# Arbitrage and Universal Pricing*

April 2001

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**Abstract**

This paper considers two methods for pricing assets and examines the relations between them. The first method is based on the principle of no-arbitrage, which asserts that introduction of the new asset should not create an arbitrage in a market that was before arbitrage free. This condition is satisfied by prices for the new asset between specific lower and upper limits, determined as the values of certain linear programming problems. The duals of these problems determine a pricing random variable. In the second method of pricing, a new asset is priced so that a given utility maximizing investor will include this asset in his or her portfolio at a zero level. The corresponding price is called the zero-level price. A zero-level price is universal for a class of utility functions if it is a zero-level price for every utility function within that class. This paper shows that universal zero-level prices exist in several important situations.

*JEL classification:* G10; G12

*Keywords:* Arbitrage pricing, Zero-level pricing

## 1 Introduction

A fundamental problem of finance is that of pricing a new asset introduced into a existing, well-functioning market. The new asset may be a new security, a derivative security, a project in a firm, or a new venture. The price determined, should in some appropriate sense be consistent with prices in the market and with investor characteristics.

One method of determining a suitable price is by use of the no-arbitrage principle. If the existing market is assumed to be arbitrage-free, it is reasonable to require that introduction of the new asset, at the assigned price, will not create an arbitrage opportunity. This is a powerful and general criterion that is used frequently in modern finance. However, in many cases, the no-arbitrage criterion is not strong enough to produce a single price; rather a range of prices satisfy the condition. To obtain a unique price, additional criteria must be introduced. Like most researchers, we favor criteria that are relatively simple, possess a strong degree of logic, and do not rely on arbitrary parameter choices.

We go beyond the no-arbitrage principle by considering zero-level pricing, which has been introduced earlier in Luenberger (1998), Smith and Nau (1995), and Holtan (1997). In this method, the price is determined such that an investor with a specific utility function will elect to include the new asset in his or her portfolio at the zero level. This has the advantage of being a linear pricing scheme, with the price of a combination of two assets being the corresponding combination of the two individual prices. We show that zero-level prices exist and give conditions

*This paper is dedicated to David Kendrick, whose creative work forged a valuable link between two disciplines.
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for such a price to be unique. This theory uses the special characterization of arbitrage-free prices obtained in the first part of the paper.

A zero-level price has the apparent disadvantage of being dependent on the particular utility function used for its derivation. To address this, we say that a zero-level price is universal if it is a zero-level price for all utility functions within a large class. Universal zero-level prices are therefore independent of particular parameter choices. We show that in many situations zero-level prices are in fact universal.

A simple situation that illustrates some of the issues associated with non-marketed assets is that of pricing a coin flip. Suppose that the coin flip pays ten thousand dollars for an outcome of heads and zero for an outcome of tails. Assume that the coin is known to be fair, and that the rate of interest over the period between betting on the flip and its payment is zero. What should be the price of a flip?

The principle of no-arbitrage is of little help for this example. It implies only that the price must lie between zero and ten thousand dollars. Zero-level pricing determines the price at which a particular risk-averse investor would "purchase" the coin flip asset at zero level. That is, it is assumed that a fractional share of the coin flip proposition can be purchased, and the price is assigned so that a specific investor would purchase the asset at zero level. For the coin flip, this price is five thousand dollars, and most importantly, this price is (essentially) independent of the investor's utility function and level of wealth. In this sense, five thousand dollars is a universal zero-level price. Because of the universality, this price is free from assumptions on parameter values or functional forms. Universality holds for several important classes of assets.

## 2 Assets, Prices, and Arbitrage

We assume that the collection of marketed assets defines a perfect market in the sense that: there are no transactions costs, it is possible to divide any asset arbitrarily, it is possible to short any asset and receive its price immediately, and the activity of any single individual does not influence prices. These are idealizations, but they have the agreeable property of being general structural assumptions rather than specific assumptions about parameter values or functional forms. We adhere to these assumptions throughout.

In addition to these assumptions concerning the perfect nature of transactions, we assume that the prices of marketed assets are such that arbitrage is impossible. (We make this precise later in this section.) This assumption provides the basis for a good portion of modern pricing theory and it is useful as a starting point for analysis of non-marketed assets.

We define an asset payoff space, say $X$, in which the payoffs of all relevant assets exist. In our framework $X$ is a space of random variables on a probability space (consisting of a set $\Omega$ of underlying possibilities together with a probability measure on $\Omega$). We assume that the elements of $X$ are square-integrable over the probability space, meaning that the expected value of the square exists and is finite. We consider two elements in $X$ to be identical if they differ only on a set of probability zero. We also assume that $X$ is complete, in the sense of a norm as will be discussed later in this section.

The linear span of the marketed payoffs is the linear space generated by linear combinations of the marketed payoffs. In general, the linear span of the marketed assets is a subset of the given $X$. However, if the linear span of marketed assets is $X$, any payoff in $X$ can be achieved as a combination of the marketed assets, and the market is said to be complete (with respect to $X$). Any new asset with payoff in $X$ can then be priced by linear pricing: pricing a combination of assets by the corresponding combination of prices.

Assets live for a common single period of time and have payoffs at the end of the period which are in $X$. If such an asset is a marketed asset it also has a unit price, which is paid at the beginning of the period to acquire one unit, and scaled proportionately for other levels.

There is a set of $n$ marketed assets with payoffs (final prices) $d_1, d_2, \ldots, d_n$ and corresponding prices $p_1, p_2, \ldots p_n$. We define $D = [d_1, d_2, \cdots, d_n]$ as the linear mapping that for an $n$-dimensional vector of amounts $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ gives the random variable $D\alpha = x = \alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_n$. Hence, $D : R^n \to X$.

Likewise, we define the price vector $p = (p_1, p_2, \ldots, p_n)$. Marketed assets are priced linearly, with the price of an asset with payoff $x = D\alpha$ being $p \cdot \alpha \equiv \sum_{i=1}^{n} p_i \alpha_i$.

An *arbitrage* is a combination of assets that produces a (random) payoff that is nonnegative and yet has a nonpositive price, one of these conditions being nontrivial. An arbitrage is therefore defined by a vector $\alpha$ satisfying

$$D\alpha \geq 0 \quad (1)$$

$$p \cdot \alpha \leq 0, \quad (2)$$

where either the top inequality is nontrivial in the sense that there is a set of positive probability where $D\alpha > 0$ or the lower inequality is strict. We usually assume that there is no possibility of arbitrage among the marketed assets.

If no arbitrage is possible, then, in particular, if there is a non-zero $\alpha$ satisfying $D\alpha = 0$, it follows that $p \cdot \alpha = 0$, for otherwise either $\alpha$ or $-\alpha$ would be an arbitrage. From this it immediately follows that if $d_k = \sum_{i \neq k} \alpha_i d_i$, then $p_k = \sum_{i \neq k} \alpha_i p_i$, which is the linear pricing rule.

Consider now a new asset with payoff $e \in X$ that is not marketed and not priced. We wish to assign a price $p_e$ to this asset. This is the basic pricing problem.

If $e$ is in the linear span of the $d_i$'s and if we require that the price $p_e$ of $e$ not introduce an arbitrage, we deduce immediately that $e$ is also priced linearly; that is, if $e$ is a linear combination of the $d_i$'s, then $p_e$ is the same linear combination of the $p_i$'s. Hence, extending pricing to assets in the linear span of the marketed assets is straightforward under the no-arbitrage principle.

When the market is incomplete, there are assets in $X$ that cannot be priced by linearity. Still, we can use the no-arbitrage principle as a first step of analysis by finding the range of prices such that inclusion of a particular new asset along with the marketed assets does not lead to the possibility of arbitrage. Such price bounds have been considered previously; see Harrison and Kreps (1979). In Holtan (1997) it is shown that the bounds are related to linear programming problems as expressed in our somewhat more general setting by the following lemma which is proved in the Appendix.

**Lemma 1** *Assume there is no arbitrage among the marketed assets. There are $p_e^l$ and $p_e^u$ defining lower and upper bounds of the price $p_e$ that can be assigned to $e$ such that no arbitrage is possible for $p_e \in (p_e^l, p_e^u)$. Arbitrage is possible for $p_e \notin [p_e^l, p_e^u]$. The bounds are given by*

$$p_e^l = \sup \{p \cdot \alpha : D\alpha - e \leq 0\} \quad (3)$$

$$p_e^u = \inf \{p \cdot \alpha : D\alpha - e \geq 0\}. \quad (4)$$

Note that the constraints may not be feasible, $p_e^l$ or $p_e^u$ may not be finite, and even if they are finite, the inf and sup in the lemma may not be achieved. Furthermore, although it is clear from their definitions that $p_e^l \leq p_e^u$, it is not obvious that the prices of lemma 1 satisfy this inequality[1].

We can obtain results that are more definitive and provide additional insight by considering the dual of each of the linear programs in lemma 1. However, since the original linear programs have constraint sets that are in (possibly) infinite-dimensional spaces, we must develop the dual programs carefully.

The dual programs involve the adjoint of the operator $D$. As stated earlier $D : R^n \to X$. We assume that $X$ is complete under the norm defined by $\|x\| = \sqrt{\mathrm{E}(x^2)}$. We therefore consider $X$ to be a Hilbert space with the inner product of two elements $w \in X$, $v \in X$ defined by $w \cdot v = \mathrm{E}(wv)$. The adjoint of $D$ maps from $X$ into $R^n$. Applied to an element $w \in X$ the adjoint produces $wD$ defined as satisfying $wD \cdot \alpha = w \cdot D\alpha$ for all $\alpha \in R^n$.

With these notions we have dual characterizations of both $p_e^l$ and $p_e^u$ expressed by the following two theorems, which are proved in the appendix.

**Theorem 1** *Assume there is no arbitrage among the marketed assets and that $e \geq 0$, $e \neq 0$. Then $p_e^l$ is finite and has the two dual characterizations:*

$$p_e^l = \begin{array}{cc} \max\limits_{\alpha} & p \cdot \alpha \\ \text{sub to} & D\alpha - e \leq 0. \end{array} \tag{5}$$

*and*

$$p_e^l = \begin{array}{cc} \min\limits_{w} & w \cdot e \\ \text{sub to} & wD = p \\ & w \geq 0. \end{array} \tag{6}$$

**Theorem 2** *Assume there is no arbitrage among the marketed assets, that there is an $\alpha$ such that $D\alpha \geq 1$, and $e$ is bounded. Then $p_e^u$ is finite and has the two dual characterizations:*

$$p_e^u = \begin{array}{cc} \min\limits_{\alpha} & p \cdot \alpha \\ \text{sub to} & D\alpha - e \geq 0. \end{array} \tag{7}$$

*and*

$$p_e^u = \begin{array}{cc} \max\limits_{w} & w \cdot e \\ \text{sub to} & wD = p \\ & w \geq 0. \end{array} \tag{8}$$

The assumption in theorem 1 that $e$ be nonnegative is usually satisfied in applications. The assumption in theorem 2 that there is an $\alpha$ such that $D\alpha > 1$ is satisfied, for example, by the existence of a risk-free asset.

The characterizations (6) and (8) of $p_e^l$ and $p_e^u$ based on duality clearly demonstrate the fundamental inequality $p_e^l \leq p_e^u$ since these are the minimum and maximum over the same constraint set.

[1]However, a short proof assuming feasibility of the constraints is possible: Assume that $p_e^u < p_e^l$. Let $\alpha_l$, $\alpha_u$ satisfy $D\alpha_l \leq e$, $D\alpha_u \geq e$, and define $p_l = p \cdot \alpha_l$, $p_u = p \cdot \alpha_u$. Then $D(\alpha_u - \alpha_l) \geq 0$. And $p_u - p_l \leq p_e^u - p_e^l < 0$. Thus $\alpha_u - \alpha_l$ is an arbitrage of the marketed assets, which is a contradiction.

The dual characterizations also provide an alternative interpretation of the price bounds. A random variable $w$ satisfying the dual constraints is a pricing variable in the sense that any payoff in the linear span of the marketed assets is priced by taking the expected value of its product with $w$. Specifically, a portfolio defined by weights $\alpha$ of marketed assets, has price $p_\alpha = p \cdot \alpha = wD \cdot \alpha = w \cdot D\alpha = \mathrm{E}(wD\alpha)$.

If we find a random variable $w$ that correctly prices the market assets by $p_i = \mathrm{E}(wd_i)$, for each $i = 1, 2, \ldots n$, we may attempt to price a payoff $e$ outside the market span by the same formula: $p_e = \mathrm{E}(we)$. To avoid arbitrage possibilities for all $e \in X$ we must require $w \geq 0$, for otherwise a positive payoff could have a negative price. Hence the dual programs (6) and (8) imply that the price bounds are the bounds obtained with respect to all possible arbitrage-free pricing variables $w$ that correctly price the marketed assets.

A final technical point of interest is the case where $p_e^l = p_e^u$. A strong conclusion can be inferred.

**Theorem 3** *Assume that $p_e^l = p_e^u$ and both are finite. Then either $e$ is linearly dependent on the marketed assets, there is a dependency among the marketed assets, or there is an arbitrage among the marketed assets.*

*Proof:* Lemma A.5 (in the appendix) implies that there is an $\alpha_l$ that solves (5), the primal for the lower bound. We have $p \cdot \alpha_l = p_e^l$ and $D\alpha_l - e \leq 0$. Likewise, there is $\alpha_u$ that solves (7), the primal for the upper bound. Then $p \cdot \alpha_u = p_e^u$ and $D\alpha_u - e \geq 0$. Subtracting the lower primal relations from the upper primal relations and accounting for $p_e^l = p_e^u$ yields $p \cdot (\alpha_u - \alpha_l) = 0$ and $D(\alpha_u - \alpha_l) \geq 0$. If $\alpha_u \neq \alpha_l$ then either $\alpha_u - \alpha_l$ represents an arbitrage among the marketed assets or it represents a linear dependency among those assets. If $\alpha_l = \alpha_u$, then $D\alpha_l = e$ which means that $e$ is linearly dependent on the other assets. ▮

# 3 Zero-Level Pricing

The idea of zero-level pricing of a nonmarketed payoff $e$ is to find the price $p_e$ such that a certain investor will elect to neither purchase nor short it. At the price $p_e$ the investor is indifferent to the inclusion of $e$. This section shows that this concept is well defined in the sense that such a $p_e$ exists.

Consider an investor having a utility function $U$ defined for positive values of final wealth. Positive random payoffs $x \in X$ are ranked by their expected utility $\mathrm{E}[U(x)]$. We assume that $U$ is continuous, strictly increasing, and strictly concave. The investor has initial wealth $W > 0$ that is to be allocated among the available assets.

As before, the $n$ marketed assets are defined by their payoffs $d_i \in X$, $i = 1, 2, \ldots, n$. There is an additional asset with payoff $e$. We assume that there is no linear dependency among the marketed assets.

Given a price $p_e$ the investor seeks to solve the following problem, which is standard in the finite-state case; see e.g. Luenberger (1998).

$$\begin{aligned} \max_{\alpha,\ \alpha_e} \quad & \mathrm{E}[U(x)] && (9) \\ \text{subject to} \quad & p \cdot \alpha + p_e \alpha_e \leq W \\ & D\alpha + e\alpha_e = x \\ & x \geq 0. \end{aligned}$$

**Theorem 4** *If there is no arbitrage and all assets are linearly independent, there is a unique solution to problem (9).*

*Proof:* Without loss of generality we take $e = 0$ and $p_e = 0$ (and $\alpha_e = 0$). Consider the constraint set $A = \{\alpha : D\alpha \geq 0,\ p \cdot \alpha \leq W\}$. We will show that $A$ is bounded. Assume not. Then there is, by convexity of $A$, a sequence of $\alpha_k$'s in $A$ with $||\alpha_k|| = k$. Let $\beta_k = \alpha_k/k$. Then $||\beta_k|| = 1$ for each $k$. Therefore, there is a subsequence of the $\beta_k$'s converging to a $\beta$. For this limit point $\beta$, we have $D\beta \geq 0,\ p \cdot \beta \leq 0$. If either of these inequalities is strict, there would be an arbitrage. If $D\beta = 0$ there would be a linear dependency. Since neither of these is possible, $A$ is bounded.

It is also clear that $A$ is closed. Hence it follows from continuity that (9) has a solution.

To prove that the solution is unique, assume that $\alpha_1$ and $\alpha_2$ are solutions, both with value $V$. By the convexity of the constraints it follows that $\alpha_0 = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2$ is feasible. By the concavity of $U$ it follows that $\mathrm{E}[U(D\alpha_0)] \geq \frac{1}{2}\mathrm{E}[U(D\alpha_1)] + \frac{1}{2}\mathrm{E}[U(D\alpha_2)] = V$. By strict concavity we must have $D\alpha_1 = D\alpha_2$, for otherwise $\mathrm{E}[U(D\alpha_0)] > V$ which is impossible. However, $D\alpha_1 = D\alpha_2$ implies $\alpha_1 = \alpha_2$ since otherwise $\alpha_1 - \alpha_2$ would represent a dependency among the assets which is ruled out. Hence the solution is unique. ∎

The solution to the utility maximization problem (9) defines a mapping $F$ from a price $p_e$ to a purchase level $\alpha_e = F(p_e)$. Under appropriate conditions, as spelled out in the next theorem, there is a $p_e$ that yields $\alpha_e = 0$. This $p_e$ is the zero-level price for the investor.

In theorem 5 below, we require a differentiability assumption for uniqueness. For technical simplicity we take this to mean that functions of the form $\mathrm{E}[U(\alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_n)]$ are continuously differentiable with respect to the $\alpha_i$'s.

**Theorem 5** *Assume no arbitrage is possible among the $n$ marketed securities, and that $e \geq 0$, $e$ is bounded above, and $e$ is independent of the marketed assets. Then there is a (zero-level) price $p_e^0$ such that $F(p_e^0) = 0$. This price is unique if: there is a marketed portfolio defined by $\alpha_0$ such that[2] $D\alpha_0 > 1$, utility is continuously differentiable, and the solution $x^*$ to (9) with $e = 0$, $p_e = 0$ has $x^* > 0$.*

*Proof:* According to theorems 1 and 2, $p_e^l$ and $p_e^u$ are finite and their corresponding primal problems achieve their extreme values. Assume first that $p_e^l < p_e^u$. Let $\alpha_l$ solve (5), the primal problem for the lower limit. It is clear that $e - D\alpha_l \geq 0$ and that this inequality is not satisfied as equality, since $e$ is linearly independent of the marketed assets. Therefore there is $\varepsilon > 0$ and $\Delta > 0$ such that on a set of probability $\Delta$, there holds $e - D\alpha_l \geq \varepsilon$.

Define $\delta_1 = \frac{1}{2}[p_e^u - p_e^l]$. Let $V$ be the maximum expected utility that can be achieved using nonpositive levels of $e$ in the portfolio for $p_e = p_e^l + \delta_1 \equiv \frac{1}{2}[p_e^l + p_e^u]$. ($V$ exists by theorem 4.) There is a positive constant $c$ such that $\Delta U(c\varepsilon) > V$. Set $p_e = p_e^l + \delta$ where $0 < \delta < \min(W/c,\, \delta_1)$. At this price there is no arbitrage (since $p_e^l < p_e < p_e^u$), and the portfolio defined by the payoff $x = ce - cD\alpha_l$ has total price $cp_e - cp \cdot \alpha_l = c[p_e - p_e^l] + cp_e^l - cp \cdot \alpha_l = c[p_e - p_e^l] = c\delta < W$. The expected utility of this (feasible) portfolio is $\mathrm{E}[U(x)] = \mathrm{E}[U(ce - cD\alpha_l)] \geq \Delta U(c\varepsilon) > V$. Hence, at price $p_e$, there is a positive level of $e$ that produces expected utility above $V$.

Since reducing $p_e$ cannot increase the value obtained with nonpositive values of $e$, it follows that for $\delta$ the optimal solution will have $\alpha_e > 0$. Hence, $F(p_e) > 0$ for values of $p_e > p_e^l$ sufficiently close to $p_e^l$.

[2] A risk-free asset would qualify.

Likewise let $\alpha_u$ solve (7), the primal for the upper bound $p_e^u$. An argument similar to that above shows that $p_e = p_e^u - \delta$ for sufficiently small positive $\delta$ will lead to optimal solutions of (9) with a negative amount of $e$. Thus $F(p_e) < 0$ for values of $p_e < p_e^u$ sufficiently close to $p_e^u$.

Since $U$ is continuous and the budget constraint is linear, it follows (see Berge (1963, p116)) that $F$ is continuous on $(p_l, p_u)$. Hence from the intermediate-value theorem there is $p_e^0$ such that $F(p_e^0) = 0$. In other words there is a zero-level price.

We now show that $p_e^0$ is unique, and for this we assume the existence of the portfolio $\alpha_0$ with $D\alpha_0 > 1$, differentiability of $U$, and that the optimal solution (of the marketed system) has $x^* > 0$. The necessary conditions for the solution $x^*$ to (9) when $p_e = p_e^0$ are

$$\mathrm{E}[U'(x^*)d_i] = \lambda p_i \tag{10}$$

for each $i = 1, 2, \ldots, n$, and

$$\mathrm{E}[U'(x^*)e] = \lambda p_e^0. \tag{11}$$

The budget constraint will be met with equality because it is always advantageous to spend surplus wealth on the portfolio $\alpha_0$. Accordingly, owing to the concavity of $U$ and the inequality in the budget constraint, the Lagrange multiplier $\lambda$ is positive; see Luenberger (1969).

Suppose that there were two zero-level prices. Since the asset $e$ is held at zero level, a solution pair $x^*, \lambda$ for one of these values must be a solution pair for the other. From (11) it follows immediately that $p_e^0$ is unique.

Now consider the case $p_e^l = p_e^u$. From theorem 3 we know that $e$ is in the linear span of marketed assets and is consistent with the prices of those assets. This is a zero-level price, since any solution having $e$ at a non-zero level can be converted to a solution having $e$ at zero level by substituting the combination of marketed assets that duplicate $e$. $\blacksquare$

If there is a marketed risk-free asset with return $R$, we may use it in (10) to obtain

$$\mathrm{E}[U'(x^*)R] = \lambda.$$

Hence (11) may be written as

$$p_e^0 = \frac{1}{R}\frac{\mathrm{E}[U'(x^*)e]}{\mathrm{E}[U'(x^*)]}. \tag{12}$$

The condition that $x^* > 0$ will be automatically satisfied for a $U$ that severely penalizes final wealth values close to zero. For example, the utility functions $U(x) = \ln x$ and $U(x) = -x^{-\gamma}$ for $\gamma > 0$ have this property. It can be argued that utility functions of this form are likely to represent the preferences of sophisticated investors. See Luenberger (1993).

The results of theorems 4 and 5 can be easily extended to the case where $U$ is defined over both positive and negative values of final wealth and the constraint $x \geq 0$ is eliminated. In particular, the necessary conditions are then the same as above.

Combining these conditions, we say that a utility function is in class $\mathcal{U}$ if it is continuously differentiable and $x^* > 0$ is known to hold or the constraint $x* \geq 0$ is not present[3]).

## 3.1 Linearity and Arbitrage-Free Pricing

Under the uniqueness assumptions of theorem 5, we have from (11)

$$p_e^0 = \mathrm{E}[U'(x^*)e]/\lambda. \tag{13}$$

[3] Strictly speaking, therefore, membership in $\mathcal{U}$ depends on the constraint as well as $U$.

This can be written as

$$p_e^0 = \mathrm{E}[we], \tag{14}$$

where $w = U'(x^*)/\lambda$.

It is clear that if $x^*$ is optimal when the price of $e$ is $p_e^0$, then it is also optimal when a new asset $e'$ is adjoined with a price of $p_{e'}^0$. In other words, $p_{e'}^0 = \mathrm{E}[we']$ as well. Indeed, once $w$ is found, the formula $p_e^0 = \mathrm{E}[we]$ produces the zero-level price for any payoff $e$ in $X$. Furthermore, this is linear pricing, since for any two payoffs $e$ and $e'$ there holds $p_{\alpha e+\beta e'}^0 = \alpha p_e^0 + \beta p_{e'}^0$.

Since $U$ is strictly increasing, $U'(x^*) > 0$. Also $\lambda > 0$. It follows that $w > 0$, which means that $w$ is a positive pricing variable, and this means that the prices that $w$ produces lead to an overall system that is arbitrage free (since $e \geq 0$ implies $p_e^0 \geq 0$). We state these observations as a corollary.

**Corollary 1** *Under the uniqueness conditions of theorem 5, zero-level pricing is linear and arbitrage free.*

# 4 Universality

Zero-level pricing provides an arbitrage-free price for a new asset, but it has the apparent disadvantage of being dependent on the utility function and the wealth of the individual. This is not a disadvantage for a particular individual who wishes to determine the threshold value for purchase of the asset, but it lacks the definiteness associated with a robust pricing theory. Fortunately, there are several situations in which the zero-level price is independent of the utility function and wealth. The most important case is when utility functions are restricted to the class $\mathcal{U}$. Accordingly, we say the zero-level price is *universal* if it is the same for all utility functions in $\mathcal{U}$ and all positive wealth levels. The simplest case of universality (within $\mathcal{U}$ is when the marketed assets are complete. We discuss other important cases below.

## 4.1 Statistical Independence

Suppose $e$ is statistically independent of all marketed assets and there is a risk-free asset with return $R$. Then (12) reduces to

$$\begin{aligned} p_e^0 &= \frac{1}{R}\frac{\mathrm{E}[U'(x^*)e]}{\mathrm{E}[U'(x^*)]} \\ &= \frac{1}{R}\mathrm{E}(e), \end{aligned} \tag{15}$$

which is independent of $U$ and $W$, and hence is universal.

The coin flip example falls into this category. Its outcome is independent of all marketed assets and hence the zero-level price is its expected value, discounted by the risk-free rate. This is true regardless of the utility function and wealth. Thus $p_e^0$ is universal for the coin flip example.

This result can be used to price any asset subject to private uncertainty—uncertainty that is independent of any marketed asset—for example, projects that are subject only to isolated technical uncertainties. Any such asset can be priced according to the discounted expected value of its payoff, and this price is a universal zero-level price.

## 4.2 Normal Random Variables

Consider next the case where the payoffs of all assets are normally distributed. The random variable $e$ can then be written as $e = y + z$ where $y$ is a linear combination of the $d_i$'s (that is, $y = D\alpha$ for some $\alpha$) and $z$ is uncorrelated with (and hence independent of) the $d_i$'s. The zero-level price of $e$ is then the sum of the zero-level prices of $y$ and $z$. The zero-level price of $y$ is $p \cdot \alpha$ (for the $\alpha$ for which $y = D\alpha$) and the zero-level price of $z$ is $\mathrm{E}(z)/R$. The total price is independent of the utility function and hence is universal.

## 4.3 Partially Complete Markets

Roughly speaking, a single-period market is partially complete if it is complete with respect to all (measurable) functions of market payoffs, although there are other assets whose payoffs are not functions of market payoffs. See Smith and Nau (1995) and Holtan (1997). We present the concept here for the single-period case with $n$ linearly independent marketed assets with payoffs $d_i$, $i = 1, 2, \ldots, n$.

Let $\Omega$ (as defined earlier) be the underlying set of possible outcomes. Let $\mathcal{F}$ be a partition of $\Omega$ into a finite number $k$ of measurable sets such that: (1) the payoff of each marketed asset is constant on every set in $\mathcal{F}$, (that is, the payoffs are *adapted* to $\mathcal{F}$), and (2) $\mathcal{F}$ is the coarsest partition with property (1), (that is, $k$ is minimal). We say $\mathcal{F}$ is the partition *generated* by the market. The market is said to be *partially complete* (with respect to $\mathcal{F}$) if $n = k$ and there are other asset payoffs in $X$ that are not adapted to $\mathcal{F}$.

Consider a finite-state model, where $\Omega$ consists of a finite number $S$ of elements. If the $n$ marketed assets are linearly independent, a partially complete market has $n = k < S$. If $n = S$ the market is complete.

**Example 1** To illustrate, suppose there are 4 states $s_1, s_2, s_3, s_4$ and two marketed assets: a stock with payoff represented by the 4-tuple $(u, u, d, d)$ (with $u =$ "up" and $d =$"down"), and a bond with payoff $(1, 1, 1, 1)$. The generated partition $\mathcal{F}$ is $\{s_1, s_2\}, \{s_3, s_4\}$. Hence $k = 2 = n < S = 4$ so the market is partially complete if there are additional assets not adapted to this partition.

We have the following theorem (which is a variation of a result in Holtan (1997)).

**Theorem 6** *Suppose the market is partially complete. Then the zero-level price of a payoff $e$ is universal.*

*Proof:* Let $\mathcal{F}$ be the partition generated by the marketed assets, and suppose that $p_e^0$ is the zero-level price determined by the uniqueness criteria of theorem 5. We have

$$p_e^0 = \mathrm{E}[U'(x^*)e/\lambda].$$

Since $U'(x^*)$ is adapted to $\mathcal{F}$, we may use the tower property of conditional expectation to write

$$\begin{aligned} p_e^0 &= \mathrm{E}\big\{\mathrm{E}[(U'(x^*)/\lambda)e|\mathcal{F}]\big\} \\ &= \mathrm{E}\{(U'(x^*)/\lambda)\mathrm{E}[e|\mathcal{F}]\}. \end{aligned}$$

Now $\mathrm{E}[e|\mathcal{F}]$ is adapted to $\mathcal{F}$ and hence by partial completeness, it can be represented as a linear combination of the marketed payoffs, and thus it is priced uniquely according to that linear combination. Its price does not depend on $U'(x^*)/\lambda$. Hence $p_e^0$ is independent of $U$ and the wealth $W$. ∎

As a special case, suppose the risk-free asset is the only marketed asset. We may then take $\mathcal{F} = \Omega$ and the price of any payoff $e$ is $p_e^0 = \mathrm{E}[e]/R$, which is universal. In general, if the market is partially complete, only the expected value of $e$ on each of the elements of $\mathcal{F}$ is required to determine the universal zero-level price.

## 4.4 Quadratic Utility

We can show that the zero-level price of $e$ is the same for all individuals with quadratic utility, independent of their level of wealth and the specific parameters of the quadratic.

Quadratic utility is problematic, of course, because it is not increasing everywhere. However, we can treat it formally and in practice assume that it is employed only over the range where it is increasing.

Concave quadratic utility functions have the form $U(x) = a + bx - cx^2$ where $c > 0$. It is clear that a constant can be added to $U$ and $U$ can be multiplied by a positive constant without changing the relative assessment of payoffs. Hence, it is sufficient to consider the family $U(x) = bx - \frac{1}{2}x^2$, or equivalently, $U'(x) = b - x$. Let us first formally take the case $b = 0$; (we say *formally* because with $b = 0$ the quadratic decreases for positive $x$). We assume that a risk-free asset with return $R$ is marketed.

If $\alpha_0$ is the optimal portfolio of marketed assets for a level of wealth $W = 1$, $\alpha_0$ satisfies the necessary conditions

$$\mathrm{E}[-D\alpha_0 \times d_i] = \lambda_0 p_i \tag{16}$$

for all $i = 1, 2, \ldots n$ and some $\lambda_0$. It must also satisfy $p \cdot \alpha_0 = 1$.

For $b \neq 0$ and $W \neq 1$ consider the portfolio defined by amounts $\alpha_b = \gamma\alpha_0$ and an additional amount $b$ of the risk-free asset with payoff 1, where $\gamma$ will be selected later. The payoff is $x_b = D\alpha_b + b$. Let $\lambda_b = \gamma\lambda_0$. These choices satisfy the necessary conditions

$$\mathrm{E}[U'(x_b)d_i] = \mathrm{E}[(b - x_b)d_i] = \mathrm{E}[(b - D\alpha_b - b)d_i] = \gamma\mathrm{E}[-D\alpha_0 \times d_i] = \gamma\lambda_0 p_i = \lambda_b p_i$$

for each $i$. We select $\gamma$ to satisfy the budget constraint

$$p \cdot \gamma\alpha_0 + b/R = W,$$

giving $\gamma = W - b/R$. Then $x_b$ is optimal for the values $b$ and $W$.

We show that all these solutions assign the same price to $e$. We find

$$
\begin{aligned}
p_e^0 &= \mathrm{E}[(b - D\alpha_b - b) \cdot e]/\lambda_b \\
&= \mathrm{E}[-\gamma D\alpha_0 \cdot e]/(\gamma\lambda_0) \\
&= \mathrm{E}[-D\alpha_0 \cdot e]/\lambda_0
\end{aligned}
$$

which is independent of $b$ and $W$. Hence $p_e^0$ is universal with respect to all quadratic utility functions.

# 5 Conclusions

The range of arbitrage-free prices for a new asset consists of an interval whose endpoints can be expressed as values of certain linear programming problems. This range may be so broad that it is of little use for determining a price. However, the structure of these linear programs and their duals provide valuable analytic information. The range of acceptable prices is reduced considerably by requiring that a price be a zero-level price for some utility function. And it is the analytic structure of the arbitrage-free interval that provides the means for establishing the existence of zero-level prices.

If the range of acceptable prices under the zero-level requirement shrinks to a single point, that price is termed a universal zero-level price. Universal zero-level prices exist in many situations, and such prices have great appeal as the single logical price to assign to a new asset.

# A Appendix: Arbitrage Proofs

**Lemma A.1** *Assume there is no arbitrage among the marketed assets. There are $p_e^l$ and $p_e^u$ defining lower and upper bounds of the price $p_e$ that can be assigned to $e$ such that no arbitrage is possible for $p_e \in (p_e^l, p_e^u)$. Arbitrage is possible for $p_e \notin [p_e^l, p_e^u]$. The bounds are given by*

$$
\begin{aligned}
p_e^l &= \sup\{p \cdot \alpha : D\alpha - e \leq 0\} && \text{(A.1)} \\
p_e^u &= \inf\{p \cdot \alpha : D\alpha - e \geq 0\}. && \text{(A.2)}
\end{aligned}
$$

*Proof:* Following Holtan (1997), arbitrage is possible only in a portfolio with $e$ included at some nonzero level. First suppose that $e$ is included at a negative level, which without loss of generality can be taken to be the level $-1$. An arbitrage is then defined by an $\alpha$ satisfying

$$
\begin{aligned}
D\alpha - e &\geq 0 && \text{(A.3)} \\
p \cdot \alpha - p_e &\leq 0, && \text{(A.4)}
\end{aligned}
$$

with one of these inequalities being nontrivial as described earlier. If a value of $p_e$ satisfies this condition, then clearly any larger value satisfies it as well. Hence,

$$p_e^u = \inf_\alpha \{p \cdot \alpha : D\alpha - e \geq 0\}.$$

If $D\alpha - e \geq 0$ is not feasible, we set the inf to $+\infty$.

To establish the formula for $p_e^l$ we consider $e$ at the $+1$ level and use $-\alpha$ to represent positions in the marketed assets. Hence an arbitrage is an $\alpha$ satisfying

$$
\begin{aligned}
D\alpha - e &\leq 0 && \text{(A.5)} \\
-p \cdot \alpha + p_e &\leq 0, && \text{(A.6)}
\end{aligned}
$$

with one of these inequalities being nontrivial. If a value of $p_e$ satisfies this condition, then clearly any smaller value also satisfies it. Hence,

$$p_e^l = \sup_\alpha \{p \cdot \alpha : D\alpha - e \leq 0\}.$$

If $D\alpha - e \leq 0$ is not feasible, we set the sup to $-\infty$. $\blacksquare$

**Theorem A.1** *Assume there is no arbitrage among the marketed assets and that $e \geq 0$, $e \neq 0$. Then $p_e^l$ is finite and has the two dual characterizations:*

$$\begin{aligned} p_e^l = \quad & \max_\alpha \quad p \cdot \alpha \\ & \text{sub to} \quad D\alpha - e \leq 0. \end{aligned} \tag{A.7}$$

*and*

$$\begin{aligned} p_e^l = \quad & \min_w \quad w \cdot e \\ & \text{sub to} \quad wD = p \\ & \qquad\qquad w \geq 0. \end{aligned} \tag{A.8}$$

**Theorem A.2** *Assume there is no arbitrage among the marketed assets and that there is an $\alpha$ such that $D\alpha > 1$ and $e$ is bounded. Then $p_e^u$ is finite and has the two dual characterizations:*

$$\begin{aligned} p_e^u = \quad & \min \quad p \cdot \alpha \\ & \text{sub to} \quad D\alpha - e \geq 0. \end{aligned} \tag{A.9}$$

*and*

$$\begin{aligned} p_e^u = \quad & \max \quad w \cdot e \\ & \text{sub to} \quad wD = p \\ & \qquad\qquad w \geq 0. \end{aligned} \tag{A.10}$$

To prove theorems A.1 and A.2 we proceed in steps defined by a series of lemmata that assure that the linear programs of theorems are well-defined dual pairs and achieve their extreme values.

**Lemma A.2** *The dual programs (A.8) and (A.10) are feasible.*

*Proof:* Suppose not. Then $p \notin S \equiv \{wD : w \geq 0\}$. $S$ is a convex cone in $R^n$. $S$ is closed since $D$ is continuous, and hence its adjoint is continuous as well. Thus there is a hyperplane separating $p$ and $S$. See Luenberger (1969). That is, there is a non-zero $\alpha$ such that $p \cdot \alpha < wD \cdot \alpha$ for all $w \geq 0$. Setting $w = 0$ gives $p \cdot \alpha < 0$. Then $wD \cdot \alpha \equiv w \cdot D\alpha \geq 0$ for all $w \geq 0$; for if there were $w' \geq 0$ with $w' \cdot D\alpha < 0$, then $w'' = cw'$ would have $w'' \cdot D\alpha < p \cdot \alpha$ for large $c > 0$. Next notice that $w \cdot D\alpha \geq 0$ for all $w \geq 0$ implies $D\alpha \geq 0$ almost everywhere. Thus we have $p \cdot \alpha < 0$ and $D\alpha \geq 0$, showing that $\alpha$ is an arbitrage among the marketed assets, contrary to our assumption. $\blacksquare$

Next we state the standard inequality that holds between feasible solutions of a linear program and its dual.

**Lemma A.3** *Suppose $\alpha$ is feasible for (A.7) and $w$ is feasible for (A.8). Then $p \cdot \alpha \leq w \cdot e$.*

*Proof:* We have

$$p \cdot \alpha = wD \cdot \alpha = w \cdot D\alpha \leq w \cdot e.$$

$\blacksquare$

We now show that the dual program (A.8) achieves a minimum.

**Lemma A.4** *Assume that* $p_e^l$ *is finite. Then there is* $w \geq 0$ *with* $wD = p$ *and* $w \cdot e = p_e^l$.

*Proof:* Suppose not. Let

$$\hat{S} = \{(w \cdot e, wD) : w \geq 0\} \subset R^{n+1}.$$

$\hat{S}$ is a closed, convex cone. The half-line $\{(p_e, p) : p_e \leq p_e^l\}$ is not in $\hat{S}$. Hence there is a hyperplane separating $\hat{S}$ and the half-line. That is, there is a non-zero $(y, \alpha)$ with $yp_e + p \cdot \alpha > yw \cdot e + wD \cdot \alpha$ for all $w \geq 0$ and all $p_e \leq p_e^l$. Clearly $yp_e^l + p \cdot \alpha > 0$. Suppose $y = 0$. According to lemma A.2 there is $w \geq 0$ with $p = wD$ and this $w$ would violate the hyperplane inequality. Hence $y \neq 0$.

Setting $w = 0$ shows that $yp_e + p \cdot \alpha > 0$ for all $p_e \leq p_e^l$. Hence $y < 0$ and we may take $y = -1$. Then from the right hand side of the hyperplane inequality we have $-e + D\alpha \leq 0$. Thus $\alpha$ is feasible for (A.7), and as stated earlier $yp_e^l + p \cdot \alpha > 0$ which means $p \cdot \alpha > p_e^l$ which is impossible. ∎

Using the construction employed in the proof of lemma A.4, we can establish that the primal achieves its maximum.

**Lemma A.5** *Assume that* $p_e^l$ *is finite. Then there is* $\alpha$ *with* $p \cdot \alpha = p_e^l$, $D\alpha - e \leq 0$. *That is, the maximum is achieved in (A.7).*

*Proof:* We know that $(p_e^l, p)$ is a boundary point of $\hat{S} = \{(w \cdot e, wD) : w \geq 0\}$. By the construction in the proof of lemma A.4 there is a hyperplane supporting $\hat{S}$ of the form $(-1, \alpha)$. Thus $-p_e^l + p \cdot \alpha \geq -e \cdot w + wD \cdot \alpha$ for all $w \geq 0$. Hence $p \cdot \alpha \geq p_e^l$ and $D\alpha - e \leq 0$. So $p \cdot \alpha = p_e^l$.
∎

Combining these lemmata we obtain the strong characterizations of the main arbitrage theorems.

*Proof of theorem A.1:* That (A.7) is feasible is seen by noting that $\alpha = 0$ is feasible. That (A.8) is feasible follows from lemma A.2. Both (A.7) and (A.8) are then finite by lemma A.3. Lemma A.4 shows that (A.8) achieves a minimum of $p_e^l$. Lemma A.5 shows that (A.7) achieves a maximum of $p_e^l$. ∎

*Proof of theorem A.2* Feasibility of the primal (A.9) is obtained by scaling $\alpha$ of the assumption so that $D\alpha \geq e$. The dual is feasible by lemma A.2. The remainder of the proof follows from the analogs of the lemmata derived earlier. ∎

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