = R_f$. The optimal portfolio has payoff y^* of the form $y^* = \alpha_1 y_1 + \alpha_2 y_2 + \cdots \alpha_n y_n + \alpha_{n+1} R_f$ which satisfies $\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_{n+1} p_{n+1} = e$ (with $p_{n+1} = 1$).

Suppose a new asset with payoff $y_{n+2} = x$ is introduced. Let $\overline{S}$ be the subspace of H spanned by all $n+2$ payoffs. We wish to assign a price p_x to x. The *zero-level* price is the price p_x such the optimal portfolio constructed with the possibility of including this asset, as well as the others, would include this asset at the zero level. In other words, the optimal portfolio would not change. (See Refs. 9 and 10.)

The necessary conditions for the optimal portfolio (which are sufficient as well) are

$$\mathrm{E}[U'(y^*) y_i] = \lambda p_i, \quad \text{for } i = 1, 2, \ldots n+2. \tag{28}$$

for some $\lambda > 0$. The quantity $\mathrm{E}[U'(y^*)y]$ is a linear functional with respect to $y \in H$. Hence, according to the Riesz–Fréchet theorem, there is a vector $h \in H$ such that $\mathrm{E}[U'(y^*)y]/\lambda = (h|y)$ for all $y \in H$. This h is a pricing vector for all $n+2$ assets. It is clear that the projection of h onto the $n+1$ dimensional subspace S spanned by the first $n+1$ asset payoffs is the pricing vector g in that subspace. In other words, h is equal to g plus a vector orthogonal to S.

In minimum norm pricing the overall pricing vector is g itself; the extra orthogonal vector is zero. Hence, minimum norm pricing is a special case of zero-level pricing where the vector h defined by the functional $U'(y^*)$ lies in the subspace S.

One case where the price can be uniquely determined is when x is independent of all payoffs in S. In this case the necessary condition (28) applied to R_f and to x gives

$$\begin{aligned} \mathrm{E}[U'(y^*)]R_f &= \lambda \\ \mathrm{E}[U'(y^*)]\mathrm{E}[x] &= \lambda p_x. \end{aligned}$$

Hence,

$$p_x = \frac{\mathrm{E}[x]}{R_f}. \tag{29}$$

We note that in the notation of Section 5, x is orthogonal to M'. Hence p_x defined by (29) is the same as p_x defined by the projection of x onto $N + M' = S$. It follows that $h = g$ in this case. In other words, if x is independent of the y_i's, (orthogonal) projection pricing is equivalent to zero-level pricing for any utility function U.

If all payoffs are normal random variables, x can be written as $x = s + t$ where $s \in S$ and t is orthogonal to S. The price of x is then equal to the price of s which is well-defined since S is the subspace of marketed payoffs. Hence, projection pricing is equivalent to zero-level pricing for any utility function U when payoffs are jointly normal.

References

[1] Markowitz, H. M., *Portfolio Selection*, Journal of Finance, 7, pp 77–91, 1952.

[2] Holtan, H. M., *Asset Valuation and Optimal Portfolio Choice in Incomplete Markets*, Ph.D. Dissertation, Department of Engineering-Economic Systems, Stanford University, Stanford, California, August 1997.

[3] Luenberger, D. G., *Optimization by Vector Space Methods*, Wiley, 1969.

[4] Tobin, J., *Liquidity Preference as Behavior Towards Risk*, Review of Economic Studies, 26, pp. 65–86 1958.

[5] Sharpe, W. F. *Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk*, Journal of Finance, 19, pp. 425–442, 1964.

[6] Lintner, J., *The Valuation of risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets*, Review of Economics and Statistics, 47, pp. 13–37, 1965.

[7] Mossin, J., *Equilibrium in a Capital Asset Market*, Econometrica, 34, pp. 768–783, 1966.

[8] Huang, Chi-fu and R. Litzenberger, *Foundations for Financial Economics*, North-Holland, New York, 1988.

[9] Smith, J. E. and R. F. Nau, *Valuing Risky Projects: Options Pricing Theory and Decision Analysis*, Management Science, 41, pp. 795–816, 1995.

[10] Luenberger, D. G. *Investment Science*, Oxford University Press, New York, 1998.

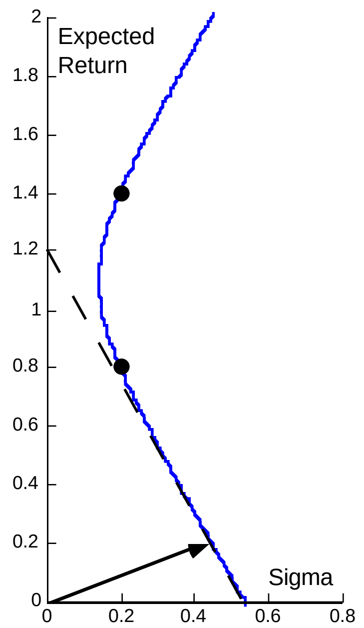

Figure 1: Feasible Set and Minimum Norm Vector

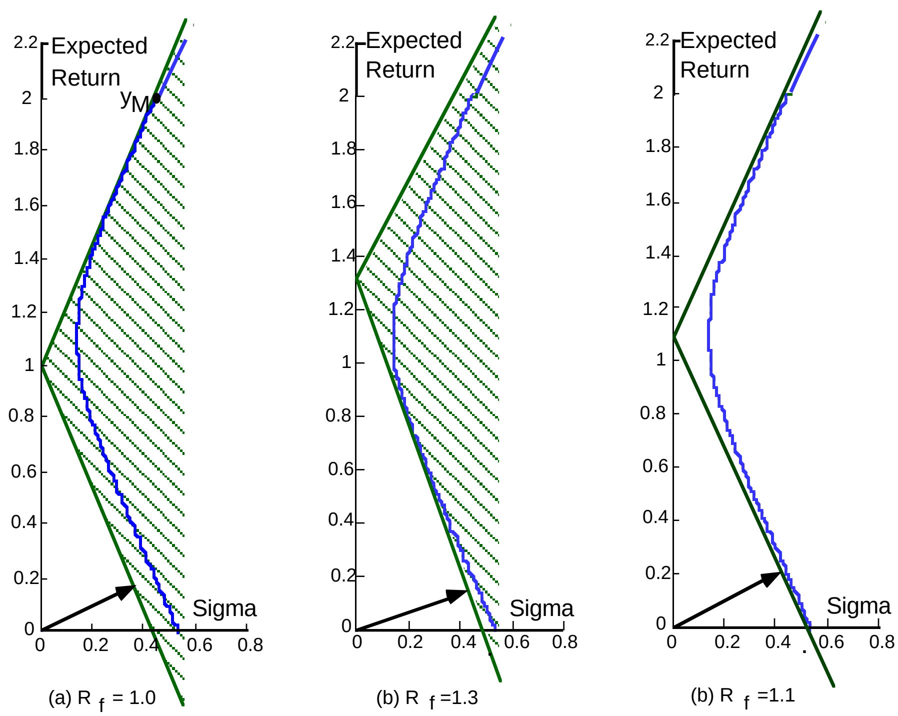

Figure 2: Feasible Set For Various Cases