Optimal Controller Synthesis for Decentralized Systems over Graphs via Spectral Factorization

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Abstract

In this paper, controller synthesis algorithms are developed for decentralized control problems. The distributed systems considered here are represented by graphs, which impose sparsity constraints on the set of allowable controllers. A spectral factorization approach is used to construct the optimal decentralized controllers. Explicit state-space solutions are provided for this class of systems, which establishes the order for the optimal policies. In addition, this work provides an intuitive understanding of the optimal solution. In particular, the standard separation principle does not hold for these decentralized problems, and the controllers must do more than simply estimate their states.

I Introduction

Decentralized control problems, consisting of multiple subsystems interacting over a network with limited communication, have seen rapidly growing interest in recent years. While centralized control algorithms are now well-understood, decentralized control has a much less complete theory, and many important practical problems require a decentralized approach. The simplest example is formation flight for teams of vehicles, where each vehicle has its own local controllers. While small teams may allow for complete communication between subsystems, so that centralized policies may be utilized, this approach does not scale well to larger teams. Moreover, large spatially distributed automated systems, such as the internet or power grid, can make complete communication infeasible. Additionally, wireless networks may be naturally decentralized if communication interference precludes the ability to efficiently transmit information between subsystems.

Unfortunately, it has been shown that finding optimal control policies for decentralized systems is, for certain specific formulations, a computationally intractable problem [1]. Moreover, for many decentralized problems, linear control policies may be strictly suboptimal compared to nonlinear policies, even when the underlying system dynamics are linear and time-invariant. This was beautifully illustrated in the well-known Witsenhausen counterexample [2].

A general formulation capturing many decentralized control problems is

$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12}\mathcal{K}(I - P_{22}\mathcal{K})^{-1}P_{21}\| \\ \text{subject to} \quad & \mathcal{K} \text{ stabilizing} \\ & \mathcal{K} \in \mathcal{S} \end{aligned} \tag{1}$$

where P_{ij} are transfer functions representing the dynamics of the overall system and $\mathcal{K}$ is a block matrix transfer function, each of whose blocks is a local controller. The set $\mathcal{S}$ imposes the decentralization constraints on the controller. This set might typically impose sparsity constraints on $\mathcal{K}$ in order to achieve a controller $\mathcal{K}$ which represents a decentralized set of individual controllers.

In general, the optimization problem (1) poses a number of difficulties. First, the question of stabilization needs to be answered. In the centralized problem, the stabilizability of a system is equivalent to the system dynamics being stabilizable and detectable. This can be easily checked and leads to a simple expression for the set of stabilizing controllers. However, for decentralized systems, there may exist *fixed modes* in the system which can make characterizing the set of stabilizing controllers much more difficult [3].

Additionally, the objective function involves a linear fractional transformation on $\mathcal{K}$, which is non-convex. This is typically overcome by a Youla parametrization $Q = h(\mathcal{K})$, which transforms the objective to an affine function in the Youla parameter Q. However, this simply pushes the non-convexity to the decentralization constraint, since we now require $Q \in h(\mathcal{S})$.

As a consequence of these issues, much of the research in decentralized control has been aimed at classifying those systems for which the above issues can be mitigated and the resulting optimization problem becomes tractable [4–6]. One of the most well-known classes of problems is called *partially nested* [7]. Basically, this class requires that system i communicates with system j if the dynamics of system i can influence the dynamics of system j. More recently, these results have been

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unified and generalized with the concept of *quadratic invariance* [8]. In short, a quadratically invariant system allows the decentralization constraint to be expressed as $Q \in h(\mathcal{S}) = \mathcal{S}$, so that the system is once again convex.

In this paper, we consider decentralized systems with a type of partially nested structure. We assume that there are no communication delays, unlimited bandwidth, and that each subsystem can measure its entire state. Thus, the set $\mathcal{S}$ represents sparsity constraints on the overall controller. It has been shown that these systems admit convex representations [9]. Similar results have been obtained using a poset-based framework in [10], and in [11], in which a construction for the optimal individual controllers are included.

While convexity is certainly desired for these systems, the optimization problem is still infinite-dimensional, since we are searching over the set of transfer functions. Consequently, a standard approximation would be via a finite set of basis functions for the impulse response of the Youla parameter [12]. This is in contrast to the centralized case for which explicit state-space solutions can be analytically constructed. Consequently, our focus in this paper is the explicit state-space controller synthesis for these decentralized problems. Such formulae offer the practical advantages of computational reliability and simplicity, as well as provide understanding and interpretation of the controller structure. Also, it establishes the order of the optimal controller for this system, which is an open problem for general decentralized systems, even in the simplest cases.

I-A Prior Work

This work follows a series of papers on decentralized controller synthesis. As two-player systems are the simplest decentralized problems, most results focus on these systems. As a result, the two-player, finite-horizon, state-feedback $\mathcal{H}_2$ case was solved in [13]. The infinite-horizon version was then provided in [14]. The generalization to a partial output feedback structure has now been presented in [15]. The results in all of the above demonstrated a separation of the optimal controller into a controller and an estimator. This work represents the natural extension of the two-player, state-feedback case to general networks which are tractable, as discussed above. This body of work can also be found in [16].

For controller synthesis, there are basically three different approaches. The approach used herein is based on spectral factorization; namely, the optimization problem is found by solving an equivalent optimality condition. Solving this optimality condition, as will be shown, involves decoupling an equation based on the spectrum of its operators. The basic approach has been suggested, but not implemented in [17]. For the $\mathcal{H}_2$ problem considered here, an analytic solution could also be obtained by first vectorizing the objective function. However, this approach greatly increases the state dimension of the system, and does not scale well with an increase of subsystems.

Another approach to controller synthesis is via dynamic programming. Again, for the two-player problem of our previous work, a dynamic programming approach was provided in [18]. This approach utilized an augmented state-space in order to achieve the recursive structure needed for this approach. Another approach was taken for the continuous-time two-player problem in [19], which achieved analogous results to [14]. For more general decentralized systems, a suboptimal approach has been proposed in [20]. In short, this method simply neglected terms which prevented the recursive algorithm from working.

Finally, the method of semi-definite programming (SDP) has seen some results in this area. For the $\mathcal{H}_2$ problem considered here, an SDP approach has been presented in [21]. In cases where the dynamics of the subsystems are decoupled, the results in [22] provide a solution for the output feedback problem. For the $\mathcal{H}_\infty$ version of these types of problems, [23] provides necessary and sufficient conditions to find the optimal controller. In addition, [24] provides some sufficient conditions for general networks. However, though these SDP approaches are finite-dimensional, none of them provide analytic solutions, and thus cannot provide information about the order of the optimal controllers or any intuition about the estimation structure.

Most recently, the result in [25] provides a solution similar to those achieved here. The approach taken therein is based on the decoupling scheme seen in [13, 14], and a similar approach is taken in this paper. While the results are basically the same seen here, the work in this paper focuses on the spectral factorization results required to achieve the solution. Additionally, the estimation structure is detailed in this work, which provides significant intuition to the optimal policies.

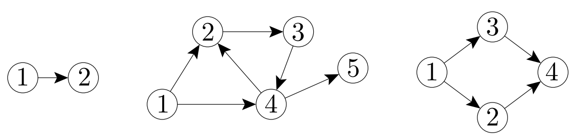
I-B Outline

This paper is organized as follows. Section II establishes some preliminary notation that will be used throughout the paper. In Section III, we provide some basic analysis results which detail the class of problems that are solved herein. The optimal controller solution for these problems is provided in Section IV. Following the statement of the main results, the following sections provide the derivations for these results. This development takes the following steps. Section V derives the optimality conditions for our decentralized control problems. In Section VI, these conditions are solved via a spectral factorization approach. Having established the optimal controller for our problem, some further intuition to this result is provided in Section VII. Namely, we show that the optimal control laws have a specific estimation structure, which does not follow the standard separation principle. Finally, some examples which utilize these results are provided in Section VIII.

II Preliminaries

II-A Graph Notation.

We represent a directed graph $\mathcal{G}$ by the set of N vertices $\mathcal{V} = \{v_1, \dots, v_N\}$ and the set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. There is a directed edge from vertex v_i to vertex v_j if $(v_i, v_j) \in \mathcal{E}$. We assume that $\mathcal{G}$ has no self-loops; that is, $(v_i, v_i) \notin \mathcal{E}$ for all vertices $v_i \in \mathcal{V}$.



(a) (b) (c)

Figure 1: Directed Graph Examples

For a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we define the *transitive closure* of the graph, the set of all paths, as the matrix $\mathcal{M}^{\mathcal{G}}$, where $\mathcal{M}^{\mathcal{G}}_{ij} = 1$ if there exists a directed path from vertex j to vertex i, or $i = j$, and $\mathcal{M}^{\mathcal{G}}_{ij} = 0$ otherwise.

For each vertex $i \in \mathcal{V}$, we define the set of its *ancestors* $\mathbf{i_A}$, and the set of its *descendants* $\mathbf{i_D}$ as

$$\mathbf{i_A} = \{j \in \mathcal{V} \mid \mathcal{M}^{\mathcal{G}}_{ij} \neq 0\} \qquad \mathbf{i_D} = \{j \in \mathcal{V} \mid \mathcal{M}^{\mathcal{G}}_{ji} \neq 0\}$$

In other words, $j \in \mathbf{i_A}$ if and only if there exists a directed path from j to i, or $i = j$. Similarly, $j \in \mathbf{i_D}$ if and only if a directed path exists from i to j, or $i = j$. Note that we always have $i \in \mathbf{i_A}$ and $i \in \mathbf{i_D}$. Consequently, we define $\mathbf{i'_A}$ and $\mathbf{i'_D}$ by removing i from these sets, so that

$$\mathbf{i'_A} = \mathbf{i_A} \setminus \{i\} \qquad \mathbf{i'_D} = \mathbf{i_D} \setminus \{i\}$$

To illustrate this notation, consider the example in Figure 1(b). Using these definitions, the following sets would be defined

$$\begin{aligned} \mathbf{1_A} &= \{1\} & \mathbf{1_D} &= \mathcal{V} \\ \mathbf{1'_A} &= \emptyset & \mathbf{1'_D} &= \{2,3,4,5\} \\ \mathbf{2_A} = \mathbf{3_A} = \mathbf{4_A} &= \{1,2,3,4\} & \mathbf{5_A} &= \mathcal{V} \\ \mathbf{2_D} = \mathbf{3_D} = \mathbf{4_D} &= \{2,3,4,5\} & \mathbf{5_D} &= \{5\} \end{aligned}$$

II-B Sparsity Structures.

Since we will be dealing with systems represented by directed graphs, we will need to deal with matrices that satisfy certain sparsity constraints. For any $m \times n$ matrix A and ring R, the set of (block) matrices in $R^{m\times n}$ with a similar sparsity structure to A is defined by

$$\text{Sparse}(A; R) = \{B \in R^{m\times n} \mid B_{ij} = 0 \text{ if } A_{ij} = 0\}$$

Typically, R is the ring of reals or the ring of stable transfer functions. Note that, by design, there is flexibility built into this notation. In particular, $\text{Sparse}(A; R)$ does not explicitly define the size of the individual (block) elements. For instance, if A is a matrix with scalar elements, $\text{Sparse}(A; R)$ may include matrices with block elements of arbitrary size; the size of the block elements will be implied by the context.

For (block) matrices indexed by $\mathcal{V} \times \mathcal{V}$, it will be convenient to define (block) submatrices based on subsets of $\mathcal{V}$. To this end, if M is an $N \times N$ block matrix, then for any sets $S, T \subset \mathcal{V}$, we construct the $|S| \times |T|$ submatrix M_{ST} by eliminating the rows of M not in S and the columns of M not in T. Note that this is simply a generalized notation for the standard indexing of individual elements M_{ij} of a matrix. Also, in cases where $S = T$, we let $M_S = M_{SS}$. This notational convention similarly applies to constructing subvectors from vectors indexed by $\mathcal{V}$.

Lastly, the notation $\text{diag}(F_i)$ will be used to denote elements of $\text{Sparse}(I; R)$ (block diagonal matrices with diagonal blocks $F_1, \dots, F_N$).

To help illustrate some of this notation, consider the graph in Figure 1(a). For this system, the following matrices may be defined.

$$\mathcal{M}^{\mathcal{G}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{R})$$
$$A_{\mathbf{1_D}\mathbf{2'_A}} = \begin{bmatrix} A_1 \\ A_{21} \end{bmatrix} \qquad A_{\mathbf{2_D}} = A_2$$

Again, the sizes of the blocks of A are not being specified; in fact, another matrix $B \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{R})$ might have block elements which differ in dimension from those of A.

II-C Hardy Spaces.

The real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The complex open unit disc is $\mathbb{D}$, and its boundary, the unit circle, is $\mathbb{T}$. As is standard, the set $\mathcal{L}_2(\mathbb{T})$ is the Hilbert space of Lebesgue measurable functions on $\mathbb{T}$, which are square integrable, and $\mathcal{H}_2$ denotes the Hardy space of functions analytic outside the closed unit disc, and at infinity, with square-summable power series. The set $\mathcal{H}_2^{\perp}$ is the orthogonal complement of $\mathcal{H}_2$ in $\mathcal{L}_2$. Also, $\mathcal{L}_\infty(\mathbb{T})$ denotes the set of Lebesgue measurable functions bounded on $\mathbb{T}$. Similarly, $\mathcal{H}_\infty$ is the subspace of $\mathcal{L}_\infty$ with functions analytic outside of $\mathbb{T}$, and $\mathcal{H}_\infty^-$ is the subspace of $\mathcal{L}_\infty$ with functions analytic inside $\mathbb{T}$.

The set $\mathcal{RP}$ denotes the set of proper real rational functions, and the prefix $\mathcal{R}$ then indicates a subset of $\mathcal{RP}$. That is, both $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$ represent the set of rational functions with poles in $\mathbb{D}$; we will use these spaces interchangeably.

The following useful facts about these sets will be used throughout this paper [26]:

- if $G \in \mathcal{L}_\infty$, then $G\mathcal{L}_2 \subset \mathcal{L}_2$
- if $G \in \mathcal{H}_\infty$, then $G\mathcal{H}_2 \subset \mathcal{H}_2$
- if $G \in \mathcal{H}_\infty^-$, then $G\mathcal{H}_2^\perp \subset \mathcal{H}_2^\perp$

For transfer functions $F \in \mathcal{RP}$, we use the notation

$$F = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(zI - A)^{-1}B + D$$

Its adjoint operator, denoted F^*, is $F^* = B^T(z^{-1}I - A^T)^{-1}C^T + D^T$. For the set $\mathcal{S} = \text{Sparse}(\mathcal{M}^\mathcal{G}; \mathcal{RH}_2)$, its orthogonal complement $\mathcal{S}^\perp$ is given by

$$G \in \mathcal{S}^\perp \qquad \Leftrightarrow \qquad \begin{array}{ll} G_{ij} \in \mathcal{H}_2^\perp & \text{if } \mathcal{M}_{ij}^\mathcal{G} \neq 0 \\ G_{ij} \in \mathcal{L}_2 & \text{if } \mathcal{M}_{ij}^\mathcal{G} = 0 \end{array}$$

Lastly, we define $\mathrm{P}_{\mathcal{H}_2} : \mathcal{L}_2 \to \mathcal{H}_2$ as the orthogonal projection onto $\mathcal{H}_2$.

II-D Networked System.

Our problem setup is as follows. Each vertex in $\mathcal{V}$ represents a separate plant with corresponding controller. In reality, a networked system consists of two underlying graph structures: a graph $\mathcal{G}^P = (\mathcal{V}, \mathcal{E}^P)$ determining the dynamic coupling of the plants, and a graph $\mathcal{G}^K = (\mathcal{V}, \mathcal{E}^K)$ indicating the allowable communication channels of the controllers.

For general graph structures $\mathcal{G}^P$ and $\mathcal{G}^K$, decentralized control is currently intractable. Thus, for our work here, we must restrict attention to a class of decentralized systems for which we can find solutions. Specifically, the graph of the plant $\mathcal{G}^P$ must be contained in the transitive closure of $\mathcal{G}^K$; more commonly, this is referred to as a partially nested structure. Consequently, for simplicity we assume throughout the paper that $\mathcal{M}^{\mathcal{G}^P} = \mathcal{M}^{\mathcal{G}^K}$, and we will henceforth drop the superscripts P and K.

For each subsystem $i \in \mathcal{V}$, the state $x_i(t) \in \mathbb{R}^{n_i}$ evolves according to

$$x_i(t+1) = \sum_{j \in \mathbf{i_A}} \big(A_{ij}x_j(t) + B_{ij}u_j(t)\big) + H_i w_i(t) \qquad (2)$$

where $u_j(t) \in \mathbb{R}^{m_j}$ are control inputs and $w_i(t) \in \mathbb{R}^{n_i}$ are exogenous, independent noise inputs to the systems with zero mean and unit covariance. As a result, the overall discrete time state-space system is given by

$$x(t+1) = Ax(t) + Bu(t) + Hw(t) \qquad (3)$$

where $x(t) = \begin{bmatrix} x_1(t)^T & \cdots & x_N(t)^T \end{bmatrix}^T$ and similarly for $u(t)$ and $w(t)$. Moreover, the block matrices A, B, and H satisfy

$$A \in \text{Sparse}(\mathcal{M}^\mathcal{G}; \mathbb{R}) \qquad B \in \text{Sparse}(\mathcal{M}^\mathcal{G}; \mathbb{R})$$

and H is block diagonal. We will assume that H is also invertible; note that this simply implies that no component of the state evolves noise-free. This assumption merely simplifies our presentation, while not fundamentally affecting our results.

Our goal is to minimize the expected cost

$$\lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} \mathrm{E}\big\|Cx(t) + Du(t)\big\|_2^2$$

To this end, we define the vector

$$z(t) = \begin{bmatrix} C_1 & \cdots & C_N \end{bmatrix} x(t) + \begin{bmatrix} D_1 & \cdots & D_N \end{bmatrix} u(t)$$

where, for simplicity, we will assume that $D^TD > 0$; this is a fairly standard assumption in optimal control, and it implies that every control action will incur some cost to the system. Note that no sparsity constraints are assumed for C or D, so this formulation allows for coupling of all states and actions in the cost. Consequently, our plant is expressed as the matrix $P \in \mathcal{RP}$, where

$$\begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \text{ and } P = \left[\begin{array}{c|cc} A & H & B \\ \hline C & 0 & D \\ I & 0 & 0 \end{array}\right] \qquad (4)$$

State feedback is assumed throughout. Additionally, no communication delays or bandwidth restrictions will be assumed, so that player i makes decision u_i based on the history of x_j's, for each j which is an ancestor of vertex i in $\mathcal{G}$. As a result, we are looking for controllers of the form

$$\begin{aligned} q_i(t+1) &= A_{K_i}q_i(t) + B_{K_i}x_{\mathbf{i_A}}(t) \\ u_i(t) &= C_{K_i}q_i(t) + D_{K_i}x_{\mathbf{i_A}}(t) \end{aligned} \qquad (5)$$

for each $i \in \mathcal{V}$. Equivalently, we want a transfer function $\mathcal{K} \in \text{Sparse}(\mathcal{M}^\mathcal{G}; \mathcal{RP})$, such that $u = \mathcal{K}x$. Figure 2 illustrates the overall feedback system, with the plant P given by equation (4) and the controller $\mathcal{K} \in \text{Sparse}(\mathcal{M}^\mathcal{G}; \mathcal{RP})$. A basic requirement for any controller $\mathcal{K}$ is that it is internally stabilizing; this means that the plant and controller states $x(t), q_i(t) \to 0$ as $t \to \infty$. Since we will deal with state-space realizations throughout this work, we will formalize stability in the following standard sense.

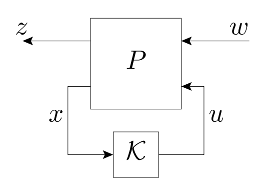


Figure 2: Overall Feedback System

Definition 1. *Given the plant P in (4), the controller $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RP})$ is* internally stabilizing *if it has a realization (A_K, B_K, C_K, D_K) such that*

$$A_{cl} = \begin{bmatrix} A + BD_K & BC_K \\ B_K & A_K \end{bmatrix} \tag{6}$$

has all eigenvalues contained in $\mathbb{D}$.

Note that this definition is dependent on the particular realization for P. In all cases of interest we will have (A, B) stabilizable, and in this case the realization dependence goes away. Then $\mathcal{K}$ is internally stabilizing if and only if

$$\begin{bmatrix} I & -\mathcal{K} \\ -P_{22} & I \end{bmatrix}^{-1} \in \mathcal{RH}_\infty$$

and so internal stability is equivalent to closed-loop input-output stability under this assumption.

Lastly, we define $\mathcal{F}(P, \mathcal{K})$ as the linear fractional transformation

$$\mathcal{F}(P, \mathcal{K}) = P_{11} + P_{12}\mathcal{K}(I - P_{22}\mathcal{K})^{-1}P_{21}$$

Consequently, our objective function is the $\mathcal{H}_2$ norm of the closed-loop transfer function from w to z. In other words, we have the following optimization problem.

$$\begin{aligned} \text{minimize} \quad & \|\mathcal{F}(P, \mathcal{K})\|_2 \\ \text{subject to} \quad & \mathcal{K} \text{ is internally stabilizing} \\ & \mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RP}) \end{aligned} \tag{7}$$

III Analysis

Before trying to find the optimal controllers, we state some results that will help to simplify our approach. The first of these results discusses the particular graph structures that will be considered in this paper. In Section III-B, we address the conditions required for a stabilizing solution to exist. Lastly, algebraic Riccati equations are considered in Section III-C.

III-A Graph Structures

As noted above, we have restricted attention to graph structures satisfying $\mathcal{M}^{\mathcal{G}^P} = \mathcal{M}^{\mathcal{G}^K}$. This is motivated by the following result, which was shown in [11, 16].

Lemma 2. *Suppose $\mathcal{G}^P$ and $\mathcal{G}^K$ are directed graphs. Let P_{22} be defined as in (4). Then,*

$$\begin{aligned} \{\mathcal{K}(I - P_{22}\mathcal{K})^{-1} \mid \mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RP})\} \\ = \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RP}) \end{aligned}$$

if and only if $P_{22} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RP})$.

Lemma 2 provides a classification for the graph structures that correspond to tractable decentralized control problems; those for which $P_{22} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RP})$. Hence, the work presented here is aimed at solving this class of problems.

The next result concerns graphs containing directed cycles. To this end, for each $i \in \mathcal{V}$, let

$$\mathcal{C}_i = \mathbf{i_A} \cap \mathbf{i_D}$$

It is clear that the graph has a directed cycle containing vertex $i \in \mathcal{V}$ if and only if $\mathcal{C}_i \neq \{i\}$; we note that $i \in \mathcal{C}_i$ for all $i \in \mathcal{V}$. Consequently, we will say that a directed graph is *acyclic* if $\mathcal{C}_i = \{i\}$ for all $i \in \mathcal{V}$. Some additional properties of cycles are considered in the following lemma. We omit the proofs, which follow directly from the definitions.

Lemma 3. *For all $i, j \in \mathcal{V}$, the following properties hold:*

i) $i \in \mathcal{C}_j$ if and only if $j \in \mathcal{C}_i$

ii) $i \in \mathbf{j_A}$ if and only if $i \in \mathbf{k_A}$ for all $k \in \mathcal{C}_j$

iii) $i \in \mathbf{j_D}$ if and only if $i \in \mathbf{k_D}$ for all $k \in \mathcal{C}_j$

From Lemma 3, the $\mathcal{C}_i$ form equivalence classes on $\mathcal{V}$, where $i \sim j$ if $i \in \mathcal{C}_j$. As such, these cycles generate a unique partition of $\mathcal{V}$. In other words, given a graph with cycles, we can group the cycles together as single entities; that is, considering the graph $(\mathcal{V}_{\text{new}}, \mathcal{E}_{\text{new}})$ whose vertices are the directed cycles of the original graph.

$$\begin{aligned} \mathcal{V}_{\text{new}} &= \{\mathcal{C}_i \mid i \in \mathcal{V}\} \\ \mathcal{E}_{\text{new}} &= \{(\mathcal{C}_i, \mathcal{C}_j) \mid \mathcal{C}_i \neq \mathcal{C}_j \\ &\qquad \text{and } \exists x \in \mathcal{C}_i, y \in \mathcal{C}_j \text{ with } (x, y) \in \mathcal{E}\} \end{aligned}$$

Note that this equivalent graph is acyclic. Finally, given a controller $\mathcal{K}$ satisfying the sparsity constraints of this acyclic graph, we must be able to construct the individual control laws for the original graph. The details of this last step may be found [11]. In short, one may think of the individual controllers as rows of the controller $\mathcal{K}$. Thus, we can assume, without loss of generality, that our graph is acyclic, and we will assume this for the remainder of the paper.

III-B Stabilization

As noted above, we are considering decentralized systems represented by directed acyclic graphs and state feedback. Under these assumptions, we must first establish when stabilization of the plant in (4) is possible. The following lemma provides the necessary and sufficient conditions for the existence of a stabilizing controller.

Lemma 4. *Suppose $\mathcal{G}$ is a directed acyclic graph. There exists a controller $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RP})$ which internally stabilizes P in (4) if and only if (A_i, B_i) is stabilizable for all $i \in \mathcal{V}$.*

Proof Since $\mathcal{G}$ is acyclic, it is straightforward to show that there exists a numbering for the vertices in $\mathcal{V}$ such that $\mathcal{M}^{\mathcal{G}}$ is lower triangular; see [16] for a proof. Consequently, we assume that the vertices are ordered in this fashion. With this in mind, we now prove the lemma.

($\Rightarrow$) If (A_i, B_i) is stabilizable for all $i \in \mathcal{V}$, then there exist matrices F_i such that $A_i + B_iF_i$ is stable. Consequently, the controller $\mathcal{K} = F = \text{diag}(F_i)$ produces the closed-loop matrix A_{cl} of (6), given by $A + BF$. Since $A, B, F \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{R})$, it follows that $A+BF \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{R})$. Thus, $A + BF$ is lower triangular, with diagonal entries equal to $A_i + B_iF_i$, for all $i \in \mathcal{V}$. Since every $A_i + B_iF_i$ is stable, then the closed-loop system is stable.

($\Leftarrow$) Suppose that (A_i, B_i) is not stabilizable, for some $i \in \mathcal{V}$. Then, there exists a transformation U_i such that

$$U_i^{-1}A_iU_i = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \qquad U_i^{-1}B_i = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}$$

where the a_{11} block has at least one unstable eigenvalue. Now, suppose $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RP})$ has the realization (A_K, B_K, C_K, D_K). Consequently, the A_{cl} matrix of (6) must then have each of its four blocks in $\text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RP})$. Since each of these blocks are lower triangular, we can permute the indices to get a new lower triangular matrix, with diagonal blocks equal to

$$\begin{bmatrix} A_j + B_jD_{K_j} & B_jC_{K_j} \\ B_{K_j} & A_{K_j} \end{bmatrix}$$

As a result, consider a change of coordinates under the transformation matrix, $T = \text{diag}(I, \ldots, U_i, \ldots, I)$ with U_i being the ith diagonal block. Under this transformation, the ith diagonal block becomes

$$\left[\begin{array}{c:c} \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} D_{K_i}U_i & \begin{bmatrix} 0 \\ b_2 \end{bmatrix} C_{K_i} \\ \hdashline B_{K_i}U_i & A_{K_i} \end{array}\right]$$

$$= \left[\begin{array}{cc:c} a_{11} & 0 & 0 \\ * & * & * \\ \hdashline \multicolumn{2}{c:}{B_{K_i}U_i} & A_{K_i} \end{array}\right]$$

where $*$ represent irrelevant entries. Consequently, it is clear that the eigenvalues of this matrix include the unstable eigenvalue of a_{11}, for any choice of matrices $A_{K_i}, B_{K_i}, C_{K_i}, D_{K_i}$. As a result, A_{cl} cannot be stabilized with any controller $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RP})$. ∎

Note that the stability condition in Lemma 4 is not equivalent to (A, B) stabilizable. Thus, this lemma shows that there exists a decentralized controller which stabilizes the overall system if and only if each individual subsystem can be stabilized.

III-C Algebraic Riccati Equations

It is well-known that solutions to the classical, centralized $\mathcal{H}_2$ problem involve the stabilizing solution to an algebraic Riccati equation. Similarly, it will be shown that Riccati equations are involved in the solutions for these decentralized problems. Thus, the necessary and sufficient conditions for the existence of a stabilizing solution to the algebraic Riccati equation is presented here.

Lemma 5. *Suppose $D^TD > 0$. Then, there exists $X \geq 0$, such that*

$$\begin{aligned} X = C^TC + A^TXA - (A^TXB + C^TD) \\ \times (D^TD + B^TXB)^{-1}(B^TXA + D^TC) \end{aligned} \quad (8)$$

and

$$\rho\big(A - B(D^TD + B^TXB)^{-1}(B^TXA + D^TC)\big) < 1$$

if and only if (A, B) is stabilizable and

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \quad (9)$$

has full column rank for all $\lambda \in \mathbb{T}$.

Proof For a proof, see [26] and [27]. ∎

For convenience, we will denote the stabilizing solution to the Riccati equation (8) as

$$X = \text{Ric}(A, B, C, D)$$

Note that this X is unique. Thus, in addition to (A, B) stabilizable, the classical problem also requires the rank condition of (9).

In the decentralized problem considered here, stabilizing solutions to multiple Riccati equations are required. In particular, we need stabilizing solutions X_i satisfying

$$X_i = \text{Ric}(A_{\mathbf{i_D}}, B_{\mathbf{i_D}}, C_{\mathbf{i_D}}, D_{\mathbf{i_D}}) \quad (10)$$

for each $i \in \mathcal{V}$. Consequently, we will need $(A_{\mathbf{i_D}}, B_{\mathbf{i_D}})$ stabilizable and

$$\begin{bmatrix} A_{\mathbf{i_D}} - \lambda I & B_{\mathbf{i_D}} \\ C_{\mathbf{i_D}} & D_{\mathbf{i_D}} \end{bmatrix}$$

having full column rank for all $\lambda \in \mathbb{T}$, for all $i \in \mathcal{V}$. Checking these conditions for each $i \in \mathcal{V}$ can be computationally expensive in large, highly connected graphs. However, this can be simplified with the following lemma.

Lemma 6. *Suppose $D^TD > 0$, (A_i, B_i) is stabilizable for all $i \in \mathcal{V}$, and*

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \quad (11)$$

has full column rank for all $\lambda \in \mathbb{T}$. Then, for all $i \in \mathcal{V}$, $D_{\mathbf{i_D}}^TD_{\mathbf{i_D}} > 0$, $(A_{\mathbf{i_D}}, B_{\mathbf{i_D}})$ is stabilizable, and

$$\begin{bmatrix} A_{\mathbf{i_D}} - \lambda I & B_{\mathbf{i_D}} \\ C_{\mathbf{i_D}} & D_{\mathbf{i_D}} \end{bmatrix} \quad (12)$$

has full column rank for all $\lambda \in \mathbb{T}$.

Proof We omit the proof for space; see [16] for a complete proof. ■

While Lemma 6 provides a sufficient condition for the existence of stabilizing solutions to all of the Riccati equations (10), in general these conditions are not necessary. However, by only requiring one rank condition to be met, this lemma provides a computationally cheaper approach to checking for the existence of stabilizing solutions to (10) for all $i \in \mathcal{V}$,

One special case to note here is when there exists an $i \in \mathcal{V}$ such that $\mathbf{i_D} = \mathcal{V}$. In this case, the rank condition of (11) is necessary for the existence of the stabilizing solution X_i; by Lemma 6, this would then imply that stabilizing solutions exist to all of the Riccati equations (10).

IV Main Results

To summarize our results of the previous section, we will make the following assumptions throughout this work.

A1) $\mathcal{M}^{\mathcal{G}^P} = \mathcal{M}^{\mathcal{G}^K}$

A2) (A_i, B_i) is stabilizable for all $i \in \mathcal{V}$

A3) $D^T D > 0$ and $HH^T > 0$

A4) $\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{T}$

As noted in Lemma 6, Assumption A4 is a sufficient condition for the existence of stabilizing solutions to the requisite Riccati equations in (10) and could be replaced by requiring that (12) has full column rank for all i, which is both necessary and sufficient. As stated however, condition A4 only requires checking the rank of one matrix, instead of N matrices.

We are now ready to present the main results of this paper. The proof of the following theorem will be developed in the next few sections.

Theorem 7. *Suppose $\mathcal{G}$ is a directed acyclic graph, and let P be defined by (4). Furthermore, suppose that Assumptions A2–A4 hold. For each $i \in \mathcal{V}$, let $X_i = \operatorname{Ric}(A_{\mathbf{i_D}}, B_{\mathbf{i_D}}, C_{\mathbf{i_D}}, D_{\mathbf{i_D}})$ be the stabilizing solution to the corresponding Riccati equations, and let*

$$K_i = (D_{\mathbf{i_D}}^T D_{\mathbf{i_D}} + B_{\mathbf{i_D}}^T X_i B_{\mathbf{i_D}})^{-1} (B_{\mathbf{i_D}}^T X_i A_{\mathbf{i_D}} + D_{\mathbf{i_D}}^T C_{\mathbf{i_D}})$$

Lastly, define $A_K, B_K, C_K, D_K \in \operatorname{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathbb{R})$ by

$$(A_K)_{ij} = \begin{cases} A_{\mathbf{i'_D}} - B_{\mathbf{i'_D}\mathbf{i_D}} K_i I_{\mathbf{i_D}\mathbf{i'_D}} & i = j \\ -(A_{\mathbf{i'_D}i} - B_{\mathbf{i'_D}\mathbf{i_D}} K_i I_{\mathbf{i_D}i}) I_{i\mathbf{j'_D}} & i \in \mathbf{j'_D} \\ 0 & \textit{otherwise} \end{cases}$$

$$B_K = \operatorname{diag}(A_{\mathbf{i'_D}i} - B_{\mathbf{i'_D}\mathbf{i_D}} K_i I_{\mathbf{i_D}i})$$

$$(C_K)_{\mathcal{V}j} = -I_{\mathcal{V}\mathbf{j_D}} K_j I_{\mathbf{j_D}\mathbf{j'_D}} + \sum_{k \in \mathbf{j'_D}} I_{\mathcal{V}\mathbf{k_D}} K_k I_{\mathbf{k_D}k} I_{k\mathbf{j'_D}}$$

$$(D_K)_{\mathcal{V}j} = -I_{\mathcal{V}\mathbf{j_D}} K_j I_{\mathbf{j_D}j}$$

Then, a unique optimal $\mathcal{K} \in \operatorname{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RP})$ exists for (7) given by

$$\mathcal{K} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \tag{13}$$

Moreover, this controller corresponds to the policies:

$$\begin{aligned} q_i(t+1) &= (A_K)_{\mathbf{i_A}} q_i(t) + (B_K)_{\mathbf{i_A}} x_{\mathbf{i_A}}(t) \\ u_i(t) &= (C_K)_{i\mathbf{i_A}} q_i(t) + (D_K)_{i\mathbf{i_A}} x_{\mathbf{i_A}}(t) \end{aligned} \tag{14}$$

for each $i \in \mathcal{V}$.

Having established the optimal decentralized policies, a few remarks are in order. The policies in (14) provide a state space representation for each individual controller. As stated though, this representation lacks any intuition into the estimation structure of these policies. To provide this intuition, let the state of $\mathcal{K}$ in (13) be η. Combining this with (3), the closed-loop system has states $\begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}$. As will be shown later, we can use a transformation to get a new state $\xi(t)$ such that the closed-loop state evolution decouples as

$$\xi_i(t+1) = (A_{\mathbf{i_D}} - B_{\mathbf{i_D}} K_i) \xi_i(t) + I_{\mathbf{i_D}i} H_i w_i(t)$$

for each $i \in \mathcal{V}$. Moreover, the optimal policies become

$$u_i(t) = -\sum_{j=1}^{N} I_{i\mathbf{j_D}} K_j \xi_j(t) = -\sum_{j \in \mathbf{i_A}} I_{i\mathbf{j_D}} K_j \xi_j(t)$$

In addition, we will show that the states $\xi_i(t)$ can be represented as

$$\xi_i(t) = \operatorname{E}\big(x_{\mathbf{i_D}}(t) \mid x_{\mathbf{i_A}}(0{:}\, t)\big) - \operatorname{E}\big(x_{\mathbf{i_D}}(t) \mid x_{\mathbf{i'_A}}(0{:}\, t)\big)$$

where $x(0{:}\, t)$ is shorthand notation for $x(0), \ldots, x(t)$. In other words, the first term in $\xi_i(t)$ represents the estimate of plant state x_i and its descendants, conditioned on knowing the histories of plant state x_i and its ancestors; the second term is a similar estimate, except it is conditioned on knowing only the ancestors of vertex i and not x_i itself. Thus, ξ_i is the estimation correction in updating the conditioning information to include state x_i.

Lastly, these results establish the order of the optimal controller, as seen in the following corollary; recall that each subsystem $i \in \mathcal{V}$ has a state dimension of n_i.

Corollary 8. *For the decentralized control problem (7), the optimal controller $\mathcal{K} \in \operatorname{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RP})$ has order of at most*

$$\sum_{\substack{i \in \mathcal{V} \\ j \in \mathbf{i'_D}}} n_j$$

It should be noted that Corollary 8 provides the optimal controller order for the overall controller in (13).

However, in practice, individual controllers may be implemented as in (14). While the individual controller might have a smaller state dimension than $\mathcal{K}$, the total computational effort required would be higher in this case. Intuitively, this is due to repetition in estimation among players; in other words, multiple players may be running the same state estimators.

In contrast, to reduce this overlap in estimation and minimize the total computation across all controllers, it might be better for each player to compute ξ_i and pass that information to descendant players. Thus, there is a trade-off that can be made between the amount of computation required by each player and the amount of information that is passed along communication channels.

V Optimality Conditions

We turn now to establishing the results of the previous section. To this end, we begin by using the standard Youla parametrization to simplify our optimization problem (7). In what follows, we use $\mathcal{S} = \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RH}_2)$.

Lemma 9. *Suppose $\mathcal{G}$ is a directed acyclic graph, and let P be defined by (4). Suppose (A_i, B_i) is stabilizable for all $i \in \mathcal{V}$, and let F_i be matrices, such that $A_i + B_i F_i$ has stable eigenvalues. Lastly, let $F = \mathrm{diag}(F_i)$. Then, the set of all internally stabilizing controllers is parametrized by*

$$\{\mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RP}) \mid \mathcal{K} \text{ internally stabilizing}\} = \{Q(I+MQ)^{-1} + F \mid Q \in \mathcal{S}\} \quad (15)$$

where $M = (zI - (A+BF))^{-1}B$. Moreover, the set of stable closed-loop transfer functions satisfies

$$\{\mathcal{F}(P,\mathcal{K}) \mid \mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RP}),\ \mathcal{K} \text{ int. stabilizing}\} = \{N_{11} + N_{12} Q N_{21} \mid Q \in \mathcal{S}\} \quad (16)$$

where $N_{12} = z^{-1}((C+DF)(zI - (A+BF))^{-1}B + D)$ and

$$\begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} = \left[\begin{array}{c|c} A+BF & H \\ \hline C+DF & 0 \\ A+BF & H \end{array}\right]$$

Proof Suppose $\mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RP})$ internally stabilizes P and has realization (A_K, B_K, C_K, D_K). By definition, this implies that (6) is stable. Let us define $Q \in \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RP})$ by

$$Q = (I - (\mathcal{K} - F)M)^{-1}(\mathcal{K} - F)$$

so that $\mathcal{K} = Q(I+MQ)^{-1} + F$. Then, algebraic manipulations (which must be omitted here for space, but can be found in [16]) show that

$$Q = \left[\begin{array}{cc|c} A + BD_K & BC_K & B(D_K - F) \\ B_K & A_K & B_K \\ \hline D_K - F & C_K & D_K - F \end{array}\right]$$

Since A_{cl} in (6) is stable, then $Q \in \mathcal{S}$.

Conversely, suppose that $Q \in \mathcal{S}$ has the realization (A_Q, B_Q, C_Q, D_Q). Then, it can be shown that

$$\mathcal{K} = Q(I+MQ)^{-1} + F = \left[\begin{array}{cc|c} A + B(F - D_Q) & BC_Q & -BD_Q \\ -B_Q & A_Q & -B_Q \\ \hline D_Q & -C_Q & D_Q + F \end{array}\right]$$

Consequently, the closed-loop matrix A_{cl} is given by

$$A_{cl} = \begin{bmatrix} A + B(D_Q + F) & BD_Q & -BC_Q \\ -BD_Q & A + B(F - D_Q) & BC_Q \\ -B_Q & -B_Q & A_Q \end{bmatrix}$$

Under a particular transformation, it turns out that A_{cl} is similar to the matrix

$$\begin{bmatrix} A+BF & BD_Q & -BC_Q \\ 0 & A+BF & 0 \\ 0 & -B_Q & A_Q \end{bmatrix}$$

Since $A+BF$ and A_Q are stable by assumption, then A_{cl} is stable. Thus, we have shown that (15) holds. Obtaining the set of closed-loop maps in (16) follows directly from plugging the set of controllers parametrized by (15) into $\mathcal{F}(P,\mathcal{K})$. ∎

As in the classical case, the Youla parametrization translates the difficult optimization problem (7) into an affine optimization problem. However, since we have state feedback in our problem, we can simplify the problem even further.

Lemma 10. *For the system in (4), let N be defined as in Lemma 9. Suppose Q is optimal for*

$$\begin{aligned} &\text{minimize} && \|N_{11} + N_{12}Q\|_2 \\ &\text{subject to} && Q \in S \end{aligned} \quad (17)$$

Then, there exists $\hat{Q} \in S$, such that $Q = \hat{Q}N_{21}$, and $\hat{Q}$ is optimal for

$$\begin{aligned} &\text{minimize} && \|N_{11} + N_{12}\hat{Q}N_{21}\|_2 \\ &\text{subject to} && \hat{Q} \in S \end{aligned} \quad (18)$$

Conversely, if $\hat{Q} \in S$ is optimal for (18), then $Q = \hat{Q}N_{21}$ is optimal for (17).

Proof This follows from the fact that $N_{21}, N_{21}^{-1} \in S$, so that $Q \in S$ if and only if $\hat{Q} \in S$. ∎

In order to solve the optimization problem in (17), which is infinite-dimensional, it is convenient to find an equivalent optimality condition, which the following lemma provides.

Lemma 11. *Suppose $G_1, G_2 \in \mathcal{RH}_\infty$. Then, $Q \in \mathcal{S}$ minimizes*

$$\begin{aligned} &\text{minimize} && \|G_1 + G_2 Q\|_2 \\ &\text{subject to} && Q \in \mathcal{S} \end{aligned}$$

if and only if

$$G_2^* G_1 + G_2^* G_2 Q \in \mathcal{S}^\perp \tag{19}$$

Proof This result is a version of the classical projection theorem; see for example [28]. ∎

While Lemma 11 provides the optimality condition for (17), it does not guarantee the existence of an optimal controller. The existence of such a controller will be shown by explicit construction in the following sections.

VI Spectral Factorization

Our goal is now to find a solution $Q \in \mathcal{S}$ which satisfies the optimality condition

$$N_{12}^* N_{11} + N_{12}^* N_{12} Q \in \mathcal{S}^\perp \tag{20}$$

To this end, we have the following result.

Lemma 12. *Suppose $G_1, G_2 \in \mathcal{RH}_\infty$. Then, $Q \in \mathcal{S}$ satisfies*

$$G_2^* G_1 + G_2^* G_2 Q \in \mathcal{S}^\perp \tag{21}$$

if and only if

$$(G_2^* G_1)_{\mathbf{i_D} i} + (G_2^* G_2)_{\mathbf{i_D}} Q_{\mathbf{i_D} i} \in \mathcal{H}_2^\perp$$

for all $i \in \mathcal{V}$.

Proof The optimality condition (21) can be equivalently written as

$$G_2^* G_1 + G_2^* G_2 Q = \Lambda \tag{22}$$

where $\Lambda \in \mathcal{S}^\perp$. Note that Λ satisfies

$$\Lambda_{ij} \in \begin{cases} \mathcal{H}_2^\perp & i \in \mathbf{j_D} \\ \mathcal{L}_2 & i \notin \mathbf{j_D} \end{cases}$$

Thus, (22) is satisfied if and only if it is satisfied for each (i,j) such that $i \in \mathbf{j_D}$. Breaking this up by column, we have for each $j \in \mathcal{V}$

$$\begin{aligned} I_{\mathbf{j_D}\mathcal{V}}(G_2^* G_1 + G_2^* G_2 Q) I_{\mathcal{V}j} &= I_{\mathbf{j_D}\mathcal{V}} \Lambda I_{\mathcal{V}j} \\ (G_2^* G_1)_{\mathbf{j_D} j} + (G_2^* G_2)_{\mathbf{j_D}} Q_{\mathbf{j_D} j} &= \Lambda_{\mathbf{j_D} j} \end{aligned}$$

Since $Q_{\mathbf{j_D} j} \in \mathcal{RH}_2$ and $\Lambda_{\mathbf{j_D} j} \in \mathcal{H}_2^\perp$, the result follows. ∎

VI-A Centralized Case

From Lemma 12, we must now solve N optimality conditions, one for each column of (20). However, each of these equations can be solved via spectral factorization, as in the classical centralized case. We summarize these results with the following two lemmas.

Lemma 13. *Suppose $G_1, G_2 \in \mathcal{RH}_\infty$ have the realizations*

$$\begin{aligned} G_1 &= C(zI - A)^{-1} H \\ G_2 &= z^{-1}(C(zI - A)^{-1} B + D) \end{aligned}$$

Suppose there exists a stabilizing solution X to the algebraic Riccati equation, $X = \mathrm{Ric}(A, B, C, D)$. Let $W = D^T D + B^T X B$, and $K = W^{-1}(B^T X A + D^T C)$, and $L \in \mathcal{RH}_\infty$ satisfying

$$L = \left[\begin{array}{c|c} A & B \\ \hline W^{\frac{1}{2}} K & W^{\frac{1}{2}} \end{array}\right]$$

Then, $L^{-1} \in \mathcal{RH}_\infty$, $L^{-*} \in \mathcal{H}_\infty^-$, and

$$L^* L = G_2^* G_2$$

Moreover,

$$\begin{aligned} L^{-*} G_2^* G_1 = W^{-\frac{1}{2}} B^T (z^{-1} I - (A - BK)^T)^{-1} X H \\ + z W^{\frac{1}{2}} K (zI - A)^{-1} H \end{aligned}$$

Proof This result follows from algebraic manipulations of the Riccati equation. A simple proof follows the approach in [29]. ∎

With the above spectral factorization, we can now solve the optimality condition of the centralized case, given by

$$G_2^* G_1 + G_2^* G_2 Q \in \mathcal{H}_2^\perp$$

with $Q \in \mathcal{RH}_2$.

Lemma 14. *Let $G_1, G_2 \in \mathcal{RH}_\infty$ be defined as in Lemma 13. Suppose there exists a stabilizing solution X to the algebraic Riccati equation, $X = \mathrm{Ric}(A, B, C, D)$, and let K and L be defined as in Lemma 13. Then, the unique $Q \in \mathcal{RH}_2$ satisfying*

$$G_2^* G_1 + G_2^* G_2 Q \in \mathcal{H}_2^\perp$$

is given by

$$Q = -zK(zI - (A - BK))^{-1} H$$

Proof From Lemma 13, we have the spectral factorization $G_2^* G_2 = L^* L$. Since $L^{-*} \in \mathcal{H}_\infty^-$, then $L^{-*} \mathcal{H}_2^\perp \subset \mathcal{H}_2^\perp$. Hence, the optimality condition is equivalent to

$$L^{-*} G_2^* G_1 + LQ \in \mathcal{H}_2^\perp$$

Since $LQ \in \mathcal{RH}_2$, we can project the optimality condition onto $\mathcal{H}_2$ to obtain

$$\mathrm{P}_{\mathcal{H}_2}(L^{-*} G_2^* G_1) + LQ = 0$$

From Lemma 13, we have

$$\mathrm{P}_{\mathcal{H}_2}(L^{-*} G_2^* G_1) = z W^{\frac{1}{2}} K (zI - A)^{-1} H$$

Consequently, we have

$$\begin{aligned} Q &= -L^{-1} \mathrm{P}_{\mathcal{H}_2}(L^{-*} G_2^* G_1) \\ &= -zK(zI - (A - BK))^{-1} H \end{aligned}$$

∎

Having established the solution for the centralized problem, we are now ready to solve our decentralized problem.

VI-B Decentralized Case

To solve the decentralized optimality condition (20), we must apply our results from the centralized case N times, as seen in the following lemma.

Lemma 15. *For the system in (4), let $A^F = A + BF$, and $C^F = C + DF$, and*

$$N_{11} = C^F(zI - A^F)H$$
$$N_{12} = z^{-1}(C^F(zI - A^F)^{-1}B + D)$$

Furthermore, for each $i \in \mathcal{V}$, suppose there exists a stabilizing solution X_i to the Riccati equation

$$X_i = \mathrm{Ric}(A^F_{\mathbf{i_D}}, B_{\mathbf{i_D}}, C^F_{\mathbf{i_D}}, D_{\mathbf{i_D}}) \tag{23}$$

and define

$$K^F_i = (D^T_{\mathbf{i_D}} D_{\mathbf{i_D}} + B^T_{\mathbf{i_D}} X_i B_{\mathbf{i_D}})^{-1}(B^T_{\mathbf{i_D}} X_i A^F_{\mathbf{i_D}} + D^T_{\mathbf{i_D}} C^F_{\mathbf{i_D}}) \tag{24}$$

Then, the unique $Q \in \mathcal{S}$ satisfying

$$N^*_{12}N_{11} + N^*_{12}N_{12}Q \in \mathcal{S}^\perp \tag{25}$$

is given by

$$Q = \sum_{i\in\mathcal{V}} I_{\mathcal{V}\mathbf{i_D}} Q_{\mathbf{i_D}i} I_{i\mathcal{V}}$$

where

$$Q_{\mathbf{i_D}i} = -zK^F_i(zI - (A^F_{\mathbf{i_D}} - B_{\mathbf{i_D}}K^F_i))^{-1} I_{\mathbf{i_D}i}H_i \tag{26}$$

Proof From Lemma 12, we know that $Q \in \mathcal{S}$ satisfies (25) if and only if

$$(N^*_{12}N_{11})_{\mathbf{i_D}i} + (N^*_{12}N_{12})_{\mathbf{i_D}} Q_{\mathbf{i_D}i} \in \mathcal{H}_2^\perp$$

for all $i \in \mathcal{V}$. This is equivalent to

$$G^*_2G_1 + G^*_2G_2Q_{\mathbf{i_D}i} \in \mathcal{H}_2^\perp$$

where

$$G_1 = C^F_{\mathbf{i_D}}(zI - A^F_{\mathbf{i_D}})I_{\mathbf{i_D}i}H_i$$
$$G_2 = z^{-1}(C^F_{\mathbf{i_D}}(zI - A^F_{\mathbf{i_D}})^{-1}B_{\mathbf{i_D}} + D_{\mathbf{i_D}})$$

Applying Lemmas 13 and 14, we find that $Q_{\mathbf{i_D}i}$ satisfies (26). The result follows from concatenating the columns of Q given by $I_{\mathcal{V}\mathbf{i_D}}Q_{\mathbf{i_D}i}$. ∎

Thus, we have found the optimal $Q \in \mathcal{S}$ for (25). However, if we now want to apply this spectral factorization approach to our problem, our Riccati equations would be in terms of the pre-compensator F. However, this difficulty can be avoided with the following result.

Lemma 16. *Suppose $X \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{m\times n}$. Then, X is the stabilizing solution to $X = \mathrm{Ric}(A, B, C, D)$, if and only if X is the stabilizing solution to $X = \mathrm{Ric}(A^F, B, C^F, D)$, where $A^F = A + BF$ and $C^F = C + DF$.*

Proof By substitution of A^F and C^F, it can be readily shown that the two Riccati equations are equivalent. ∎

Thus, our ability to solve the optimality condition (20) is independent of our choice of the pre-compensator F.

Now, to find the optimal $\mathcal{K} \in \mathrm{Sparse}(\mathcal{M}^\mathcal{G}; \mathcal{RP})$, we must invert the various transformations made along the way. From Lemma 10, we first solve for $\hat{Q} = QN_{21}^{-1}$. Then, inverting the Youla parametrization in Lemma 9, we must compute

$$R = (I + \hat{Q}M)^{-1}\hat{Q}$$

Lastly, we have $\mathcal{K} = R + F$. These computations are performed in the following results.

Lemma 17. *For the system in (4), let N_{21} and M be defined as in Lemma 9, and suppose Q is defined as in Lemma 15. Lastly, define $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ as*

$$\tilde{A}_{km} = \begin{cases} A^F_{\mathbf{k'_D}} - B_{\mathbf{k'_D}\mathbf{k_D}}K^F_k I_{\mathbf{k_D}\mathbf{k'_D}} & k = m \\ -(A^F_{\mathbf{k'_D}k} - B_{\mathbf{k'_D}\mathbf{k_D}}K^F_k I_{\mathbf{k_D}k})I_{k\mathbf{m'_D}} & k \in \mathbf{m'_D} \\ 0 & \textit{otherwise} \end{cases}$$

$$\tilde{B} = \mathrm{diag}(A^F_{\mathbf{k'_D}k} - B_{\mathbf{k'_D}\mathbf{k_D}}K^F_k I_{\mathbf{k_D}k})$$

$$\tilde{C}_{\mathcal{V}m} = -I_{\mathcal{V}\mathbf{m_D}}K^F_m I_{\mathbf{m_D}\mathbf{m'_D}} + \sum_{p\in\mathbf{m'_D}} I_{\mathcal{V}\mathbf{p_D}}K^F_p I_{\mathbf{p_D}p}I_{p\mathbf{p'_D}}$$

$$\tilde{D}_{\mathcal{V}m} = -I_{\mathcal{V}\mathbf{m_D}}K^F_m I_{\mathbf{m_D}m}$$

Then,

$$R = (I + QN_{21}^{-1}M)^{-1}QN_{21}^{-1} = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] \tag{27}$$

Proof Let us define the intermediate variables, $\hat{A} = \mathrm{diag}(A^F_{\mathbf{i_D}} - B_{\mathbf{i_D}}K^F_i)$ and

$$\hat{B} = \begin{bmatrix} I_{\mathbf{1_D}\mathbf{1_A}}B_{\mathbf{1_A}}I_{\mathbf{1_A}\mathcal{V}} \\ \vdots \\ I_{\mathbf{N_D}\mathbf{N_A}}B_{\mathbf{N_A}}I_{\mathbf{N_A}\mathcal{V}} \end{bmatrix}$$
$$\hat{C} = \begin{bmatrix} -I_{\mathcal{V}\mathbf{1_D}}K^F_1 & \cdots & -I_{\mathcal{V}\mathbf{N_D}}K^F_N \end{bmatrix}$$

Note that $\hat{A}, \hat{B}, \hat{C}$ are partitioned into blocks indexed by $\mathcal{V}$. Thus, our subscripting notation will be used for these matrices as well. To avoid confusion when using the identity matrix, we will use $\hat{I}$ to indicate the identity matrix corresponding to matrices with the larger blocks.

It is straightforward to show that

$$(I + QN_{21}^{-1}M)^{-1} = \left[\begin{array}{c|c} \hat{A} - \hat{B}\hat{C} & \hat{B} \\ \hline -\hat{C} & I \end{array}\right]$$

Note that $\hat{A} - \hat{B}\hat{C} \in \mathrm{Sparse}(\mathcal{M}^\mathcal{G}; \mathbb{R})$. Consequently,

$$I_{i\mathcal{V}}(I + QN_{21}^{-1}M)^{-1}I_{\mathcal{V}\mathbf{k_D}} = \left[\begin{array}{c|c} \hat{A}_{\mathbf{i_A}\cap\mathbf{k_D}} - \hat{B}_{\mathbf{i_A}\cap\mathbf{k_D}}\hat{C}_{\mathbf{i_A}\cap\mathbf{k_D}} & \hat{B}_{\mathbf{i_A}\cap\mathbf{k_D}}I_{\mathcal{V}\mathbf{k_D}} \\ \hline -I_{i\mathcal{V}}\hat{C}_{\mathbf{i_A}\cap\mathbf{k_D}} & I_{i\mathbf{k_D}} \end{array}\right]$$

since

$$\hat{B} I_{\mathcal{V}\mathbf{k_D}} = \hat{I}_{\mathcal{V}\mathbf{k_D}} \hat{B}_{\mathbf{k_D}} I_{\mathcal{V}\mathbf{k_D}} \qquad I_{i\mathcal{V}} \hat{C} = I_{i\mathcal{V}} \hat{C}_{\mathbf{i_A}} \hat{I}_{\mathbf{i_A}\mathcal{V}}$$

Multiplying this expression by $z^{-1} Q_{\mathbf{k_D}k}$ and removing an uncontrollable state yields

$$\begin{aligned} & z^{-1} I_{i\mathcal{V}} (I + Q N_{21}^{-1} M)^{-1} I_{\mathcal{V}\mathbf{k_D}} Q_{\mathbf{k_D}k} \\ &= \left[\begin{array}{c|c} \hat{A}_{\mathbf{i_A}\cap\mathbf{k_D}} - \hat{B}_{\mathbf{i_A}\cap\mathbf{k_D}} \hat{C}_{\mathbf{i_A}\cap\mathbf{k_D}} & \hat{I}_{\mathbf{i_A}\cap\mathbf{k_D}k} (I_{\mathbf{k_D}k} H_k) \\ \hline I_{i\mathcal{V}} \hat{C}_{\mathbf{i_A}\cap\mathbf{k_D}} & 0 \end{array}\right] \end{aligned}$$

Let us define the shorthand notation,

$$\mathbb{A}_{ik} = \hat{A}_{\mathbf{i_A}\cap\mathbf{k_D}} - \hat{B}_{\mathbf{i_A}\cap\mathbf{k_D}} \hat{C}_{\mathbf{i_A}\cap\mathbf{k_D}} \qquad \mathbb{C}_{ik} = I_{i\mathcal{V}} \hat{C}_{\mathbf{i_A}\cap\mathbf{k_D}}$$

As a result, we have

$$\begin{aligned} R_{ij} &= \sum_k I_{i\mathcal{V}} (I + Q N_{21}^{-1} M)^{-1} I_{\mathcal{V}\mathbf{k_D}} Q_{\mathbf{k_D}k} I_{k\mathcal{V}} N_{21}^{-1} I_{\mathcal{V}j} \\ &= \left[\begin{array}{c|c} \mathbb{A}_{ij} & \mathbb{A}_{ij} \hat{I}_{\mathbf{i_A}\cap\mathbf{j_D}j} I_{\mathbf{j_D}j} - \hat{E}_{\mathbf{i_A}\cap\mathbf{j_D}j} \\ \hline \mathbb{C}_{ij} & -I_{i\mathbf{j_D}} K_j^F I_{\mathbf{j_D}j} \end{array}\right] \end{aligned}$$

where we have defined

$$\hat{E}_{\mathcal{V}j} = \begin{bmatrix} I_{\mathbf{1_D}1} A_{1j}^F \\ \vdots \\ I_{\mathbf{N_D}N} A_{Nj}^F \end{bmatrix}$$

Note that the direct feedthrough term above is equal to $\tilde{D}_{ij}$. Similarly, algebraic manipulations, which must be omitted for space, show that the above expression simplifies to (27). ■

We are now ready to prove Theorem 7.

Proof of Theorem 7. From Lemmas 5 and 6 and Assumptions A2–A4, there exist matrices X_i satisfying the Riccati equations (10). From Lemma 16, this implies that there exist stabilizing solutions to (23). As a result, define K_i^F as in (24). From Lemmas 9, 10, and 11, we know that $\mathcal{K}$ is optimal for (7) if and only if Q is optimal for (20) and $\mathcal{K} = (I + Q N_{21}^{-1} M)^{-1} Q N_{21}^{-1} + F$. Since the optimal Q was found in Lemma 15, the optimal controller $\mathcal{K}$ is given by $\mathcal{K} = R + F$, with R given in Lemma 17.

Notice that

$$A_{\mathbf{k'_D}}^F - B_{\mathbf{k'_D}\mathbf{k_D}} K_K^F I_{\mathbf{k_D}\mathbf{k'_D}} = A_{\mathbf{k'_D}} - B_{\mathbf{k'_D}\mathbf{k_D}} K_k I_{\mathbf{k_D}\mathbf{k'_D}}$$

which implies that $\tilde{A} = A_K$. Similarly, we can show that $\tilde{B} = B_K$, $\tilde{C} = C_K$, and $\tilde{D} = D_K - F$. Thus, $\mathcal{K} = R + F$ satisfies (13). The individual controllers of (14) come from simply computing $u_i = \mathcal{K}_{i\mathcal{V}} x = \mathcal{K}_{i\mathbf{i_A}} x_{\mathbf{i_A}}$. ■

Note that, despite the need for the pre-compensator F to handle unstable plants, the optimal controller, and its existence, is independent of the choice of F.

VII Estimation Structure

While Theorem 7 provides the optimal controller $\mathcal{K} \in \text{Sparse}(\mathcal{M}^{\mathcal{G}}; \mathcal{RP})$, the resulting controller is not particularly intuitive. To better understand our results, let us first consider the classical, centralized problem. In that case, the optimal controller satisfies

$$u(t) = K \hat{x}(t)$$

where $\hat{x}(t)$ is the minimum mean-square error (MMSE) estimate of the state. Thus, the optimal controller separates nicely into the LQR gain K and the Kalman filter dynamics for estimating the state.

However, this is not the case for our decentralized problems. In fact, consider the following policy, which is the natural analog of the centralized result

$$u_i(t) = K_{i\mathcal{V}} \operatorname{E}\big(x(t) \mid x_{\mathbf{i_A}}(0\colon t)\big)$$

That is, each player uses the same LQR gain multiplied by its own estimate of the state conditioned on its known information $x_{\mathbf{i_A}}$. It is straightforward to show that implementing such a policy for a decentralized problem may actually destabilize the system.

Nevertheless, there does exist an intuitive estimation structure for these decentralized problems, and we will demonstrate this in this section. To begin, we find that the closed-loop system dynamics has a particularly nice representation.

Lemma 18. *Let P and $\mathcal{K}$ be defined by (4) and (13), respectively. Then, the closed-loop state dynamics satisfy*

$$\xi(t+1) = A_{cl} \xi(t) + B_{cl} w(t) \tag{28}$$

where

$$A_{cl} = \operatorname{diag}(A_{\mathbf{i_D}} - B_{\mathbf{i_D}} K_i) \qquad B_{cl} = \operatorname{diag}(I_{\mathbf{i_D}i} H_i)$$

Proof Combining the dynamics of (3) and (13), the closed-loop dynamics are given by

$$\begin{bmatrix} x(t+1) \\ \eta(t+1) \end{bmatrix} = \begin{bmatrix} A + B D_K & B C_K \\ B_K & A_K \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} H \\ 0 \end{bmatrix} w(t)$$

The result follows by a straightforward state transformation. ■

Thus, from Lemma 18, we can rewrite the closed-loop dynamics in terms of the states ξ, whose dynamics are decoupled since A_{cl} and B_{cl} are block diagonal. As a consequence of this transformation, the new state ξ satisfies

$$\xi_i(t) = \begin{bmatrix} x_i(t) - \sum_{j \in \mathbf{i'_A}} I_{i\mathbf{j'_D}} \eta_j(t) \\ \eta_i(t) \end{bmatrix} \tag{29}$$

for each $i \in \mathcal{V}$. We can then rewrite the optimal controller in terms of these new states as

$$\begin{aligned} u(t) &= \begin{bmatrix} D_K & C_K \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} \\ &= \begin{bmatrix} -I_{\mathcal{V}\mathbf{1_D}} K_1 & \cdots & -I_{\mathcal{V}\mathbf{N_D}} K_N \end{bmatrix} \xi(t) \end{aligned}$$

This implies that each individual controller is

$$u_i(t) = -\sum_{j \in \mathbf{i_A}} I_{i\mathbf{j_D}} K_j \xi_j(t) \tag{30}$$

While this is a very clean result for the optimal control policy, we can gain additional intuition to it with the following result.

Theorem 19. *Suppose ξ satisfies the dynamics of (28) and (29), with $w_i(t)$ independent, zero-mean random variables. Then,*

$$\xi_i(t) = \mathrm{E}\big(x_{\mathbf{i_D}}(t) \mid x_{\mathbf{i_A}}(t)\big)$$

Proof From Lemma 18, it is clear that $\xi_i(t)$ are independent. Since $\mathbf{i_D} \cap \mathbf{i'_A} = \emptyset$, this implies that

$$\mathrm{E}\big(\xi_{\mathbf{i_D}}(t) \mid \xi_{\mathbf{i'_A}}(0\colon t)\big) = 0$$

As a result, from (29), we have

$$\begin{aligned}
\mathrm{E}\big(x_{\mathbf{i_D}}(t) \mid \xi_{\mathbf{i'_A}}(0\colon t)\big) &= \sum_{\substack{j,k \\ k \,\in\, \mathbf{i_D} \cap \mathbf{j'_D}}} I_{\mathbf{i_D}k} I_{k\mathbf{j'_D}} \, \mathrm{E}\big(\eta_j(t) \mid \xi_{\mathbf{i'_A}}(0\colon t)\big) \\
&= \sum_{j \in \mathbf{i'_A}} I_{\mathbf{i_D}\mathbf{j'_D}} \eta_j(t)
\end{aligned}$$

where $\mathrm{E}\big(\eta_j(t) \mid \xi_{\mathbf{i'_A}}(0\colon t)\big) = \eta_j(t)$ if $j \in \mathbf{i'_A}$, and zero otherwise. In particular, note that

$$\mathrm{E}\big(x_i(t) \mid \xi_{\mathbf{i'_A}}(0\colon t)\big) = \sum_{j \in \mathbf{i'_A}} I_{i\mathbf{j'_D}} \eta_j(t)$$

and

$$\mathrm{E}(x_{\mathbf{i'_D}}(t) | \xi_{\mathbf{i'_A}}(0\colon t)) = \sum_{j \in \mathbf{i'_A}} I_{\mathbf{i'_D}\mathbf{j'_D}} \eta_j(t)$$

By the same construction, we can also show that

$$\mathrm{E}(x_{\mathbf{i'_D}}(t) | \xi_{\mathbf{i_A}}(0\colon t)) = \eta_i(t) + \sum_{j \in \mathbf{i'_A}} I_{\mathbf{i'_D}\mathbf{j'_D}} \eta_j(t)$$

Lastly, note that conditioning on $\xi(0\colon t)$ is equivalent to conditioning on $x(0\colon t)$, $\eta(0\colon t)$. However, since η is simply a function of x, then conditioning on $x(0\colon t)$, $\eta(0\colon t)$ is equivalent to conditioning on $x(0\colon t)$. Thus,

$$\begin{aligned}
\xi_i(t) &= \begin{bmatrix} x_i(t) - \mathrm{E}\big(x_i(t) \mid \xi_{\mathbf{i'_A}}(0\colon t)\big) \\ \mathrm{E}\big(x_{\mathbf{i'_D}}(t) \mid \xi_{\mathbf{i_A}}(0\colon t)\big) - \mathrm{E}\big(x_{\mathbf{i'_D}}(t) \mid \xi_{\mathbf{i'_A}}(0\colon t)\big) \end{bmatrix} \\
&= \mathrm{E}\big(x_{\mathbf{i_D}}(t) \mid x_{\mathbf{i_A}}(0\colon t)\big) - \mathrm{E}\big(x_{\mathbf{i_D}}(t) \mid x_{\mathbf{i'_A}}(0\colon t)\big)
\end{aligned}$$

where it is clear that $x_i(t) = \mathrm{E}\big(x_i(t) \mid x_{\mathbf{i_A}}(0\colon t)\big)$. ∎

From Theorem 19, it is clear that the optimal controllers in (30) can be alternately expressed as

$$u_i(t) = -\sum_{j \in \mathbf{i_A}} I_{i\mathbf{j_D}} K_j \big(\hat{x}_{\mathbf{j_D}|\mathbf{j_A}}(t) - \hat{x}_{\mathbf{j_D}|\mathbf{j'_A}}(t)\big)$$

where we have defined

$$\begin{aligned}
\hat{x}_{\mathbf{j_D}|\mathbf{j_A}}(t) &= \mathrm{E}\big(x_{\mathbf{j_D}}(t) \mid x_{\mathbf{j_A}}(0\colon t)\big) \\
\hat{x}_{\mathbf{j_D}|\mathbf{j'_A}}(t) &= \mathrm{E}\big(x_{\mathbf{j_D}}(t) \mid x_{\mathbf{j'_A}}(0\colon t)\big)
\end{aligned}$$

Thus, we see that, in contrast to the centralized case, the optimal decentralized policies are linear combination of the estimation errors between players.

VIII Examples

VIII-A Two Player System

In [13, 14], the solution for the two player decentralized problem, with one-way communication, was provided. This corresponds to the graph in Figure 1(a). Using the notation of this paper, we have that $\mathbf{1_A} = \{1\}$, $\mathbf{1_D} = \mathbf{2_A} = \mathcal{V} = \{1,2\}$, $\mathbf{2_D} = \{2\}$. As a consequence, it immediately follows from the results here that the optimal controllers satisfy

$$\begin{aligned}
u_1(t) &= -I_{1\mathcal{V}} K_1 \big(\hat{x}_{\mathcal{V}|1}(t) - \hat{x}_{\mathcal{V}|\emptyset}\big) \\
&= -(K_1)_{11} x_1(t) - (K_1)_{12} \hat{x}_{2|1}(t) \\
u_2(t) &= -I_{2\mathcal{V}} K_1 \big(\hat{x}_{\mathcal{V}|1}(t) - \hat{x}_{\mathcal{V}|\emptyset}\big) - K_2 \big(\hat{x}_{2|\mathcal{V}}(t) - \hat{x}_{2|1}(t)\big) \\
&= -(K_1)_{21} x_1(t) - (K_1)_{22} \hat{x}_{2|1}(t) \\
&\qquad - K_2 \big(x_2(t) - \hat{x}_{2|1}(t)\big)
\end{aligned}$$

As expected, these results match the solutions of those previous works.

VIII-B Communication Trade-off

In the 2-player case of [14], there are only three possible communication schemes (when the two subsystems are identical). These correspond to the decentralized cases where either no communication or one-way communication is allowed, and the centralized system with two-way communication.

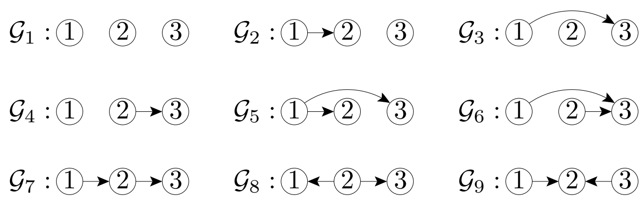
In the 3-player case, there are many more possible communication structures. In this section, we examine the trade-off curves for a number of different communication structures. To this end, we consider the subsystems, given by

$$\begin{bmatrix} x_i^1(t+1) \\ x_i^2(t+1) \end{bmatrix} = \begin{bmatrix} 0.98 & 0.45 \\ -0.09 & 0.80 \end{bmatrix} \begin{bmatrix} x_i^1(t) \\ x_i^2(t) \end{bmatrix} + \begin{bmatrix} 0.12 \\ 0.45 \end{bmatrix} u_i(t) + 0.1 w_i(t)$$

Here, x_i^1 and x_i^2 represent the position and velocity of a simple point mass, respectively. In order for all communication graph structures to admit tractable solutions, we assume that the dynamics of the subsystems are decoupled. In other words, $P_{22} \in \mathrm{Sparse}(I; \mathcal{RP}) \subset \mathrm{Sparse}(\mathcal{M}^{\mathcal{G}^K}; \mathcal{RP})$, for any graph $\mathcal{G}^K$, so that Lemma 2 applies for any communication structure. The cost to minimize is given by

$$\sum_{i=1,2} (x_i^1(t) - x_{i+1}^1(t))^2 + \sum_{i=1,2,3} 0.01 \|x_i(t)\|^2 + \mu u_i(t)^2$$

The first term represents the positional errors between the masses and mass 2; the second term ensures that the states tend toward the origin; and the third term is the total input energy required for control. Thus, we are essentially trading off the input effort with the positional errors between the masses, with a weighting factor of μ. The following nine graph structures are considered.



Note that all of these graph structures are acyclic. Of course, adding cycles is possible, and there are many other possible graphs for this 3-player system. However, some interesting behavior is observed with the above acyclic structures.

By minimizing our cost over various μ, we obtain the following trade-off curves for these graph structures, shown in Figure 3.

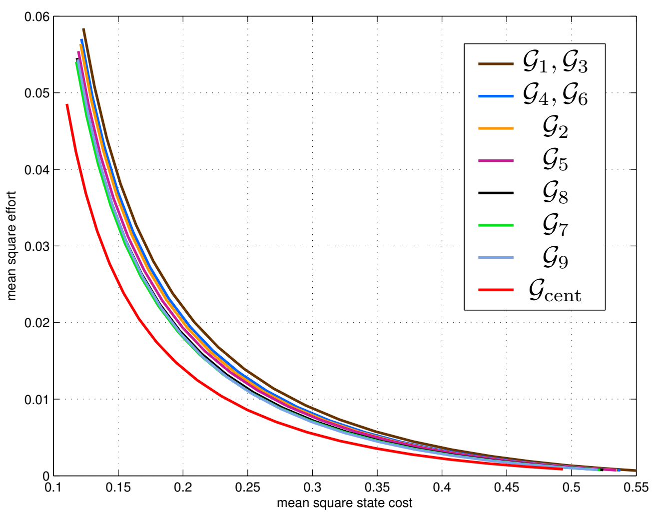


Figure 3: Three-Player Trade-off Curves

For comparison, the centralized solution is included as $\mathcal{G}_{\text{cent}}$. Also, the ordering in the legend is from graphs with higher cost to graphs with lowest cost, with $\mathcal{G}_{\text{cent}}$ having the lowest cost as expected.

Some interesting results can be obtained from Figure 3. First, $\mathcal{G}_1$ and $\mathcal{G}_3$ have the same cost, and $\mathcal{G}_4$ and $\mathcal{G}_6$ have the same, slightly lower cost. In other words, for both $\mathcal{G}_1$ and $\mathcal{G}_4$, there is no benefit to adding a line of communication $1 \to 3$. However, when adding the path $1 \to 3$ to graph $\mathcal{G}_2$, there is an added benefit. The remaining three graphs are also interesting. For all values of μ, graph $\mathcal{G}_8$ always has higher cost than $\mathcal{G}_7$. However, the relative ranking of $\mathcal{G}_9$ varies over μ. In particular, for low values of μ, $\mathcal{G}_9$ has higher cost than $\mathcal{G}_8$, while for higher values of μ, $\mathcal{G}_9$ has lower cost than $\mathcal{G}_7$.

This behavior is very interesting, and not immediately obvious from the graph structures alone. In particular, this analysis would be very useful in cases where communication itself is costly. For instance, if only one channel of communication can be afforded, it is clear that the non-trivial answer would be to use graph $\mathcal{G}_2$. Thus, there is a clear practical use for the analysis that our results provide. Moreover, our results are applicable to much more general settings than the simple system considered here.

IX Conclusion

In this paper, a decentralized control algorithm was developed for systems connected over graphs. Explicit state-space solutions were provided. The approach taken involved a spectral factorization approach, whereby the underlying decentralized problem was decoupled into multiple centralized ones. Upon solving each of these, which involved separate Riccati equations, the optimal decentralized solution could be constructed. By providing analytic results, the order of the optimal controller was established. Moreover, an intuitive view of the optimal controllers was provided in terms of estimation processes. Lastly, a numerical example was provided which demonstrated the power of our results in trading off the number of allowable communication channels.

This work represents one step in a fully general development of optimal decentralized control. In particular, it was assumed here that state feedback was used and communication was immediate and perfect. While some of our work has already extended our results to cases without state feedback [15], future work will investigate the full output feedback case, as well as system with communication delays, or even different objective norms.

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