

Brief paper

# Structured semidefinite programs for the control of symmetric systems<sup>☆</sup>

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## Abstract

In this paper we show how the symmetry present in many linear systems can be exploited to significantly reduce the computational effort required for controller synthesis. This approach may be applied when controller design specifications are expressible via semidefinite programming. In particular, when the overall system description is invariant under unitary coordinate transformations of the state space matrices, synthesis semidefinite programs can be decomposed into a collection of smaller semidefinite programs.

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## 1. Introduction

This paper focuses on the use of symmetry to reduce computational requirements for the design of controllers for a wide range of control objectives. Many systems with symmetries arise in practice. An example is the design of active dampers for drilling towers which have a rotational symmetry with respect to the locations of the dampers (Fagnani & Willems, 1993). Another example, discussed in Hazewinkel and Martin (1983), consists of a fleet of identical ships with dynamic coupling induced by information exchange. The methods described in this paper can be applied to these problems, as well as many others, to simplify the process of synthesizing controllers.

Specifically, in this paper we focus on controller synthesis problems for which an equivalent formulation as a semidefinite program (SDP) exists. There has been significant research on the properties and decomposition of linear systems with

symmetry, and it is known that in many cases the symmetry leads to decomposition of the dynamics into a collection of smaller uncoupled subsystems, for which one may perform controller synthesis directly (Fagnani & Willems, 1993, 1994; Hazewinkel & Martin, 1983; Sundareshan & Elbanna, 1991). This in particular occurs when the performance objective is either an  $H_2$  or an  $H_\infty$  norm. In this paper we show further that, when using semidefinite programming for controller synthesis for symmetric systems, the resulting SDPs are highly structured, which leads to significant computational benefits.

### 1.1. Previous work

The branch of group theory known as *representation theory* and the associated notions of symmetry have been well known and applied, most often in chemistry and quantum mechanics, for over half a century (Weyl, 1950). Techniques for exploiting symmetry in semidefinite programs have appeared in several recent papers. In Kanno et al. (2001), it is shown that symmetry of solutions is preserved when applying interior point algorithms to symmetric semidefinite programs. Specifically, it is shown that the central path followed by an interior point algorithm always consists of symmetric feasible solutions in a symmetric semidefinite program. An interesting consequence of this is that, if search directions are chosen carefully, interior point algorithms will always produce a symmetric

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optimal solution, while this is not true of other nonlinear programming methods in general. In Gatermann and Parrilo (2002) symmetry in semidefinite programs is used to simplify the process of determining sum of squares decompositions of polynomials. The use of such an algebraic structure for simplifying computation is also discussed in Parrilo and Lall (2003).

The role of symmetry in dynamical systems has also been previously studied in many different contexts. An important and well-known result is the equivalence of symmetries and conservation laws in Hamiltonian dynamics, and the use of symmetry reduction for such systems (Marsden & Ratiu, 1994). Also, it has been shown how symmetries can be exploited in the study of bifurcations in nonlinear systems (Golubitsky et al., 1988). For linear systems, properties resulting from symmetry are well-known, and have been analyzed in Fagnani and Willems (1993, 1994), where the decomposition of a symmetric system into smaller uncoupled systems is discussed, and it is shown that certain properties of the symmetric system, such as stability, can be determined by examining the individual uncoupled systems. In Iwata (1993) it is shown that the  $H_\infty$  norm of a symmetric system can be determined from the  $H_\infty$  norms of the uncoupled systems. Dynamical systems which are composed of interconnected subsystems have been studied in Lunze (1986) and Sundareshan and Elbanna (1991), where the symmetry arises due to an invariance under permutation of the subsystems. Without making use of representation theory, the authors show that system stability can be analyzed by considering two significantly smaller, uncoupled systems related to the original system. In related work (Tanaka & Murota, 2000), it is shown how to use group-theoretic methods to study the fault-tolerance properties of arrays of symmetric systems.

Recent work Bamieh et al. (2002) describes how spatial invariance can be exploited in controlling distributed systems. Although the systems discussed in that paper are infinite dimensional, it is shown how the synthesis problems can be decomposed into an infinite family of finite dimensional problems. This decomposition is achieved by taking a Fourier transform with respect to spatial coordinates. The approach of Bamieh et al. (2002) makes use of a decomposition very similar to that presented in this paper, but focuses on the case where the underlying symmetry group is abelian. In another recent paper D'Andrea and Dullerud (2003), the authors consider semidefinite programming formulations of distributed control problems on integer lattices. Such systems are also symmetric with respect to an abelian group. In the follow up work Recht and D'Andrea (2004), the authors extend the results of D'Andrea and Dullerud (2003) to spatially invariant systems with non-commutative lattice structures.

In this paper we consider the general problem of exploiting symmetry to simplify control synthesis. We consider systems which may be symmetric with respect to any finite group, including non-abelian groups. For finite groups the only abelian groups are the cyclic groups, and the restriction to abelian groups would therefore exclude some useful cases, such as the dihedral symmetry of finite arrays and the permutation symmetry when identical subsystems are

completely interconnected. We show that when symmetries are present, synthesis semidefinite programs may be transformed in a straightforward way to yield highly structured semidefinite programs. These results hold for any symmetry and any control synthesis which can be posed as a semidefinite program.

It is known that, when decomposing a symmetric system into decoupled subsystems, the  $H_2$  norm of the original system is given by the sum of the  $H_2$  norms of the decoupled subsystems. Similarly, the  $H_\infty$  norm of the original system is the maximum of the  $H_\infty$  norm of the decoupled subsystems. This leads to immediate computational benefits when using any numerical approach to minimize either of those performance indices. However, for general control design objectives, such a complete decoupling does not occur. In these cases, the benefits of symmetry can still be obtained from the structure in the semidefinite program. In particular, it is in this situation that the methods in this paper offer a new benefit over previously analyzed approaches. This will be demonstrated in this paper for the problem of mixed  $H_\infty/H_2$  synthesis. Application of the methods in this paper to either  $H_2$  or  $H_\infty$  synthesis will also offer computational benefits similar to those obtained by previous decoupling techniques, and in this sense this paper unifies these different approaches.

## 2. Representation theory

Suppose  $G$  is a finite group. A map  $\theta : G \rightarrow \mathbb{C}^{n \times n}$  is called a *representation* if

$$\theta(g)\theta(h) = \theta(gh) \quad \text{for all } g, h \in G.$$

More abstractly,  $\theta$  is a group homomorphism from  $G$  into the group of invertible linear operators on  $\mathbb{C}^n$ . The fixed-point subspace of  $\theta$  is

$$S_{\text{fixed}}(\theta) = \{x \in \mathbb{C}^n \mid \theta(g)x = x \text{ for all } g \in G\}.$$

Representations  $\theta$  and  $\Omega$  are called *equivalent* if they are related by a change of coordinates, that is if there exists an invertible matrix  $U \in \mathbb{C}^{n \times n}$  such that for all  $g \in G$  we have  $U\theta(g)U^{-1} = \Omega(g)$ . This defines an equivalence relation on representations. There always exist coordinates on  $\mathbb{C}^n$  such that  $\theta(g)$  is unitary for all  $g \in G$ . A representation  $\theta$  is called *reducible* if there exist coordinates in which it is block diagonal, that is, if there exists a matrix  $U$  and representations  $\Phi$  and  $\Omega$  such that

$$U^{-1}\theta(g)U = \begin{bmatrix} \Phi(g) & 0 \\ 0 & \Omega(g) \end{bmatrix} \quad \text{for all } g \in G. \quad (1)$$

The representation  $\theta$  is called *irreducible* if it is not reducible. A subspace  $V \subset \mathbb{C}^n$  is called *invariant* under  $\theta$  if  $\theta(g)V \subset V$  for all  $g \in G$ . It is called *proper* if  $V \neq \{0\}$  and  $V \neq \mathbb{C}^n$ . A representation is reducible if and only if there exists a proper invariant subspace, as in the following result.

**Theorem 1 (Maschke's Theorem).** *Suppose  $\theta : G \rightarrow \mathbb{C}^{n \times n}$  is a unitary representation. Then  $\theta$  is reducible if and only if there exists a proper subspace  $V \subset \mathbb{C}^n$  which is invariant under  $\theta$ . In this case, the orthogonal complement  $V^\perp$  is also invariant.*

Hence if  $\theta$  is reducible, let  $V$  be a proper invariant subspace and let  $U_1$  be a matrix whose columns form an orthonormal basis for  $V$ , and choose  $U_2$  so that

$$U = [U_1 \quad U_2]$$

is unitary. This is then a unitary choice of  $U$  which block diagonalizes  $\theta$ .

The set of inequivalent irreducible representations of a group  $G$  is called the *dual group*, denoted  $\hat{G}$ . For a finite group  $G$ , the number of elements of  $\hat{G}$  is finite and equal to the number of conjugacy classes of  $G$ , and we denote this number by  $r$ . This means that every representation  $\theta$  on  $G$  is equivalent to a block-diagonal representation, in which every block is one of the  $\Omega_1, \dots, \Omega_r$ . Specifically, given a unitary representation  $\theta$ , there always exists a unitary matrix  $U$  such that

$$U^* \theta U = \text{diag}(\Omega_1, \dots, \Omega_1, \Omega_2, \dots, \Omega_2, \dots, \Omega_r, \dots, \Omega_r).$$

The basis given by the columns of  $U$  is called the *symmetry-adapted basis* for  $\theta$ . It can be shown that the multiplicity  $p_i$  of  $\Omega_i$  in  $\theta$  is given by

$$p_i = \frac{1}{|G|} \sum_{g \in G} \text{trace}(\theta(g))^* \text{trace}(\Omega_i(g)). \tag{2}$$

### 2.1. Linear maps invariant under symmetries

The above representation theory enables characterization of all linear maps which are invariant under the action of a group. Suppose  $\theta_1 : G \rightarrow \mathbb{C}^{m \times m}$  and  $\theta_2 : G \rightarrow \mathbb{C}^{n \times n}$  are representations of the same group  $G$ . A matrix  $A \in \mathbb{C}^{m \times n}$  is called *intertwining* or *equivariant* with respect to  $\theta_1$  and  $\theta_2$  if

$$\theta_1(g)A = A\theta_2(g) \quad \text{for all } g \in G. \tag{3}$$

If both representations are the same and equal  $\theta$ , we say  $A$  is equivariant with respect to  $\theta$ . The set of  $A$  equivariant with respect to given  $\theta_1$  and  $\theta_2$  is a subspace of  $\mathbb{C}^{m \times n}$ , specifically the fixed-point subspace of the conjugacy representation which for each  $g \in G$  maps  $A \mapsto \theta_1(g)A\theta_2(g)^*$ . The following result will enable explicit parametrization this subspace.

**Lemma 2 (Schur’s Lemma).** *Suppose  $\theta_1 : G \rightarrow \mathbb{C}^{m \times m}$  and  $\theta_2 : G \rightarrow \mathbb{C}^{n \times n}$  are irreducible representations of the group  $G$ . If  $A \in \mathbb{C}^{m \times n}$  is equivariant with respect to  $\theta_1$  and  $\theta_2$  then*

- (i) *If  $\theta_1$  and  $\theta_2$  are inequivalent, then  $A = 0$ .*
- (ii) *If  $\theta_1 = \theta_2$ , then there exists  $b \in \mathbb{C}$  such that  $A = bI$ .*

The proof of this result may be found, for example, in Serre (1977). This allows us to see what happens more generally, when  $\theta_1$  and  $\theta_2$  are reducible. Suppose, after choosing coordinates appropriately, we have reducible representations  $\Delta_1$  and  $\Delta_2$  in the form

$$\Delta_i = \text{diag}(\Omega_1, \dots, \Omega_1, \Omega_2, \dots, \Omega_2, \dots, \Omega_r, \dots, \Omega_r) \tag{4}$$

where  $\Omega_1, \dots, \Omega_r$  are the  $r$  inequivalent irreducible representations of  $G$ , and  $\Omega_i$  has dimensions  $d_i \times d_i$ . Let  $p_i$  be the multiplicity by which  $\Omega_i$  occurs in  $\Delta_1$ , and  $q_i$  be the multiplicity by which  $\Omega_i$  occurs in  $\Delta_2$ . Now partition the matrix  $A$  as

$$A = \begin{bmatrix} A_{11} & \dots & A_{1r} \\ \vdots & & \vdots \\ A_{r1} & \dots & A_{rr} \end{bmatrix}$$

so that  $A_{ij}$  has dimensions  $p_i d_i \times q_j d_j$ . Suppose now  $A$  is equivariant, that is  $\Delta_1(g)A = A\Delta_2(g)$  for all  $g \in G$ . If  $i \neq j$  then  $\Omega_i$  and  $\Omega_j$  are inequivalent, and so by part (i) of Schur’s lemma we have

$$A_{ij} = 0 \quad \text{if } i \neq j.$$

Now consider the  $i$ th block on the diagonal. We have

$$\begin{bmatrix} \Omega_i & & \\ & \ddots & \\ & & \Omega_i \end{bmatrix} A_{ii} = A_{ii} \begin{bmatrix} \Omega_i & & \\ & \ddots & \\ & & \Omega_i \end{bmatrix}.$$

Here the block diagonal matrix on the left has  $p_i$  copies of  $\Omega_i$ , and that on the right has  $q_i$  copies. Then part (ii) of Schur’s lemma implies that

$$A_{ii} = \begin{bmatrix} B_{i11}I & \dots & B_{i1q_i}I \\ \vdots & & \vdots \\ B_{ip_i1}I & \dots & B_{ip_iq_i}I \end{bmatrix}$$

where each  $B_{ijk}$  is a complex scalar and the identity  $I$  has dimension  $d_i$ . We can also write this as the Kronecker product

$$A_{ii} = B_i \otimes I$$

where  $B_i$  is the matrix with entries  $B_{ijk}$ . Define the permutations  $F_{mn}$  by

$$F_{mn} \text{vec}(E) = \text{vec}(E^T) \quad \text{for all } E \in \mathbb{R}^{m \times n}.$$

Then  $F_{d_i p_i} A_{ii} F_{d_i q_i}^T = \text{diag}(B_i, \dots, B_i)$ , where  $B_i$  has dimensions  $p_i$  by  $q_i$ . For convenience define the permutation

$$Z_1 = \text{diag}(F_{d_1 p_1}, \dots, F_{d_r p_r}) \tag{5}$$

and similarly for  $Z_2$ . Then a matrix  $A$  is equivariant under  $\Delta_1$  and  $\Delta_2$  if and only if

$$Z_1 A Z_2^T = \text{diag}(B_1, \dots, B_1, B_2, \dots, B_2, \dots, B_r, \dots, B_r)$$

for some  $B_1, \dots, B_r$  of the above dimensions, and with  $B_i$  having multiplicity  $d_i$ . Thus in (permutations of) the symmetry-adapted bases for  $\Delta_1$  and  $\Delta_2$ , a matrix is equivariant if and only if it is block diagonal with the blocks having the above multiplicities.

The multiplicity of the blocks which occur in this basis is a consequence of the non-scalar irreducible representations of  $\theta$ . These non-scalar blocks occur precisely when the underlying group  $G$  is non-abelian. For abelian groups, every

irreducible representation is scalar, and all blocks there occur with multiplicity 1 in the symmetry-adapted basis for an equivariant map.

More generally, suppose  $C$  is equivariant with respect to representations  $\Theta_i$  so that  $\Theta_1(g)C = C\Theta_2(g)$  for all  $g \in G$ . Let  $U_i$  be the symmetry adapted basis for  $\Theta_i$ , so that  $U_i^* \Theta_i U_i$  is block diagonal. Then

$$T_1^* C T_2 = \text{diag}(B_1, \dots, B_1, B_2, \dots, B_2, \dots, B_r, \dots, B_r)$$

where  $T_i = U_i Z_i^T$ . The matrix  $T_i$  is called the *permuted symmetry adapted basis* for the representation  $\Theta_i$ .

Computationally therefore, all that is needed is an algorithm to find the invariant subspaces of a representation. One simple algorithm to reduce a representation results from the following lemma (Dixon, 1970).

**Lemma 3.** *Suppose  $\Theta : G \rightarrow \mathbb{C}^{n \times n}$  is a unitary representation. Then  $\Theta$  is reducible if and only if there exists a nonzero Hermitian matrix  $H$  such that*

$$\Theta(g)H = H\Theta(g) \quad \text{for all } g \in G$$

and there does not exist  $b \in \mathbb{C}$  such that  $H = bI$ . Further, given such a matrix  $H$ , let  $U$  be a unitary matrix whose columns are eigenvectors of  $H$ , that is  $HU = UD$  for some diagonal matrix  $D$ . Then

$$U^* \Theta(g) U = \text{diag}(\Phi_1(g), \dots, \Phi_m(g)) \quad \text{for all } g \in G \quad (6)$$

where  $\lambda_1, \dots, \lambda_m$  are the distinct eigenvalues of  $H$  and the dimension of  $\Phi_i$  equals the multiplicity of  $\lambda_i$ .

Therefore  $\Theta$  is irreducible if and only if the set of linear constraints on  $H$

$$\Theta(g)H = H\Theta(g) \quad \text{for all } g \in G$$

$$H = H^*$$

$$\text{trace}(H) = 0$$

have a nonzero solution. A basis of eigenvectors of  $H$  in general gives only a partial reduction, since it is not guaranteed that the  $\Phi_i$  are irreducible. Hence to completely reduce a representation one applies this algorithm iteratively to all subrepresentations  $\Phi_i$  until all subrepresentations are irreducible. An alternative approach is possible when the irreducible representations of the group  $G$  are known, with a more explicit formula for the basis; see Serre (1977).

Note that although this section has been phrased in the language of groups, Schur's lemma and many of the above reducibility properties do not require the  $\Theta$  to be a group representation, and they may also be stated for finite sets of invertible matrices. However, if the set of such matrices corresponds to a representation, then the corresponding irreducible representations are known, and their multiplicities (2) and other properties dependent on orthogonality may be used.

### 3. Convex synthesis for symmetric systems

In this section we will discuss the notion of a symmetric state-space system and show that in appropriate coordinates, the system decouples into a collection of independent subsystems. Suppose  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times p}$ ,  $C \in \mathbb{R}^{m \times n}$  and  $D \in \mathbb{R}^{m \times p}$ . As standard, we refer to  $(A, B, C, D)$  as a state-space system, corresponding to the linear equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \quad (7)$$

We consider state-space systems which are symmetric in the following sense.

**Definition 4.** Suppose  $G$  is a finite group with unitary representations  $\Theta_x : G \rightarrow \mathbb{C}^{n \times n}$ ,  $\Theta_y : G \rightarrow \mathbb{C}^{m \times m}$ ,  $\Theta_u : G \rightarrow \mathbb{C}^{p \times p}$ . The state-space system  $(A, B, C, D)$  is called symmetric with respect to  $\Theta_x, \Theta_u, \Theta_y$  if

$$\begin{bmatrix} \Theta_x(g) & \\ & \Theta_y(g) \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Theta_x(g) & \\ & \Theta_u(g) \end{bmatrix}$$

for all  $g \in G$ .

That is, the matrices that define a symmetric state-space system are each equivariant with respect to a specific pair of representations. Now suppose  $G$  has  $r$  inequivalent irreducible representations, and let  $T_x, T_u$  and  $T_y$  be the permuted symmetry-adapted bases for  $\Theta_x, \Theta_u$  and  $\Theta_y$  respectively. Define  $\tilde{B} = T_x^* B T_u$  so that

$$\tilde{B} = \text{diag}(\tilde{B}_1, \dots, \tilde{B}_1, \dots, \tilde{B}_r, \dots, \tilde{B}_r).$$

As in Section 2, each of the  $\tilde{B}_i$  has multiplicity  $d_i$ , where  $d_i$  is the dimension of the  $i$ th irreducible representation of  $G$ . Similarly define  $\tilde{A}$ ,  $\tilde{C}$  and  $\tilde{D}$ . Let  $\tilde{x} = T_x^* x$  and similarly for  $\tilde{u}$  and  $\tilde{y}$ , and we have

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t) \\ \tilde{y}(t) &= \tilde{C}\tilde{x}(t) + \tilde{D}\tilde{u}(t). \end{aligned} \quad (8)$$

Notice that this dynamical system now consists of  $r$  distinct dynamical systems, where the  $i$ th block occurs with multiplicity  $d_i$ .

Notice that this is simply a change of coordinates, and so many analytical properties of the original dynamical system (7) may be verified on the system (8). For the simplest example, the original matrix  $A$  is Hurwitz if and only if the  $r$  blocks  $\tilde{A}_1, \dots, \tilde{A}_r$  are each Hurwitz. The amount of computation required in this case is reduced both as a result of the diagonalization of  $A$  as well as the multiplicities of the blocks of  $\tilde{A}$ .

#### 3.1. Semidefinite programming for control synthesis

Here we discuss a standard control synthesis SDP and use the results of the previous section to show that the presence of symmetry results in computational simplification. We show

a general procedure that may be applied to a wide class of semidefinite programs which arise in control analysis and synthesis problems.

In the example of stability analysis above, a substantial reduction is enabled by representing the system in the symmetry-adapted basis. The amount of computation is reduced partly because stability may be evaluated separately for each block of  $\tilde{A}$ . Many control synthesis problems similarly decouple, such as state-feedback synthesis. In this case, one would like to find a matrix  $K \in \mathbb{R}^{p \times n}$  such that  $A + BK$  is Hurwitz. In the symmetry-adapted basis, one need simply find  $r$  matrices  $\tilde{K}_1, \dots, \tilde{K}_r$  such that  $\tilde{A}_i + \tilde{B}_i \tilde{K}_i$  is Hurwitz. In other words, the synthesis SDP reduces to a collection of uncoupled synthesis problems. In this case, one could alternatively first decouple the state space matrices into their block diagonal form, then perform any synthesis on the individual blocks.

For another example, consider a multiobjective controller synthesis problem, where one would like to minimize an  $H_\infty$  norm subject to a constraint on the  $H_2$  norm. Even though this problem does not decouple into  $r$  separate synthesis problems, one may still achieve substantial computational benefits from the symmetry. To do this, we apply the symmetry reduction to the convex program corresponding to the desired synthesis, thereby exploiting symmetry directly in the synthesis SDP. Here we present only the continuous time case, however identical results can be obtained in the discrete time case by similar arguments.

Suppose  $\theta : G \rightarrow \mathbb{C}^n$  is a representation, and  $C \subset \mathbb{C}^n$  is a convex set. Then  $C$  is called *invariant* if

$$\theta(g)C \subset C \quad \text{for all } g \in G.$$

The following result enables the computational reduction.

**Theorem 5.** *Suppose  $C \subset \mathbb{C}^n$  is convex and  $C \neq \emptyset$ . Then  $C \cap S_{\text{fixed}}(\theta) \neq \emptyset$ .*

**Proof.** The proof is by construction. Let  $y \in C$ , and define

$$x = \frac{1}{|G|} \sum_{g \in G} \theta(g)y.$$

Then since  $C$  is invariant,  $\theta(g) \in C$  for all  $g \in G$ , and since  $x$  is a convex linear combination of elements of  $C$  we have  $x \in C$  also. One can immediately verify that this constructed  $x$  satisfies  $x \in S_{\text{fixed}}(\theta)$ .  $\square$

Hence, for a convex programs in which the feasible set is invariant under a group action  $\theta$ , we can limit the search for feasible points to the fixed-point subspace of  $\theta$ .

### 3.2. Multiobjective control

In the cases of stabilization,  $H_2$ , and  $H_\infty$  control, the synthesis SDPs completely decouple into a set of smaller independent synthesis SDPs. In these cases, one could first transform each of the system matrices into their block diagonal form, then independently apply any synthesis procedure to each block. *However, such a complete decoupling does not*

*always occur, and it is in precisely these situations where the benefits of exploiting symmetry at the level of the control synthesis SDP are realized.* Hence in the following example we consider a multiobjective problem of minimizing the system  $H_\infty$  norm subject to a constraint on the  $H_2$  norm. In this case, the synthesis SDP almost completely decouples, except for a single constraint. This example illustrates the idea that, given the choice between applying symmetry at the system level by decoupling the system matrices, or at the level of the SDP, the latter approach provides a systematic simplification in situations where the former does not.

Suppose we have a system with a state space realization

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_w w(t) + B_u u(t) \\ z(t) &= C_z x(t) + D_z u(t) \\ y(t) &= C_y x(t) + D_y u(t) \end{aligned} \tag{9}$$

where  $u$  is the control input and  $w$  is the disturbance input. We would like to determine a static state feedback control law  $u(t) = Kx(t)$  which minimizes the closed loop  $H_\infty$  norm from  $w$  to  $y$  subject to the constraint that the  $H_2$  norm from  $w$  to  $z$  is less than some specified constant  $\beta$ .

A simple convex relaxation for this problem is as follows. We solve the semidefinite program

$$\begin{aligned} &\text{minimize } \gamma \\ &\text{subject to } F(\gamma, X, Z) < 0 \\ & \quad H(W, X, Z) > 0 \\ & \quad \text{trace}(W) < \beta \\ & \quad X > 0. \end{aligned} \tag{10}$$

Here the decision variables are  $\gamma, X, Z$  and  $W$ . The matrix  $H$  is an affine function of the decision variables, given by

$$H(W, X, Z) = \begin{bmatrix} X & XC_z^* + Z^* D_z^* \\ C_z X + D_z Z & W \end{bmatrix}$$

and similarly  $F$  is the affine function

$$F(\gamma, X, Z) = \begin{bmatrix} F_{11} & F_{12} \\ F_{12}^* & F_{22} \end{bmatrix}$$

where

$$\begin{aligned} F_{11} &= AX + XA^* + B_u Z + Z^* B_u^* \\ F_{12} &= XC_y^* + Z^* D_y^* \\ F_{22} &= -\gamma I. \end{aligned}$$

Then given the optimal solution of this SDP, one constructs the controller

$$K = ZX^{-1}$$

and this controller then achieves the specifications that the  $H_\infty$  norm from  $w$  to  $y$  is less than or equal to  $\sqrt{\gamma}$  and the  $H_2$  norm from  $w$  to  $z$  is less than  $\beta$ .



Now suppose that the state-space system (9) is equivariant in the sense that

$$\begin{bmatrix} \Theta_x(g) & & \\ & \Theta_z(g) & \\ & & \Theta_y(g) \end{bmatrix} \begin{bmatrix} A & B_w & B_u \\ C_z & 0 & D_z \\ C_y & 0 & D_y \end{bmatrix} \\ = \begin{bmatrix} A & B_w & B_u \\ C_z & 0 & D_z \\ C_y & 0 & D_y \end{bmatrix} \begin{bmatrix} \Theta_x(g) & & \\ & \Theta_w(g) & \\ & & \Theta_u(g) \end{bmatrix}$$

for all  $g \in G$ . Then, as in the previous section, for each  $\gamma \in \mathbb{R}$  the feasible set satisfying the constraints (10) is invariant under the group representation  $\Gamma$ , which for each  $g \in G$  maps  $X, W, Z$  to  $X_1, W_1, Z_1$  where

$$X_1(g) = \Theta_x(g)^* X \Theta_x(g)$$

$$W_1(g) = \Theta_x(g)^* W \Theta_x(g)$$

$$Z_1(g) = \Theta_u(g)^* Z \Theta_x(g).$$

Hence, Theorem 5 implies that we can add the constraints that  $X, W, Z$  be equivariant to (10) without changing the optimal value. Next we use Schur's lemma to make these constraints explicit, since we have

$$T_x^* X T_x = \text{diag}(\tilde{X}_1, \dots, \tilde{X}_1, \dots, \tilde{X}_r, \dots, \tilde{X}_r)$$

and similarly for  $W$  and  $Z$ . Also  $F$  and  $H$  are equivariant in the following sense

$$\begin{bmatrix} \Theta_x(g) & \\ & \Theta_y(g) \end{bmatrix} F = F \begin{bmatrix} \Theta_x(g) & \\ & \Theta_y(g) \end{bmatrix} \\ \begin{bmatrix} \Theta_x(g) & \\ & \Theta_z(g) \end{bmatrix} H = H \begin{bmatrix} \Theta_x(g) & \\ & \Theta_z(g) \end{bmatrix}$$

for all  $g \in G$ . Hence we have

$$\begin{bmatrix} T_x^* & \\ & T_z^* \end{bmatrix} H \begin{bmatrix} T_x & \\ & T_z \end{bmatrix} = \text{diag}(\tilde{H}_1, \dots, \tilde{H}_1, \dots, \tilde{H}_r, \dots, \tilde{H}_r)$$

where

$$\tilde{H}_i = \begin{bmatrix} \tilde{X}_i & \tilde{X}_i \tilde{C}_{zi}^* + \tilde{Z}_i^* \tilde{D}_{zi}^* \\ \tilde{C}_{zi} \tilde{X}_i + \tilde{D}_{zi} \tilde{Z}_i & \tilde{W}_i \end{bmatrix}$$

and similarly for  $\tilde{F}$ . Therefore (10) is equivalent to the following SDP, which has repeated, smaller constraints.

$$\begin{aligned} & \text{minimize } \gamma \\ & \text{subject to } \tilde{F}_i(\gamma, \tilde{X}_i, \tilde{Z}_i) < 0 \quad \text{for } i = 1, \dots, r \\ & \tilde{H}_i(\tilde{W}_i, \tilde{X}_i, \tilde{Z}_i) > 0 \quad \text{for } i = 1, \dots, r \\ & \tilde{X}_i > 0 \quad \text{for } i = 1, \dots, r \\ & \sum_{i=1}^r d_i \text{trace}(\tilde{W}_i) < \beta. \end{aligned}$$

Given a solution  $\tilde{X}, \tilde{W}, \tilde{Z}$ , we can finally construct a real controller from complex solutions by taking  $\hat{K} = \text{Re}(\tilde{Z})\text{Re}(\tilde{X})^{-1}$ .

## 4. Conclusions

In this paper it was shown that certain types of symmetry which may be present in the state space realizations of linear systems may be exploited to significantly reduce the amount of computation required for controller synthesis. Specifically, when symmetries are present, we can decompose the synthesis SDPs into a collection of smaller, coupled SDPs. Depending on the particular symmetry of the system, we may also have repeated SDPs in the decomposition, further simplifying the synthesis of optimal controllers.

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