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Danielle Maddix

ICME Refresher Course 2015 9/14-9/15

Numerical Linear Algebra Sections:
(1) Monday 9/14: 1:00-2:15 pm
(2) Tuesday 9/15: 9:00-10:15am
(3) Tuesday 9/15: 10:30-11:45 am
Final 2 sections Lar:
(4) Wednesday 9/16: 10:30-11:45
(5) Thursday 9/17: 9:00-10:15 am

Outline

Monday: (1) Review of basic concepts from linear algebra (dim, rank, null space, linear independence, vector space subspace, rank/nullity theorem, invertible matrix theorem)

Monday: (2) Determinants/ trace

Tuesday: (3) Gaussian elimination - direct solves and factorizations (LU decomp, LDL^T, Cholesky) forward/ back substitution for triangular systems $O(n^2)$, positive definite definition

Tuesday: (4) Conditioning + perturbation for $Ax=b$ - ill-cond systems, matrix/vector norms, condition norm, consequence of using a small pivot

Tuesday: (5) orthogonalization - orthogonal basis / matrix, inner product spaces, cauchy-schwarz, gram-schmidt, QR

Lan's topics

(6) eigenvalues and eigenvectors
(7) SVD = $A = U \Sigma V^T$ (U, V orthogonal)
(8) iterative methods
(9) least squares

Introduction: Danielle Maddix (demaddix@stanford.edu - 3rd yr ICME)

- This part of course is intended to prepare you for Fall icme core course CME 302

- survey of which department/degree program everyone is from

9/14

① Review of basic concepts from lin alg:

What is a linear transformation/Operator?

$L[\alpha \vec{v_1} + \beta \vec{v_2}] = \alpha L[\vec{v_1}] + \beta L[\vec{v_2}]$

① Linear combination of vectors $\{\vec{v_j}\}_{j=1}^{n}$

$$\boxed{\alpha_1 \vec{v_1} + \cdots + \alpha_n \vec{v_n} = \sum_{j=1}^{n} \alpha_j \vec{v_j} \; , \; \alpha_j \in \mathbb{R}}$$

(weighted sum)

How can we express this in terms of matrices and vectors?

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$\boxed{n}$ = number of columns/ number of variables

$\boxed{m}$ = number of rows/eqtns

- what is $A\vec{x}$ a linear combination of?

$$A\vec{x} = \underbrace{[\vec{a_1} \cdots \vec{a_n}]}_{n \text{ column vectors}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$= \boxed{x_1 \vec{a_1} + \cdots + x_n \vec{a_n}}$ ⇒ linear combo of columns of A, with coefficients x_j (vector elements)

- A solution to $Ax = b$ exists ⇔ b is a linear combo of columns of A

Ex: $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $b_1 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$[A \mid b_1]$ <u>augmented</u> system

$$b_3 = 2\vec{a_1} + 2\vec{a_2}$$

only 1 pivot

$$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 0 & 0 \end{array}\right] \quad x_1 = 6 - 2x_2 \quad x_2 = x_2 \text{ (free variable)}$$

$$\left\{ \begin{bmatrix} 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

$\Rightarrow$ infinitely many solutions

Example: $x_2 = 1 \quad x_1 = 4 \quad \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Now, take $b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (not a linear combo of columns of A)

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right] \Rightarrow 0 \neq 1$$

no solution exists / inconsistent system

② linear independence of a set of vectors

Definition: A set of vectors $\{\vec{v_j}\}_{j=1}^{n}$ is linearly independent if none of the vectors can be expressed as linear combos of the others, i.e

$$\boxed{\sum_{j=1}^{n} \beta_j \vec{v_j} = 0 \Leftrightarrow \beta_j = 0 \quad \forall j}$$

On the contrary for linear dependence:
1 or more of the vectors can be expressed as linear combo of the others

$\exists \{\beta_j\} \neq 0$ s.t. $\sum_{j=1}^{n} \beta_j \vec{v_j} = \vec{0}$ β_j not all 0.

$$\exists k \quad v_k = \sum_{\substack{j=1 \\ j \neq k}}^{n} \beta_j \vec{v_j}$$

Then $\sum_{j=1}^{n} \beta_j \vec{v_j} = \sum_{\substack{j=1 \\ j \neq k}}^{n} \beta_j \vec{v_j} + \beta_k v_k$

$= \sum_{\substack{j \geq 1 \\ j \neq k}}^{n} \beta_j \vec{v_j} + \beta_k \sum_{\substack{j \geq 1 \\ j \neq k}}^{n} \beta_j \vec{v_j}$ $\boxed{\text{Take } \beta_k = -1}$

Question | Determine if a set of vectors is linearly independent?

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Dependent | $\boxed{\vec{v_3} = \vec{v_2} - \vec{v_1}}$

Larger case: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

How to solve: write as matrix

By definition: $\beta_1 \vec{v_1} + \beta_2 \vec{v_2} + \beta_3 \vec{v_3} = \vec{0}$

check if all $\beta_i \equiv 0$

matrix system: $\boxed{A\vec{\beta} = \vec{0}}$ (want only trivial solution)

For $\vec{\beta} = 0$, we need A to be invertible (non-singular)

check if matrix has 3 pivots (square matrix invertible if pivot in every row)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ linearly independent

③ Vector Space | Set V that is closed under addition and scalar multiplication that satisfies the following axioms

1) Commutativity: $\vec{u} + \vec{v} = v + u$
2) Associativity: $(u + v) + w = u + (v + w)$

3) Additive identity: $\exists \vec{0} \in V$ s.t. $v + \vec{0} = v \quad \forall v \in V$

4) Additive inverse: $\forall v \in V, \exists w \in V$ s.t. $v + w = 0$

5) Multiplicative identity: $1v = v$

6) Distributivity $\alpha(u+v) = \alpha u + \alpha v$

$(\alpha + \beta)v = \alpha v + \beta v$

Example: $\mathbb{R}^n$, $\mathbb{P}_n$, $\mathbb{C}^n$, $C[0,1]$

$\mathbb{R}^n$: Euclidean space
$\mathbb{P}_n$: space of polynomials deg $\leq n$
$\mathbb{C}^n$: complex
$C[0,1]$: space of continuous functions on $[0,1]$

(4) Subspace) A subspace U of a vector space V ($U \subseteq V$) is a subset of elements such that

inherits other properties from vector space

(1) $\vec{0} \in U$

(2) U is closed under addition

$(\forall u, v \in U, \; u + v \in U)$

(3) U is closed under scalar multiplication

$\forall v \in U, \alpha v \in U$

* a subspace is a vector space

Examples) (a) any line in $\mathbb{R}^2$ of the form $x_2 = \alpha x_1$ is a subspace

- subset of $\mathbb{R}^2$

1) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in U \quad 0 = \alpha 0$ ✓ line through the origin

2) closed under addition (slope α)
Let $\vec{x}, \vec{y} \in U$ Then $x_2 = \alpha x_1$
$y_2 = \alpha y_1$
$x_2 + y_2 = \alpha x_1 + \alpha y_1 = \alpha(x_1 + y_1)$ ✓
$\Rightarrow \vec{x} + \vec{y} \in U$

3) closed under scalar multiplication
$\vec{x} \in U$. want to show $c\vec{x} \in U$
Since $\vec{x} \in U$ $x_2 = \alpha x_1$
So, $\boxed{c\vec{x}_2 = \alpha(c\vec{x}_1)}$ $\Rightarrow c\vec{x} \in U$ ✓

(b) A line in $\mathbb{R}^2$ of the form $x_2 = x_1 + b$, $b \neq 0$ is not a vector space, since $x_1 = x_2 = 0$ is not on the line (doesn't pass through origin)

(c) Prove that the nullspace is a subspace of $\mathbb{R}^n$.

See formal def later

$\boxed{\{x \in \mathbb{R}^n \mid A\vec{x} = 0\}}$

1) Clearly $A\vec{0} = \vec{0}$ ✓ $\vec{0} \in N(A)$
2) Suppose $\vec{x}, \vec{y} \in N(A)$, $A \in \mathbb{R}^{m \times n}$
Then $A(\vec{x} + \vec{y}) = \underbrace{A\vec{x}}_{=0} + \underbrace{A\vec{y}}_{=0} = \vec{0}$ ✓ $\vec{x} + \vec{y} \in N(A)$
3) $A(\alpha\vec{x}) = \alpha \underbrace{A\vec{x}}_{=0} = 0$ since $\vec{x} \in N(A)$
$\Rightarrow \alpha\vec{x} \in N(A)$
(2) + (3) by linearity of A

(d) Prove the <u>range</u> is a subspace of $\mathbb{R}^m$

$\{b \in \mathbb{R}^m \mid Ax = \vec{b}\}$ $\exists v \Rightarrow 0$

1) $0 \in \text{Range}(A)$ since $A0 = 0$

2) Let w_1, w_2 be in the range of A.
By definition, $\exists v_1, v_2$ such that
$Av_1 = w_1, \quad Av_2 = w_2$

$\Rightarrow$ ~~A(v1+v2)=w1+w2~~
$Av_1 + Av_2 = w_1 + w_2$

$\Leftrightarrow A(v_1 + v_2) = w_1 + w_2 \Rightarrow w_1 + w_2 \in \text{Range}(A)$

3) $A(cv_1) = cAv_1 = \boxed{cw_1}$ ✓

* Set of vectors $\vec{x}$ s.t. $Ax = b$, $b \neq 0$ is not a vector space, since $\vec{0}$

(e) space of continuous, real-valued functions on $[0,1]$, $C([0,1])$

$U = \{f \in C([0,1]) \mid f(0) = f(1) = 0\}$

1) $0 \in U$

2) $f, g \in U$ Then $(f+g)(0)$
$= f(0) + g(0) = 0 + 0$ ✓
same for 1

3) $(cf)(0) = cf(0) = 0$ ✓

4 Spanning Set | A set of vectors $\{\vec{v}_i\}$

spans a vector space if every vector in the space can be expressed as a linear combination of v_i's

- It is a subspace of all linear combinations of these vectors:

$$\boxed{\text{span}(\{v_1, \dots, v_n\}) = \{w \mid w = \sum_{i=1}^{n} \alpha_i v_i\}}$$

Example: $\text{span}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}\right\}\right)$

$$= \left\{\begin{pmatrix}\alpha_1\\\alpha_2\\0\end{pmatrix} \middle| \alpha_i \in \mathbb{R}\right\} = \boxed{\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}}$$

⑤ Basis (Key concept)

- spans the space and is linearly independent
- maximal linearly independent set
- minimal spanning set
- Given any vector $v \in V$, we can write v in the basis $\{u_i\}_{i=1}^{n}$

$$\boxed{v = \sum_{i=1}^{n} \alpha_i u_i}$$ for some set of coefficients (α_i)

⑥ Dimension - number of vectors in a basis

Examples: $\{\vec{e_1}, \dots, \vec{e_n}\}$ is the standard basis for $\mathbb{R}^n$ $\left\{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ $\boxed{\dim(\mathbb{R}^n) = n}$

$\mathbb{R}^2$ $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ $\dim(\mathbb{R}^2) = 2$

$\left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix} \right\}$ <u>redundant</u>

Find linearly independent subset

$$\begin{bmatrix}1 & 2 & -1\\1 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 2 & -1\\0 & 1 & -2\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & 0 & 3\\0 & 1 & -2\end{bmatrix}$$

spans ⇒ pivot every row

linearly independent take columns

$A\alpha = 0$

$\left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix} \right\}$ basis - we can see not unique

- But # of vectors in a basis for the same space is always the same

Since $\dim(\mathbb{R}^2) = 2$, need <u>2</u> LI vectors in basis

Find coefficients to show dependency:

$$\boxed{\begin{matrix}\alpha_2 = 2\alpha_3 & \alpha_3 = \alpha_3\\ \alpha_1 = -3\alpha_3 & \end{matrix}}$$

Take $\alpha_3 = 1$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\Rightarrow \boxed{-3\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 = 0}$$

$$\Rightarrow \boxed{\vec{v}_3 = 3\vec{v}_1 - 2\vec{v}_2}$$ ✓ linearly dependent

<u>Example:</u> $x_2 = \alpha x_1$ is a 1D subspace of $\mathbb{R}^2$ $\vec{x} = \begin{bmatrix}x_1\\ \alpha x_1\end{bmatrix} = x_1 \begin{bmatrix}1\\ \alpha\end{bmatrix}$ $\boxed{\left\{ \begin{bmatrix}1\\ \alpha\end{bmatrix} \right\}}$

Question: Find a basis for a subspace of $\mathbb{R}^4$ spanned by vectors

$$\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- we know it spans the space, so to be a basis we must take the maximal linear independent subset

$$\begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ basis

can check that $\boxed{\vec{v_4} = \frac{1}{2}\vec{v_1} + \frac{1}{2}\vec{v_2}}$

- $\dim(V) = \#\{$vectors in a basis$\}$
- Not all vector spaces are finite-dimensional, such as $C[0,1]$

Ex: basis for $\mathbb{P}_n(x)$ $\{1, x, x^2, \dots, x^n\}$

$\dim(\mathbb{P}_n(x)) = \boxed{n+1}$ due to scalar term

Ex: Find a basis for a subspace of $\mathbb{R}^3$: $\boxed{x_1 + x_2 + x_3 = 0}$: U

1) Show subspace: $0 \in U$

$\cancel{\alpha(x_1 + x_2)}$ $\cancel{x_1 + x_2} \to (x_1 + y_1) + (x_2 + y_2)$
$+ (x_3 + y_3) = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = 0$

2) $\alpha x_1 + \alpha x_2 + \alpha x_3 = \alpha(x_1 + x_2 + x_3) = 0$ ✓

To find basis use constraint:

$\boxed{x_1 = -x_2 - x_3}$

$$\vec{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

2D subspace of $\mathbb{R}^3$ is a plane

* Important matrix subspaces

⑧ Column space - range of A

- R(A) is space spanned by the columns
- Columns of m×n all live in $\mathbb{R}^m$

$$\vec{a_n} = \begin{bmatrix} a_{1n} \\ | \\ a_{mn} \end{bmatrix} \in \mathbb{R}^m$$

- If there are <u>m</u> linearly independent columns, then columns = $\mathbb{R}^m$

$$R(A) = \{ Ax \mid x \in \mathbb{R}^n \}$$
$$= \{ x_i \vec{a_i} \mid x_i \in \mathbb{R}, 1 \le i \le n \}$$
$$= \text{span}(\{a_i\}_{i=1}^{n})$$

⑨ Row space - space spanned by rows of A.

→ $\boxed{R(A^T)}$ - rows of A are columns of A^T.

$A^T_{ij} = A_{ji}$ $\quad A^H_{ij} = A^*_{ij} = \overline{A_{ji}}$ (complex)

$\boxed{\text{subspace of } \mathbb{R}^n}$

Simple to show that span is a subspace

1) Set $\alpha_i = 0$ $\quad 0 = \sum_{i=1}^{n} \alpha_i v_i$ ✓

2) Let $w, u \in \text{span}(\{v_1, \dots, v_n\})$

Then $w + u = \sum_{i=1}^{n} \alpha_i v_i + \sum_{i=1}^{n} \beta_i v_i$

$= \sum_{i=1}^{n} (\alpha_i + \beta_i) v_i$ ✓

3) $\alpha w = \alpha \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} (\alpha\alpha_i) v_i$ ✓

In Gaussian elim $A \to U$ (upper triangular) ~~consequence of linear comb~~

- A and U have the same <u>row space</u> but not column space.

* all rows of U are linear combos of rows of A

(10) Rank of a matrix A is dim of column space / $\boxed{\dim(R(A))}$

$\boxed{\text{rank}(A) = \text{rank}(A^T)}$ $\Rightarrow$ column space & row space have same dimension

(11) Null space $N(A) = \{x \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$

- subspace in $\mathbb{R}^n$.

<u>Example</u> Find bases for row space, column space and null space.

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & -2 & 1 & 1 \\ 1 & 2 & -3 & -7 \end{bmatrix}$$

(all involve row reduction)

- column space take the LI columns from <u>A</u>

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & -2 & 1 & 1 \\ 1 & 2 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col A: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$

$\boxed{\text{rank} A = 2}$

Row space: can take linearly independent rows from U or A, since span same space

Basis for row A: $\{ [1\ 2\ 0\ 2]^T, [0\ 0\ 1\ 3]^T \}$
or
$\{ [1\ 2\ 0\ 2]^T, [-1\ -2\ 1\ 1]^T \}$

Null space: $x_2 = x_2$ (free) $\quad x_3 = -3x_4$
$x_4 = x_4 \quad x_1 = -2x_2 - 2x_4$

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

• dim = # free variably

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

$\dim(\text{Nul}(A)) = 2$

*Observe $\boxed{\text{rk} A + \dim \text{Nul}(A) = 4 = n}$

This can be formalized in the rank-nullity theorem

* $\boxed{\text{rank}(A) + \dim(N(A)) = n \text{ (\# columns)}}$

Proof: Let $\{u_i\}_{i=1}^{k}$ be a basis for $N(A)$. We can extend this to a basis for $\mathbb{R}^n$ (dim n) w/ $n-k$ linearly independent vectors, $\boxed{\{u_i\}_{i=1}^{k} \cup \{w_j\}_{j=1}^{n-k}}$

We must show $R(A)$ is $n-k$

For $x \in \mathbb{R}^n$ $x = \sum_{i=1}^{k} \alpha_i u_i + \sum_{j=k+1}^{n} \beta_j w_j$

by definition of basis.

Then, 0 nullspace

$$\boxed{Ax = \sum_{i=1}^{k} \cancel{\alpha_i} A u_i + \sum_{j=k+1}^{n} \beta_j A w_j}$$

$\Rightarrow \{A w_j\}_{j=1}^{n-k}$ is a basis for $R(A)$

$\Rightarrow$ 4 fundamental subspaces

1) $R(A) \subseteq \mathbb{R}^m$ 3) $R(A^T) \subseteq \mathbb{R}^n$
2) $N(A) \subseteq \mathbb{R}^n$ 4) $N(A^T) \subseteq \mathbb{R}^m$

- We will formalize orthogonality later, but recall $\vec{u} \perp \vec{v} \Leftrightarrow \boxed{u^T v = 0}$ (0 dot product)

- Fundamental Theorem of linear algebra

Let $A \in \mathbb{R}^{m \times n}$

1) $R(A) \perp N(A^T)$ and $R(A) \oplus N(A^T) = \mathbb{R}^m$ (direct sum)
2) $R(A^T) \perp N(A)$ and $R(A^T) \oplus N(A) = \mathbb{R}^n$

For two subspaces to be orthogonal, $V \perp U \Leftrightarrow v^T u = 0$
$\forall v \in V, u \in U$

- Show $R(A) \perp N(A^T)$

Let $v \in N(A^T)$, $w \in R(A)$
$\Rightarrow \exists u \in \mathbb{R}^n$ s.t. $\boxed{Au = w}$

$v^T w = v^T A u = (Au)^T v$ (By symmetry)
$= u^T (A^T v) = \boxed{0}$
$\underset{\text{b/c } v \in N(A^T)}{}$

Recall: $(AB)^T = B^T A^T$ (reverse order)
$(AB)^{-1} = B^{-1} A^{-1}$

Finally, we will summarize these properties:

Invertible Matrix Theorem

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix
then the following statements are equivalent:

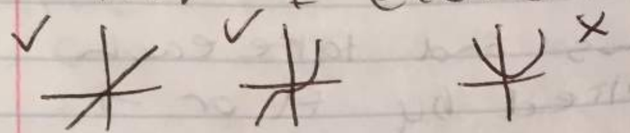
1) A is invertible (nonsingular) matrix
2) A is row-equivalent to $n \times n$ identity
3) A has n pivot positions

pivot every column
4) $Ax = 0$ has only trivial solution $x = 0$ ($N(A) = \emptyset$)
5) Columns of A form a linearly independent set
6) $x \mapsto Ax$ is 1-to-1

pivot every row
7) $Ax = b$ has at least 1 solution for each $b \in \mathbb{R}^n$
8) Columns of A span $\mathbb{R}^n$

9) $x \mapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
10) $\exists$ A^{-1} s. t. $AA^{-1} = I_n = A^{-1}A$
11) $\det(A) \neq 0$

- **1-to 1** – if $f(a) = f(b)$ then $a=b$ and no element of B is image of more than 1 element of A



- **onto** – $\forall b \in B$, $\exists\, a \in A$ s. t. $f(a)=b$.
  every element of B gets hit

Consequences] $\text{rank}(A) = \text{rank}(A^T) = n \Rightarrow$ A^T invertible
- $Ax=b \Leftrightarrow \boxed{x = A^{-1}b}$ (unique)

existness from onto + uniqueness from one-to-one

(2) Determinants / Traces (Scalar functions)

* A is invertible $\Leftrightarrow \det(A) \neq 0$

simple 2×2 case

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{ad-bc}$$

$$\boxed{\det A = ad-bc \neq 0}$$

3×3 case

ignore column + row

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \quad |A| = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Ex: cross product 2 vectors $\vec{a} \times \vec{b}$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- For general $n \times n$ matrices, we follow a similar procedure.
- loop over rows and take each element multiplied by +1 or -1 according to checkerboard pattern
- multiply each first row element by determinant of $(n-1) \times (n-1)$ matrix after deleting row & column
- continue onto $(n-2) \times (n-2) \ldots 2 \times 2$

$$\boxed{|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \, \underbrace{|M_{ij}|}_{\text{minor}}}$$

a_{ij} - element of A in row i, col j
M_{ij} - minor matrix from A by deleting row i and j

Properties

1) If elements of any row or column are multiplied by a constant k, then det(A) is multiplied by k

$\Rightarrow$ B multiplying A by nonzero k, then $\boxed{|B| = k|A|}$

2) If B is obtained by replacing 1 row of A by itself plus a multiple of another row, then $\boxed{|B| = |A|}$

(preserved under elementary row operations)

3) If B is obtained by interchanging 2 rows of A, $|B| = -|A|$

This implies an easier method to compute the determinant:
Use gaussian elimination to put A in triangular form,

$\boxed{|A| = (-1)^r |U|}$ r = # row interchanges, U = upper triangular matrix

$= (-1)^r u_{11} \cdots u_{nn}$

4) * det of triangular matrix is product of diagonal entries

$\Rightarrow \det A = \pm \prod_{i=1} \text{pivots} = \pm u_{11} \cdots u_{nn}$

5) $\boxed{\det(A+B) \neq \det(A) + \det(B)}$ (not a linear operator)

6) $\boxed{\det(AB) = \det(A)\det(B)}$

7) $\det(A) = \det(A^T)$

8) If any row or column $= 0$, then $\det(A) = 0$
$\Rightarrow$ A is singular

9) $\det(cA) = c^n \det(A)$

10) $\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$ Proof: $\det(A^{-1}A) = \det(I) = 1$
$= \det(A^{-1})\det(A)$

Determinants of special matrices

① orthogonal $\boxed{P^T P = I}$ - $\boxed{\det(P) = \pm 1}$

$\det(P^T P) = \det(I) = 1 = \det(P^T)\det(P)$
$= \det(P)^2$
$(\det(P))^2 = 1 \Leftrightarrow \boxed{\det(P) = \pm 1}$ (sqrt both sides)
$\Rightarrow$ gauranteed to be invertible $\det \neq 0$

② Projector matrices A matrix P
is a projector if $\boxed{P^2 = P}$

intuition: projects onto range. If
$x \in R(P)$ then acts as identity
operator.
Proof: $\exists v$ s.t. $Pv = x$
Then $Px = P^2 v = Pv = x$ ✓

- If P is also symmetric ($P = P^T$) $\mathbb{R}^n$
or Hermitian ($P = P^*$), then it is $\mathbb{C}^n$
an orthogonal projector (not in general orthogonal matrices)

$\det(P^2) = \det(P)\det(P) = \det(P)$
$\Leftrightarrow \boxed{\det(P)^2 - \det(P) = 0}$
$\Leftrightarrow \det(P)(\det(P) - 1) = 0$
$\Leftrightarrow \boxed{\det(P) = 0, 1}$

③ Permutation matrices - are formed
by exchanged rows of I. $\det(I) = 1$
$\Rightarrow \boxed{\det(P) = \pm 1}$ (just swapping rows)

- The trace is another function from $\mathbb{R}^{m \times n} \to \mathbb{R}$ (scalar)

- defined on square matrices.
- $\boxed{tr(A) = \sum_{i=1}^{n} A_{ii}}$ (sum of diagonal)

Properties ① $\boxed{tr(A+B) = tr(A) + tr(B)}$

$$tr(A+B) = \sum_{i=1}^{n} (A+B)_{ii} = \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii}$$

$= tr(A) + tr(B)$ (linearity) recall - doesn't hold for determinant

② $\boxed{tr(AB) = tr(BA)}$

- Recall matrix-matrix multiplication

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Now, $tr(AB) = \sum_{i=1}^{n} (AB)_{ii}$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ji} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} B_{ji} A_{ij} \right) = \sum_{j=1}^{n} (BA)_{jj}$$

swap order summation

$= \boxed{tr(BA)}$

③ invariant under cyclic permutation:

$tr(ABC) = tr(BCA) = tr(CAB)$

Proof $tr(A(BC)) = tr(B(CA)) = tr(CAB)$ ✓

- Lan will discuss eigenvalues later and we can define trace/determinants in terms of them

- $\det(A) = \prod_{i=1}^{n} \lambda_i$

$\boxed{Av = \lambda v}$

- $tr(A) = \sum_{i=1}^{n} \lambda_i$