A Correlation Pricing Formula[1]

David G. Luenberger
Stanford University

Abstract

In strict terms, the Capital Asset Pricing Model applies only to marketed assets, but the CAPM is frequently used to assign prices to nonmarketed assets as well. The Correlation Pricing Formula (CPF) is similar in form to the CAPM, and gives the same result. However, the CPF expresses the price of a nonmarketed asset in terms of a priced asset that is most correlated with the nonmarketed asset, rather than in terms of the market portfolio. This method is a rigorous version of the common practice of assigning a price to a new asset by considering prices of comparable marketed assets. The method sometimes has accuracy advantages when values in the formula must be estimated.

1 Introduction

We consider a collection of marketed assets in a single-period environment. One of these assets is risk-free with payoff R and price 1. There are n additional assets whose payoffs are, respectively, the random variables $A_1, A_2, \ldots, A_n$ with corresponding prices $p_1, p_1, \ldots, p_n$. There may be additional marketed assets as well, but the given collection of $n+1$ assets is assumed to be basic in the sense that any other marketed asset has a payoff that is a linear combination of those in the basic collection. It is further assumed that there is no possibility of arbitrage in the market.

The span of the basic $n+1$ asset payoffs is the set of all linear combinations of $A_1, A_2, \ldots, A_n, R$. We denote this span by N. With this framework, any asset whose payoff is duplicated by a linear combination of the basic $n+1$ payoffs represented by the A_i's and R is in N and can be priced by linearity—the price being equal to linear combination of prices that corresponds to the linear combination used to duplicate the payoff.

The pricing formula of the Capital Asset Pricing Model (CAPM) [1], [2] automatically prices assets in N according to this linearity rule, as long as the market portfolio used in the CAPM is the mean–variance efficient portfolio of risky assets (alternatively termed the Markowitz portfolio).

Now consider a new asset with payoff defined by the random variable B which is not in the span N of the original assets. A unique price cannot be inferred from the marketed asset prices. The market is incomplete and hence to assign a price to the new asset, a new criterion must be introduced. Several approaches have been investigated. See, for example, [3], [4].

One widely accepted method for assigning a price in this situation is to apply the CAPM formula to this asset as well, by simply entering the random payoff B into the CAPM formula the same way that it would be entered if it were in N. In fact, this procedure is commonly used to price projects within firms, evaluate companies that do not have public securities, price hypothetical new ventures, and so forth.

The price of B obtained this way (by CAPM) has a systematic relationship to the prices of the basic assets. Specifically, it is the price of the asset with payoff in N that best approximates B in the sense of minimum expected squared error. (See [5].) In a vector space framework (with

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inner product defined by expectation of the product), the payoff that best approximates the new asset payoff B is the *projection* of B onto the subspace N. Hence, one way to think about the pricing of B is to imagine that B is projected onto the space of basic assets, giving a payoff B_N. The price of this payoff is then found in terms of the basic assets by expressing B_N as a linear combination of the basic assets.

We can carry out this pricing process in an alternative (but equivalent) manner to derive an alternative to the CAPM formula. We term this new formula the Correlation Pricing Formula (CPF), and although it produces the same price as the CAPM, it is more convenient than CAPM in many practical and theoretical situations.

2 Derivation of the CPF

The projection of a payoff B onto the space of payoffs defined by $A_1, A_2, \ldots, A_n, R$ is the linear combination (with real coefficients $\overline{x}_i$ for $i = 0, 1, 2, \ldots, n$)

$$B_N = \overline{x}_0 R + \overline{x}_1 A_1 + \overline{x}_2 A_2 + \cdots + \overline{x}_n A_n \tag{1}$$

that minimizes the expected squared error

$$e = \mathrm{E}[(B - B_N)^2]$$

with respect to all combinations of the form (1). According to the classic projection theorem [6], the projection B_N is such that the error $B - B_N$ is orthogonal to each of of basic payoffs, where orthogonality is defined as the expected value of the product being zero.

An elementary transformation enormously simplifies the algebra required to carry out this approach. We choose as basic payoffs the alternative collection $R, A_1 - \overline{A}_1, A_2 - \overline{A}_2, \ldots, A_n - \overline{A}_n$, (where the overbar indicates expected value). This collection has the same span as the original collection. In terms of these payoffs the projection B_N will have the form (with different coefficients than (1))

$$P_B = x_0 R + x_1(A_1 - \overline{A}_1) + x_2(A_2 - \overline{A}_2) + \cdots + x_n(A_n - \overline{A}_n). \tag{2}$$

The orthogonality condition for the projection then leads directly to the equations[2]

$$\begin{aligned}
\mathrm{E}\big\{\big[B - x_0 R - x_1(A_1 - \overline{A}_1) - x_2(A_2 - \overline{A}_2) - \cdots - x_n(A_n - \overline{A}_n)\big]R\big\} &= 0 \\
\mathrm{E}\big\{\big[B - x_0 R - x_1(A_1 - \overline{A}_1) - x_2(A_2 - \overline{A}_2) - \cdots - x_n(A_n - \overline{A}_n)\big](A_1 - \overline{A}_1)\big\} &= 0 \\
\mathrm{E}\big\{\big[B - x_0 R - x_1(A_1 - \overline{A}_1) - x_2(A_2 - \overline{A}_2) - \cdots - x_n(A_n - \overline{A}_n)\big](A_2 - \overline{A}_2)\big\} &= 0 \\
&\vdots \\
\mathrm{E}\big\{\big[B - x_0 R - x_1(A_1 - \overline{A}_1) - x_2(A_2 - \overline{A}_2) - \cdots - x_n(A_n - \overline{A}_n)\big](A_n - \overline{A}_n)\big\} &= 0
\end{aligned}$$

In the first of these equations all products except the first two are zero, and hence this equation can be solved immediately to produce $x_0 = \overline{B}/R$.

Note also that x_0 cancels out of the remaining equations since in the i-th such equation x_0 occurs only as $\mathrm{E}[x_0 R(A_i - \overline{A}_i)]$ which is zero. Hence these remaining equations define n equations for the n unknowns $x_1, x_2, \ldots, x_n$.

[2] Alternatively, these equations can be found by writing the necessary conditions for minimization of the expected squared error.

These equations can be written in the simplified form

$$\mathrm{cov}(B, A_i) = \sum_{k=1}^{n} \mathrm{cov}(A_i, A_k) x_k \tag{3}$$

for $i = 1, 2, \ldots n$. The n equations can be solved for the x_k's in terms of the covariance matrix of the A_i's but for our purposes it is not necessary to solve them. Instead, we find an alternative interpretation of the equations by considering a new problem.

The correlation coefficient of two payoffs B and C is

$$\rho = \frac{\mathrm{cov}(C, B)}{\sigma(C)\sigma(B)}. \tag{4}$$

where σ denotes the standard deviation. The new problem we pose is that of finding an asset within the span of the original assets whose payoff is most correlated with a given payoff B. Such an asset is not unique because correlation is not affected by a positive scale factor nor by the addition of a risk-free payoff. However, if the maximum ρ is greater than zero, the maximizing asset payoff is unique to within such changes.

If in (4) we consider C to be a payoff in the span of basic assets and B to be fixed, we can maximize the correlation ρ with respect to C by maximizing the numerator while holding the denominator constant (since a scale factor cancels out). Hence we maximize $\mathrm{cov}(C, B)$ subject to $\mathrm{var}(C) = 1$. The choice of 1 for the value of the constraint is arbitrary and merely sets the scale factor.

Since correlation is not affected by the addition of a risk-free payoff, we can restrict the search to those C's with zero expected value. Specifically, for the problem of finding an asset from the basic collection that is most correlated with B we may look for an asset of the form

$$C = y_1(A_1 - \overline{A}_1) + y_2(A_2 - \overline{A}_2) + \cdots + y_n(A_n - \overline{A}_n).$$

The problem of finding a most-correlated marketed asset (or more precisely, the most-correlated portfolio of marketed assets) is then

$$\begin{array}{ll} \text{maximize} & \mathrm{cov}\big\{B, y_1(A_1 - \overline{A}_1) + y_2(A_2 - \overline{A}_2) \cdots y_n(A_n - \overline{A}_n)\big\} \\ \text{subject to} & \mathrm{var}\big\{y_1(A_1 - \overline{A}_1) + y_2(A_2 - \overline{A}_2) \cdots y_n(A_n - \overline{A}_n)\big\} = \quad 1 \end{array}$$

We can simplify this problem in a few ways. First, the expected values can be dropped from both the covariance and variance expressions. Second, the equality in the constraint can be changed to $\leq$ for with this inequality the maximum will be attained at a point of equality. Finally, the variance expression in the constraint can be expanded to show the cross product terms. With these transformations, the maximization problem becomes

$$\begin{array}{ll} \text{maximize} & \sum_{k=1}^{n} \mathrm{cov}(B, A_k) y_k \\ \text{subject to} & \sum_{j,k=1}^{n} y_j \,\mathrm{cov}(A_j, A_k)\, y_k \leq \quad 1. \end{array}$$

Introducing a Lagrange multiplier λ we may move the constraint up to the objective, and find the necessary conditions by differentiating the result with respect to each of y_i's separately. The Lagrange multiplier λ will be positive, owing to the inequality in the constraint and the fact that the function on the left is convex [7]. Setting the derivatives to zero produces

$$\mathrm{cov}(B, A_i) = 2\lambda \sum_{k=1}^{n} \mathrm{cov}(A_i, A_k) y_k \tag{5}$$

for each $i = 1, 2, \ldots, n$.

Except for the factor 2λ these equations are exactly the same as those of (3) that define the x_i's (for $1 \leq i \leq n$) of the projection B_N. We conclude that the set of y_i's is the same as the set of x_i's except for a positive scale factor.

Then

$$C = y_1(A_1 - \overline{A}_1) + y_2(A_2 - \overline{A}_2) + \cdots + y_n(A_n - \overline{A}_n),$$

where the y_i's are found as above, is a payoff in N that has maximum correlation with B. Since we know that $x_0 = \overline{B}$, the projection B_N is given by

$$B_N = \overline{B} + \beta C \tag{6}$$

for some real number β. Then from the fact that the error $B - B_N$ is orthogonal to all payoffs in the span of basic payoffs, we have

$$\mathrm{E}[(B - B_N)C] = 0.$$

Substituting (6) for B_N we have

$$\mathrm{E}[(B - \overline{B} - \beta C)C] = 0.$$

Since $\overline{C} = 0$ this becomes

$$\mathrm{cov}(B, C) - \beta \mathrm{var}(C) = 0,$$

or finally,

$$\beta = \frac{\mathrm{cov}(B, C)}{\mathrm{var}(C)}. \tag{7}$$

We know that the appropriate price p_B for B (using projection) is the price of $B_N = \overline{B} + \beta C$. Hence

$$\begin{aligned} p_B &= \frac{\overline{B}}{R} + \frac{\mathrm{cov}(B, C)}{\mathrm{var}(C)} p_C \\ &= \frac{1}{R}[\overline{B} + \beta R p_C] \end{aligned}$$

where p_C is the price of C. In this expression, C is an asset in the span N that is most correlated with B and has zero expected value. We may generalize the formula by adding a risk-free payoff to C. Addition of a risk-free component to C changes its expected value to, say $\overline{C}$, and this has price $\overline{C}/R$. However, the change does not influence beta. Hence

$$\begin{aligned} p_B &= \frac{1}{R}[\overline{B} + \beta R p_{(C - \overline{C})}] \\ &= \frac{1}{R}[\overline{B} - \beta(\overline{C} - R p_C)]. \end{aligned} \tag{8}$$

This is the final version of the Correlation Pricing Formula.

In the CPF (8), the payoff C is any payoff made up from the original marketed assets that has maximal correlation with the new payoff B. There are two degrees of freedom in the choice of C: its expected value can be changed by the addition of a risk-free asset, and it may be positively scaled. The formula remains the same for any such C.

As a simple application of the CPF let us apply it to an asset whose payoff B is in the subspace N. In this case, a most-correlated payoff in N is B itself, which means that $\beta = 1$. Hence the CPF gives

$$p_B = \frac{1}{R}[\overline{B} - (\overline{B} - Rp_B)] \equiv p_B,$$

which is of course correct.

It should be clear that the general form of the CPF is similar to that of the CAPM in certainty-equivalent form, with C replacing the return R_M of the market portfolio used in the CAPM. However, in the CAPM, the same R_M is used to price every asset, while in the CPF the choice of C depends on the asset being priced.

Beta

We introduce the notation

$$\beta_{B,C} = \frac{\mathrm{cov}(B,C)}{\mathrm{var}(C)} \tag{9}$$

to define a "beta" for any two random variables B and C (and the order is clearly important).

The beta used in the CPF has an important composition property that relates it to the standard beta of the CAPM and therefore establishes a relation between the two pricing formulas. Let X be any random payoff in the span N of basic payoffs. Let B be any random payoff (possibly outside N) and C a payoff in N most correlated with B. From (6) we have

$$B_N = \overline{B} + \beta_{B,C}(C - \overline{C}). \tag{10}$$

From the fact that $B - B_N$ is orthogonal to N we have

$$0 = \mathrm{E}[(B - B_N)X] = \mathrm{E}[\big(B - \overline{B} - (B_N - \overline{B})\big)X] = \mathrm{cov}(B,X) - \mathrm{cov}(B_N,X).$$

Hence,

$$\mathrm{cov}(B,X) = \mathrm{cov}(B_N,X).$$

Then, using (10) we have

$$\mathrm{cov}(B,X) = \beta_{B,C}\,\mathrm{cov}(C,X).$$

Dividing both sides by $\mathrm{var}(X)$ we obtain the important relation

$$\beta_{B,X} = \beta_{B,C}\,\beta_{C,X}, \tag{11}$$

which holds for any $X \in N$, any B, and a C most correlated with B.

As an application of this result, suppose that we use the correlation pricing formula to price B but we do not know the price of the correlated asset C. We find the price of C by using the CAPM formula. Hence, we have

$$p_B = \frac{1}{R}\left[\overline{B} - \beta_{B,C}\,(\overline{C} - p_C R)\right] \tag{12}$$

$$p_C = \frac{1}{R}\left[\overline{C} - \beta_{C,M}(\overline{R}_M - R)\right] \tag{13}$$

where R_M is the return on the Markowitz portfolio (possibly equal to the market portfolio) and $\beta_{C,M}$ is the beta of C andR_M. Substituting the formula for p_C into the formula for p_B, we obtain

$$\begin{aligned} p_B &= \frac{1}{R}\left[\overline{B} - \beta_{B,C}[(\overline{C} - [\overline{C} - \beta_{B,M}(\overline{R}_M - R)])]\right] \\ &= \frac{1}{R}\left[\overline{B} - \beta_{B,C}\ \beta_{B,M}(\overline{R}_M - R)\right] \\ &= \frac{1}{R}\left[\overline{B} - \beta_{B,M}(\overline{R}_M - R)\right] \end{aligned} \tag{14}$$

where the last equation uses the composition relation (11). This last formula, is of course, the CAPM formula for the price of B.

Accuracy Considerations

An advantage of the correlation pricing formula is that in some circumstances it is more convenient and more accurate than the CAPM when the relevant quantities must be estimated. This is especially true if a most-correlated asset is easily identified. Analysis of a new real estate venture, for example, may best be accomplished by comparison with similar projects in the same locale rather than by correlation with a general asset market portfolio.

Two issues concerning accuracy arise. First, there are likely to be errors in determining a most-correlated asset or the Markowitz portfolio. These errors will produce corresponding errors in the determined price. Second, even assuming that a most-correlated asset or the Markowitz portfolio is correctly identified, it is necessary to estimate the associated statistics—the means, covariances and variances—called for in the corresponding pricing formula. We explore these issue by considering the following points.

1. Pricing a known asset. As shown above, the CPF applied to a payoff B in the span N leads directly to the proper price p_B (as a tautology). The CAPM on the other hand, because of errors in the definition of the optimal portfolio and errors in beta, would likely give an incorrect price. Similarly, for payoffs that can be identified as being close to the span of N, the CPF is likely to more accurate than the CAPM.
2. Market portfolio. In the CAPM the market portfolio is frequently substituted for the Markowitz portfolio, based on the assumption that the market portfolio is mean–variance efficient. In practice this market portfolio is taken to be one of the popular market indices, such as the S&P 500. However, the evidence that the market is efficient, although plausible, is not strong and use of it in the pricing formula introduces errors.
3. Projection Price. Ideally, both CAPM and CPF assign a price to the payoff B equal to the price of the projection of B onto the span N of priced assets. For CAPM to produce this price faithfully, it is necessary that it price any most-correlated payoff C correctly. To prove this note that the critical component in the CAPM formula is $\mathrm{cov}(B, R_M)$ where R_M is the return of the market. However, from (6)

$$\mathrm{cov}(B, R_M) = \mathrm{cov}(B_N, R_M) = \beta\ \mathrm{cov}(C, R_M)$$

   where β is a constant independent of R_M. Hence, for CAPM to be accurate, it must be able to evaluate $\mathrm{cov}(C, R_M)$ accurately. On the other hand, the CPF uses C and the price of C directly.

4. Accuracy of Statistical Quantities. Assume that both the Markowitz portfolio and a most-correlated asset are accurately identified. Both the CAPM and the CPF require a beta, determined by covariances and variances. We can roughly investigate the accuracy inherent in the two methods. As a first approach, consider the problem of estimating covariance from historical data. Suppose the random variables x and y are correlated and we have n independent sample pairs $(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$. We form the estimates $\hat{\overline{x}}$ and $\hat{\overline{y}}$

$$\hat{\overline{x}} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$\hat{\overline{y}} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

The error in these estimates can be characterized by the error variance, which for $\hat{\overline{x}}$ (for example) is

$$\mathrm{var}(\hat{\overline{x}} - \overline{x}) = \frac{\sigma_x^2}{n}. \tag{15}$$

We estimate the covariance through the formula

$$\hat{\mathrm{cov}}(x, y) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\overline{x}})(y_i - \hat{\overline{y}}). \tag{16}$$

The expected value of this estimate is in fact $\mathrm{cov}(x, y)$. If the samples are drawn from a jointly normal distribution of x and y, then the sample covariance (16) is distributed according to a Wishart distribution [9], [10] and has a variance (see [11])

$$\mathrm{var}[\hat{\mathrm{cov}}(x, y)] = \sigma_x^2\,\sigma_y^2\frac{1+\rho^2}{n-1}. \tag{17}$$

We expect $\rho(R_M, B)$ to be smaller than $\rho(C, B)$ (since in fact $\rho(C, B)$ is maximal). From (17) we see that the accuracy of the estimate of covariance is relatively independent of ρ, so the $\mathrm{cov}(B, R_M)$ used by CAPM and the $\mathrm{cov}(B, C)$ used by CPF are likely to have similar absolute accuracy when estimated this way.

We can do better by estimating the correlation coefficients directly. We can estimate the ρ of x and y from samples by finding the ρ that solves

$$\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\overline{x}})(y_i - \hat{\overline{y}}) - \frac{\rho}{n-1}\left[\sum_{i=1}^{n}(x_i - \hat{\overline{x}})^2\right]^{1/2}\left[\sum_{i=1}^{n}(y_i - \hat{\overline{y}})^2\right]^{1/2} = 0. \tag{18}$$

For $\rho = 0$ the second term vanishes and the error in the first term is (from (17)) $2\sigma_x^2\sigma_y^2/(n-1)$, implying[3] a large variance in the estimate of ρ. If $\rho = 1$ then $x_i = cy_i$ for some constant c, and the error in the formula is zero implying that the correct value of ρ will be found exactly. Hence, in general, high correlation leads to lower error than low correlation. This implies that estimates may be more accurate for the CPF than for the CAPM.

[3]Of course, the error in the estimate of ρ may be larger than this, since the solution of (18) involves the coefficient of ρ as well.

The same technique can be applied directly to the estimation of beta by defining the estimate of beta as that solving

$$\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\hat{\overline{x}})(y_i-\hat{\overline{y}})-\frac{\beta}{n-1}\sum_{i=1}^{n}(y_i-\hat{\overline{y}})^2=0.$$

For $\rho = 0$ it follows that $\beta = 0$ and the variance in the equation is $2\sigma_x^2\sigma_y^2/(n-1)$. For $\rho = 1$ there is no error and hence the estimate of β will be exact. Again, high values of ρ yield more accurate results than low values.

5. Expected Payoff Estimates. The most difficult quantity to measure in asset pricing formulas of this type is the expected payoff; or more precisely, the difference between the payoff and a reference payoff. From (15) we see that the standard deviation of an error in, say $\overline{R}_M$, is $\sigma_M/\sqrt{n}$. For a market volatility of 15% and 9 years of data the error is 5% which is considerably larger than we wish; see [8] Chapter 8.

   Assuming again that the Markowitz (or market) portfolio is identified, the CAPM formula is
   $$p_B = \frac{1}{R}[\overline{B} - \beta_{B,M}(\overline{R}_M - R)].$$
   If we assume that $\beta_{B,M}$ is reasonably accurate (and we have shown above that it may not to be), we still must estimate $\overline{B}$ and $\overline{R}_M - R$, both of which are difficult. We can simplify the calculation by introducing a transformation.

   If the market portfolio is scaled so that its price is, say, p_M rather than 1, and its payoff is M, the CAPM formula is
   $$p_B = \frac{1}{R}\left[\overline{B} - \beta_{B,M}[\overline{M} - p_M R]\right],$$
   where now $\beta_{B,M} = \mathrm{cov}(B, M)/\mathrm{var}(M)$. We may select the scale factor so that $\beta_{B,M} = 1$. This scaling does not require knowledge of $\overline{M}$. With that scale factor the CAPM reduces to
   $$p_B - p_M = \frac{1}{R}[\overline{B} - \overline{M}]. \tag{19}$$
   The issue thus boils down to the estimation of $\overline{B} - \overline{M}$.

   The CPF can likewise, by scaling of C, be transformed to the form
   $$p_B - p_C = \frac{1}{R}[\overline{B} - \overline{C}]. \tag{20}$$
   When B is closely related to C we might expect (20) to be superior to (19). First, the transformation to the simple form depends on the beta of the formula and this is likely to be more accurate for the CPF. Second, if judgment used to estimate $\overline{B}-\overline{M}$ or $\overline{B}-\overline{C}$, the latter may be better when B and C are similar.

6. Identification Errors. A further issue, of course, is that a most-correlated asset C or the Markowitz portfolio M may not be perfectly identified. This will of course cause errors in the pricing formulas. We can examine the first-order effect of such mis-identification.

   We define p_B^M and p_B^C to be the prices of B estimated by the CAPM and the CPF, respectively. Also we let $q^M = \overline{B} - Rp_B^M$ and $q^C = \overline{B} - Rp_B^C$.

From the CAPM we have

$$q^M = \beta_{B,M}[\overline{M} - Rp_M]$$

where M is the payoff of the Markowitz portfolio. Now suppose that the true Markowitz portfolio is mis-indentified as $M + \epsilon\Delta$, where Δ is a random payoff and ϵ is a small constant. We let E^M be the resulting error factor defined as $\mathrm{d}q^{M+\epsilon\Delta}/\mathrm{d}\epsilon$ evaluated at $\epsilon = 0$. specifically,

$$E^M = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\beta_{B,M+\epsilon\Delta}(\overline{M+\epsilon\Delta} - Rp_{M+\epsilon\Delta})\right]\Big|_{\epsilon=0}$$

Using

$$p_{M+\epsilon\Delta} = \frac{1}{R}[\overline{M+\epsilon\Delta} - \beta_{M+\epsilon\Delta,M}(\overline{M} - Rp_M)]$$

we differentiate $q^{M+\epsilon\Delta}$ to obtain (after some algebra)

$$E^M = \left[\frac{\mathrm{cov}(B,\Delta)}{\mathrm{cov}(B,M)} - \frac{\mathrm{cov}(M,\Delta)}{\sigma_M^2}\right] q^M \tag{21}$$

Likewise, assuming that a most-correlated asset C is mis-identified as $C + \epsilon\delta$, we define

$$E^C = \frac{\mathrm{d}}{\mathrm{d}\epsilon} q^{C+\epsilon\delta} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\beta_{B,C+\epsilon\delta}(\overline{C+\epsilon\delta} - Rp_{C+\epsilon\delta})\right].$$

Using (13) for p_C and for $p_{C+\epsilon\delta}$ we find

$$E^C = \left[\frac{\mathrm{cov}(B,\delta)}{\mathrm{cov}(B,C)} + \frac{\mathrm{cov}(\delta,M)}{\mathrm{cov}(C,M)} - 2\frac{\mathrm{cov}(C,\delta)}{\sigma_C^2}\right] q^C. \tag{22}$$

Note that if Δ is proportional to M, then E^M is zero. Likewise, if δ is proportional to C, then E^C is zero (as expected, since C or M can be changed by scale factors).
It is difficult argue that one of the values (21) or (22) is typically lower than the other. We might assume that on average $\mathrm{cov}(M,\Delta)$ and $\mathrm{cov}(C,\delta)$ are small (reflecting the notion that the mis-identification error is uncorrelated with what is being identified) and that B is not strongly correlated with M but strongly correlated with C. Then

$$E^M \approx \frac{\mathrm{cov}(B,\Delta)}{\mathrm{cov}(B,M)} q^M \tag{23}$$

$$E^C \approx \frac{\mathrm{cov}(\delta,M)}{\mathrm{cov}(C,M)} q^C. \tag{24}$$

These two are essentially symmetric, and either could be superior to the other, depending on the particular nature of the mis-identifications. In practice, proxies for both M and C are often found without explicit calculation. M is frequently taken to be the payoff of a broad market index. C can be approximated by study of marketed assets whose payoffs are obviously similar to that of the asset being priced. Both of these have advantages and weaknesses.

The six points explored in this section constitute only a rough analysis. However, this analysis tends to support the notion that when a nonmarketed asset is highly correlated with a portfolio of marketed assets, the CPF offers an attractive alternative to the CAPM. The error due to mis-identification of the pricing asset (C or M respectively) is about equally troublesome for both methods, and the estimation of associated statistical quantities is perhaps less troublesome for the CPF than for the CAPM.

3 Simultaneous Pricing

The CPF can be extended to a formula that prices a collection of assets by using a common priced asset to determine the betas for each asset in the collection. The result is a combination of the CPF and the CAPM. The procedure is simple. Suppose that $B_1, B_2, \ldots B_m$ are risky assets to be priced, and let $C_1, C_2, \ldots, C_m$ be corresponding most-correlated assets in the space N of marketed assets. Assume these most-correlated assets have prices $p_1, p_2, \ldots, p_m$. We then find a portfolio of these C_i's and the risk-free asset R that is efficient (and is not the trivial risk-free asset). Call this portfolio C. Then the projection price of any asset B in the space spanned by the B_i's will be

$$p_B = \frac{1}{R}\left[\overline{B} - \beta_{B,C}[\overline{C} - p_C R]\right].$$

This formula is verified in three steps. First, although the operation of determining a most-correlated asset is not linear, the operation of orthogonal projection is linear. Hence, the orthogonal projection of the space of B_i's onto N is a linear subspace N' of N. This subspace N' is contained in the space N^0 spanned by the C_i's and R. Second, we know that the (reduced) CAPM formula derived for N^0 will price all the C_i's correctly. Finally, this restricted CAPM will also price all the B_i's correctly since it will price them according to their orthogonal projection onto N^0 which is the same as their orthogonal projection onto N.

This result might be used in the design of optimal financial projects. For example, suppose that a planned project has two distinct components whose relative weights vary in different project designs. Suppose also that the two components each have a corresponding most-correlated marketed asset. By computing the appropriate combination of the two most-correlated assets, we can write a general expression that gives the value of the project as a function of the weights of the two components. We can then optimize the project with respect to the choice of weights of the two components.

4 Conclusions

In the case of perfect estimates of all relevant quantities, the Correlation Pricing Formula and the CAPM assign exactly the same prices to assets. The two formulas are similar in structure, the only difference being that the role of the Markowitz (or market) portfolio used in the CAPM is replaced by a marketed asset most closely correlated with the asset being priced. The CPF appears to have advantages when assets with payoffs closely correlated with that of the asset to be priced can be easily identified by the nature of the situation. The CPF can be regarded as a formalization of the common practice of pricing new assets by using the prices of similar assets as points of comparison.

The CPF is especially valuable in the pricing of derivative securities when the underlying asset is not marketed. The correlation pricing formula provides a framework that can be used for theoretical purposes and as a basis for efficient computation. This application of the CPF will be reported in a subsequent report.

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