Data Tracking under Competition

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Abstract

We explore the welfare implications of data tracking technologies that enable firms to collect consumer data and potentially use it for price discrimination. The model we develop centers around two features: first, competition between firms and, second, consumers’ level of sophistication. Our baseline environment features a firm that can collect information about the consumers it transacts with in a duopoly market, which it can then use in a second monopoly market. We characterize and compare the equilibrium outcomes in three settings of interest: (i) an economy with myopic consumers, who, when making purchase decisions, do not internalize the fact that firms have the ability to track their behavior and use this information in future transactions, (ii) an economy with forward-looking consumers, who take into account the implications of data tracking when determining their actions, and (iii) an economy where no data tracking technologies are used either due to technological or regulatory constraints. We find that the absence of data tracking may lead to a decrease in consumer surplus, even when consumers are myopic. Importantly, this result relies critically on competition: consumer surplus is higher when data tracking technologies are used in the marketplace only when multiple firms offer substitutable products to consumers. Our results contribute to the debate of whether to regulate firms’ use of data tracking technologies by illustrating that their effect on consumers depends not only on their level of sophistication, i.e., the extent to which they internalize how their data may be used, but also on the degree of competition in the market. Finally, in contrast to earlier work, we show that firms may have no incentive to self-regulate their use of consumer data even when consumers fully internalize and anticipate how their data may be used.

1 Introduction

As firms adopt and employ increasingly sophisticated information technology tools, economic transactions not only involve the exchange of goods and services, but also generate potentially valuable information about consumers’ preferences. The availability of such data provides firms with the opportunity to personalize their interactions with their customers, for example, by offering personalized prices, promotions, or product recommendations.

Motivated by these observations, we study an economy with multi-product firms, which can leverage the information generated by early transactions to learn consumer-specific attributes and

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use them to personalize subsequent interactions. This dynamic arises in several settings of interest. For example, in the process of applying for a loan, customers disclose detailed information about their net worth, sources of income, and financial goals. Such data is often used by financial services firms to personalize subsequent product offerings (Kim and Wagman (2015)). As another example, the automotive industry is increasingly adopting data tracking technologies to better understand the driving behavior of its customers. Such information may be useful in several application domains ranging from predictive maintenance to insurance products tailored to individual drivers. Finally, online retailers that operate in several product markets track and may use consumer information from one market to tailor their offerings in another.

While the personalization enabled by data tracking may benefit consumers, e.g., by allowing firms to make better product recommendations, there are also potential downsides, e.g., using customer-specific data for price discrimination. Not surprisingly, this has sparked a debate among firms, consumer advocates, and regulatory authorities centered around consumers’ privacy (Goldfarb and Tucker (2019)). In fact, several firms offer “privacy,” i.e., a commitment not to share or exploit the data they collect on consumers, as an appealing feature of their products. Our primary goal in this paper is to better understand the implications of data tracking technologies primarily for consumer welfare but also for other market-wide outcomes. In particular, we focus on two potentially first-order features affecting the interaction between firms and consumers in data rich environments: (i) the structure of competition in the market, and (ii) the level of consumer sophistication, i.e., the extent to which consumers understand how information pertaining to their interactions with firms may be used in the future.

In order to study these questions, we consider a stylized model of an economy with two firms and two products. The first product market is a duopoly, whereas one of the firms (firm B) is a monopoly in the second product market. The interaction between firms and consumers takes place over two time periods. In the first period, firms compete in prices for product 1. After consumers make their purchase decisions for product 1, in the second period, firm B sets its price for product 2, which consumers choose to buy or not. We introduce data tracking in the economy by assuming that firm B collects information about the consumers it transacts with in the first product market that it can then use to offer personalized prices to those same consumers when selling the second product. In addition, we mainly consider forward-looking consumers who anticipate that buying the first product from firm B may lead to price discrimination in the second product market. Thus, consumers may have an incentive to avoid transacting with firm B in order to preserve their “privacy,” whose value in the context of our model is not exogenously given but

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1As an example of a firm moving towards this direction, General Motors recently announced plans to sell car insurance and use individual driving data to set personalized prices (https://www.wsj.com/articles/gm-wants-to-not-only-sell-cars-but-insure-them-too-11605708000).

2When Amazon acquired Whole Foods in 2017, the Wall Street Journal reported that part of the acquisition’s value would result from enabling Amazon to gather customer-specific information about grocery shopping habits (https://www.wsj.com/articles/big-prize-in-amazon-whole-foods-deal-data-1497951004).

3For example, Apple recently launched ads that emphasize the iPhone’s privacy protection features (https://www.theverge.com/2020/9/3/21420108/apple-new-over-sharing-ad-privacy-security-iphone).
can be endogenously quantified based on the purchasing decisions consumers make and the prices they are quoted. Furthermore, as a way to better understand the relationship between consumers’ understanding of how their data may be used and market-wide outcomes, we also consider a setting with consumers who act myopically, i.e., they do not internalize the future consequences of them revealing information to firm B when choosing to transact with it in the first product market.

Finally, motivated by the broader debate on privacy, we also consider whether consumers may benefit from privacy-protecting regulations, i.e., imposing constraints on firms in using the consumer data they obtain through tracking technologies. To this end, we study a restricted setting, where firm B has no access to tracking technologies and therefore its operations in the two product markets are decoupled as far as information on individual consumers is concerned.

Our results can be briefly summarized as follows. First, we establish that as long as the value of consumer data is not exceedingly high, consumers are worse off in an economy in which firms do not employ data tracking, due to technological or regulatory constraints, even when they do not internalize how their information may be used in future transactions. The intuition behind this result lies in the fact that data tracking provides an incentive for firm B to subsidize its product in the first market as a way of increasing its share of transactions and obtaining information about a larger fraction of the consumer population. In turn, competition drives prices lower for both firms in the first product market relative to a setting without tracking. Finally, we show that the benefit of lower prices in the first market offsets the potentially adverse effects of price discrimination in the second one.

Competition is a key driver of this result. Indeed, we also consider an alternative economy in which firm B is a monopolist in both product markets. Then, data tracking always leads to a decrease in consumer surplus when consumers are myopic and, typically, i.e., for a wide range of parameters, it makes forward-looking consumers worse-off as well. To some extent, this result contributes to the debate on whether regulating such technologies benefits consumers, by illustrating that their effect depends not only on the value of consumer information or the consumers’ level of sophistication, but also on how competitive the relevant market(s) are.

Furthermore, we establish that firms with data tracking ability have no incentive to self-regulate, i.e., in the context of our model, data tracking makes firm B better off even when consumers fully internalize the effect of their actions on firms’ pricing strategies. This result is in stark contrast with recent literature, which suggests that firms may find optimal to self-regulate their use of data tracking when consumers develop privacy concerns, i.e., when they are forward-looking in the context of our model (refer to Acquisti, Taylor, and Wagman (2016) for an extensive discussion). Finally, we show that the gains resulting from data tracking and the ensuing competition between the firms are unevenly distributed among consumers. In particular, we establish that consumers at the lower end of the willingness-to-pay spectrum benefit the most.
1.1 Related literature

Our work is related to the growing literature on the economics of data and privacy (for an excellent survey, refer to Acquisti et al. (2016)). A series of papers in this stream study the implications of firms having the ability to set personalized prices based on consumer information. Taylor (2004) considers the three settings we study in an economy with two products that are sold by two different monopolists. In his model, the firm that sells the first product collects information about consumers that it can then sell to the second firm. Villas-Boas (2004) studies an infinite-horizon model with a monopolist that may set different prices to new or repeating customers, while Acquisti and Varian (2005) analyze a model where a monopolist can condition prices based on consumers’ purchase histories. In addition, Conitzer, Taylor, and Wagman (2012) consider consumers that have access to “anonymizing technologies,” i.e., they can choose not to reveal their information when transacting with a firm by incurring a cost. Finally, Bonatti and Cisternas (2020) study a setting where consumers face a sequence of monopolists that have access to a score that aggregates consumers’ purchase histories and can be used for price discrimination.

There are two main features that differentiate our work from these papers. First, we study the interplay between competition and the degree of consumer awareness of firms’ data tracking practices while they focus on settings without competition. Second, they find that it may be optimal for firms to commit not to use consumer data for pricing if consumers are forward-looking. By contrast, in our setting, we establish that the firm which has access to data tracking has no such incentive, i.e., it always profits from using the data it generates even if consumers are forward-looking. Thus, in contrast to much of the prior work, we focus on a setting in which firms have no incentive to self-regulate their use of consumer data, which we believe makes our findings relevant in the context of the recent debate around consumer privacy.

A number of recent papers focus on studying competition dynamics in the context of data acquisition and privacy. In particular, Kim and Wagman (2015) study competition among financial services providers that may sell data resulting from loan applications. They provide empirical evidence to illustrate that privacy-oriented regulation may result in under-collection of information and subsequent efficiency losses. Closer to our work, Ali, Lewis, and Vasserman (2020) study a setting where consumers have the option to voluntarily disclose their data to firms, which in turn use it to price discriminate. They find that whether voluntary disclosure improves welfare depends on the flexibility of the consumers’ ability to communicate, i.e., how much information consumers can share with firms, and market competitiveness. While related, our setting differs from theirs in a number of ways. First, we consider a firm that may infer consumer information solely based on their actions, e.g., prior transactions, rather than by direct communication/disclosure. In addition, we study asymmetric competition where only one of the firms can directly benefit from gathering and exploiting data. This enables us to quantify the endogenous value of a “privacy premium” associated with consumers being able to protect their information. Finally, our model allows us to relate the implications of data tracking with the consumers’ level of awareness of how their data may be used, which is becoming increasingly relevant in the context of efforts to make firms’
practices around data collection more transparent.

Also related, a number of recent papers have focused explicitly on information acquisition in settings where third parties, i.e., data brokers, sell information to firms. In particular, Bimpikis, Crapis, and Tahbaz-Salehi (2019) study a setting where firms may acquire information from a data broker and find that the latter’s optimal selling strategy depends on the competitive structure among firms, and in particular, that it may be profitable for the data broker to restrict the sale of information to only a limited number of firms. More recently, Drakopoulos and Makhdoumi (2020) study a dynamic setting and establish conditions under which data brokers have the incentive to provide free data samples to the buyer early in the process as a way of inducing subsequent purchases. We refer the reader to Bergemann and Bonatti (2019) for a comprehensive survey on the broader literature that studies markets for information. Our work differs from this branch of the literature in several aspects. In particular, the central feature of our modeling framework is the fact that firms learn consumer information from previous transactions rather than by acquiring a dataset from a third party.

In addition, a strand of the literature has focused on the relationship between policies around privacy protection and the equilibrium outcomes they induce. For example, Casadesus-Masanell and Hervas-Drane (2015) study firms that compete along two dimensions – prices and privacy policies. Firms in their model collect revenue from consumer purchases and from selling the consumer data they generate in a secondary market. Cummings, Ligett, Pai, and Roth (2016b) study a setting where an advertiser observes a differentially private signal that results from consumer behavior and show that increasing the privacy level of this signal may lead to a decrease in consumer surplus and make the advertiser better off. Fainmesser, Galeotti, and Momot (2019) analyze firms’ choices of data collection and data protection policies, and the relationship of these choices with the firms’ revenue collection models. They argue that to induce the socially optimal outcome, regulation should not focus entirely on data protection, but also on taxing the amount of information collected by firms. Ichihashi (2020b) presents a model in which consumers may disclose personal information to a firm that may use it to generate product recommendations but also for price discrimination, and analyzes the relationship between their choices and the firm’s privacy policies. Our results contribute to this debate as we establish that limiting firms’ data tracking abilities may lead to a decrease in consumer surplus, even when consumers act myopically, i.e., they do not internalize how firms may use the data they generate from past transactions.

Furthermore, there is a large literature that focuses on decision-making in the presence of privacy concerns. In particular, the notion of differential privacy has been studied in a variety of settings (see Dwork and Roth (2014) for an introduction to the topic). Broadly, the focus is on how to perturb datasets in order to ensure that decision-makers can use them without being able to identify information about specific individuals. Recent contributions in this area include Lei, Miao, and Momot (2020), who study algorithms for personalized pricing that satisfy differential privacy.

\footnote{A related, growing literature considers fairness considerations. In particular, Cohen, Elmachtoub, and Lei (2019) study a personalized pricing problem in the presence of fairness constraints.}
privacy requirements, and Hastings, Hemenway Falk, and Tsoukalas (2020), who develop algorithms for network analytics that satisfy privacy constraints in the context of financial networks. Also related, Cummings, Echenique, and Wierman (2016a) study choice theory with consumers that have intrinsic preferences for privacy, whereas Acemoglu, Makhdoumi, Malekian, and Ozdaglar (2017) explore a network formation game where agents trade off the benefits of adding friends to their social network with the associated privacy loss resulting from sharing personal information. By and large, in this literature, the value for privacy is an exogenous modeling primitive that is taken as fixed and given. In contrast, a key driving force behind our results is that in our model consumers develop privacy concerns endogenously, i.e., due to the potential of their data being used for personalized pricing.

A series of recent papers, such as Acemoglu, Makhdoumi, Malekian, and Ozdaglar (2021), Bergemann, Bonatti, and Gan (2020), Liang and Madsen (2020), and Ichihashi (2020a), study externalities in a context where data has a social dimension, i.e., data pertaining an individual is informative about her peer group. While we do not explicitly consider such correlations, externalities arise in our model since the actions of some consumers (rather than the data itself) may provide information about others.5

Lastly, the literature has studied widely the importance of taking into account the degree of consumer sophistication in a variety of pricing and revenue management settings. For example, Cachon and Swinney (2009) study markdown policies for a retailer that faces both myopic and forward-looking consumers. Swinney (2011) explores the value of quick response production practices in the presence of strategic consumers. Besbes and Lobel (2015) provide a characterization of the optimal pricing strategy for a monopolist given a steady arrival of consumers who strategically time their purchases, whereas Cachon and Feldman (2015) establish that a firm might find it optimal to offer frequent price discounts, even in the presence of strategic consumers. Lobel, Patel, Vulcano, and Zhang (2016) study the optimal timing of new product launches in a related setting.6 We contribute to this literature by exploring the interplay between the consumers’ degree of sophistication, i.e., whether they are myopic or forward-looking in their behavior, and the welfare implications of data tracking practices.

The rest of the paper is organized as follows. We introduce our model and discuss the implications of the main assumptions in Section 2. The characterization of equilibria in the three settings we consider follows in Section 3. Then, we discuss the welfare implications of equilibrium behavior in Section 4. Finally, we conclude and discuss directions for further research in Section 5. The proofs of our main results results are presented in the Appendix, while the remaining proofs and the extension of the main model to the monopoly setting can be found in an Electronic Companion.

5Recently, Aridor, Che, and Salz (2020) provide empirical evidence to support such a dynamic in the context of the European Union’s GDPR. They show that as some consumers opted out from data collection, it became easier for firms to track and interpret the data associated to consumers that did not opt out.

6In addition, there is growing interest in applications of information design in operational settings where firms influence the decision-making of rational agents by appropriately disclosing their information. Representative papers include Kucukgul, Ozer, and Wang (2020) and Drakopoulos, Jain, and Randhawa (2021).
Nature draws the consumer’s type \( (s, \theta) \sim \mu_0 \) 

Firms set prices \( p^A_1, p^B_1 \) for product 1 

Firm B updates its beliefs on the consumer’s type and sets price \( p^B_2 \) for product 2 

Payoffs are realized

\[ t = 0 \quad t = 1 \quad t = 2 \]

Consumer decides which firm to buy product 1 from 

Consumer decides whether to buy product 2

Figure 1: Timeline of the game

2 Model

Our model economy consists of a consumer, two firms, A and B, and two products, 1 and 2. Both firms sell differentiated variants of product 1, while only firm B sells product 2.\(^7\) The interaction between the consumer and the firms proceeds as follows. First, the consumer decides from which firm to buy product 1 based on the prices chosen by the firms and her type, which captures both her preference for firm A’s product 1 relative to firm B’s (modeled as the consumer’s location on a Hotelling line) and her valuation for product 2. Then, firm B sets a price for product 2, which may depend on the consumer’s purchase decision for product 1, and the consumer chooses whether to buy the product. Figure 1 illustrates the timing of events in the model economy we consider.

A key feature in our framework is that firm B has the ability to observe the type of the consumer after she buys product 1 from it. In addition, firm B can use this information to set its (personalized) price for product 2. We refer to this as data tracking. Although we mainly focus on an environment where firm B employs data tracking and the consumer is forward-looking, i.e., she internalizes the implications of her actions (purchase decisions) on firms’ future pricing, we also consider two additional benchmarks: (i) a restricted setting, where data tracking is not possible due to technological or regulatory constraints, and (ii) a setting where the consumer is myopic, i.e., she makes purchase decisions solely to maximize her per-period utilities.

In what follows, we provide a formal description of our framework in the presence of data tracking and forward-looking consumers and also briefly describe the additional settings we consider in our analysis.

**Consumer** The consumer’s type is two-dimensional and consists of: (i) her location in the Hotelling line for product 1, which we denote by \( s \in [0, 1] \) with the endpoints of the interval representing the locations of firms A and B, respectively, and (ii) her valuation for product 2, which we denote by \( m\theta \) with \( \theta \in [0, 1] \), where \( m \) is a parameter meant to capture the value of consumer data to firms as will become evident in what follows.\(^8\) Thus, the consumer can be

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\(^7\)To ease exposition, we assume that firms face no production costs. Our results extend to the case where both firms have equal and constant marginal costs for product 1.

\(^8\)For an intuitive interpretation of \( m \), note that since we assume that \( \theta \) is uniformly distributed in \([0, 1]\), the prior distribution of the consumer’s valuation for product 2 is uniform in \([0, m]\). Therefore, the increase in firm B’s expected profit that results from observing the consumer’s type before setting its price for product 2, i.e., the value firm B attaches to consumer data, is increasing in \( m \).
represented by her type $\tau = (s, \theta) \in \mathcal{T} = [0, 1] \times [0, 1]$, which we assume is ex ante known to her but not to the firms. Throughout the paper, we assume that the consumer’s type $(s, \theta)$ is drawn from the uniform distribution in the unit square, which, in turn, implies that $s, \theta$ are independent.

First, the consumer interacts with the firms in product market 1. In particular, she observes their prices, $p_A^1$ and $p_B^1$, and decides whether to buy product 1 from either firm. When buying product 1, the consumer derives a baseline utility of $\bar{u} > 0$ and faces linear transportation costs, where the unit transportation cost is normalized to $1/2$. To simplify exposition, we assume that the consumer always buys product 1 from one of the firms, i.e., we explicitly restrict the consumer’s actions in period 1 to be whether she buys product 1 from firm A or B.\footnote{The characterization of equilibria in our model extends to the setting where consumers are allowed to forgo buying product 1 (for a payoff of zero) if we assume, as is common in the literature, that $\bar{u}$ in (2.1) is large enough so that all consumers buy product 1 in equilibrium. In particular, it can be shown that $\bar{u} \geq 4/5$ suffices to achieve this.}

We denote the consumer’s action in the first period by $a_1 \in \{0, 1\}$, where $a_1 = 1$ denotes buying from firm B. Thus, the consumer’s utility associated with product 1 given her action, her type, and the prices is equal to

$$u_1(a_1; s, \theta, p_A^1, p_B^1) = (1 - a_1) \left( \bar{u} - s/2 - p_A^1 \right) + a_1 \left( \bar{u} - (1 - s)/2 - p_B^1 \right).$$

(2.1)

Then, in period 2, the consumer observes firm B’s price $p_B^2$ and decides whether to buy product 2 ($a_2 = 1$) or not ($a_2 = 0$). Thus, the consumer’s utility associated with product 2 can be written succinctly as

$$u_2(a_2; s, \theta, p_B^2) = a_2 \left( m\theta - p_B^2 \right),$$

(2.2)

where recall that $m\theta$ is the valuation for product 2 for a consumer with type $(s, \theta)$. Finally, the consumer’s aggregate utility is then the sum of the utilities she derives from each product.

**Firms** At the beginning of period 1, both firms set prices $p_A^1, p_B^1 \in \mathbb{R}$. Given product 1 prices, the consumer’s action $a_1$, and assuming that production costs are normalized to zero, the profits associated with product 1 for firms A and B are, respectively,

$$\pi_A^1(p_A^1, p_B^1, a_1) = (1 - a_1) p_A^1 \quad \text{and} \quad \pi_B^1(p_A^1, p_B^1, a_1) = a_1 p_B^1.$$

(2.3)

Firm A participates only in product market 1, so its total payoff is $\pi_A^1$. At the beginning of period 2, firm B sets a price $p_B^2$ for product 2, and given the consumer’s action $a_2$, generates profit

$$\pi_B^2(p_B^2, a_2) = a_2 p_B^2.$$

(2.4)

Much of our analysis centers on the implications of data tracking, i.e., the ability of firms to collect/infer consumer information in a transaction and use it in subsequent transactions (e.g., to set personalized prices). Given our focus, we describe in detail the timeline of the game and the corresponding histories and information available to firms and the consumer at each stage.
Timeline, histories, and information structure The timeline of the game is illustrated in Figure 1. The game begins with period \((t = 0)\) when the consumer’s type is drawn, followed by periods \((t = 1, 2)\) during which firms and the consumer transact. Each of these periods begins with firms setting prices (with only firm B in period 2), followed by the consumer’s purchasing decision.

In what follows, we formally define the histories and information available to players at each stage of the game. First, we introduce some notation. We denote a generic history by \(h\). Given that the consumer knows her type and observes the firms’ actions, i.e., prices, the information she has at her disposal when making purchase decisions is the full history \(h\). Note that this is generally not true for firms, which may set prices based on incomplete information. Thus, given a history \(h\) where one or both firms move, we denote the information available to them by \(\mathcal{J}(h)\). In other words, \(\mathcal{J}(h)\) represents the components of history \(h\) that are observed by firms when setting their prices. We now describe the histories at each stage of the game. Note from Figure 1 that, within each period \(t = 1, 2\), the consumer’s purchasing decision follows firms’ pricing decision. Thus, in what follows, we denote the collection of possible histories before the consumer moves in period \(t\) by \(H^c_t\). Similarly, we use \(H^f_t\) for the collection of histories that immediately precede the firms’ move in period \(t\).

Period 0: The game begins with the empty history \(h = \emptyset\). At time \(t = 0\), Nature draws the consumer’s type \((s, \theta) \in \mathcal{T} = [0, 1] \times [0, 1]\) according to the uniform distribution over the unit square, and the consumer observes her type.

Period 1: The history at the beginning of period 1 consists only of the realization of the consumer’s type. Thus, we can write the history at this stage as \(h = (s, \theta)\). We denote the collection of all such histories by \(H^f_1\). Next, we describe the firms’ and the consumer’s sequential actions in period 1:

(i) Firms. Both firms set their product 1 prices \(p^A_1\) and \(p^B_1\) simultaneously at the beginning of period 1. We assume that firms do not know the consumer’s type when making this decision: given history \(h \in H^f_1\), firms observe \(\mathcal{J}(h) = \emptyset\) (i.e., no information) and choose product 1 prices.\(^{10}\)

(ii) Consumer. The consumer then makes her purchasing decision for product 1 after observing firms’ prices for product 1, i.e., after histories of the form \(h = (s, \theta, p^A_1, p^B_1)\). We denote the collection of all such histories by \(H^c_1\).

Period 2: The history at the beginning of period 2 consists of the consumer’s type, firms’ prices for product 1, and the consumer’s purchasing decision for product 1, i.e., we can write the history at the beginning of period 2 as \(h = (s, \theta, p^A_1, p^B_1, a_1)\). We let \(H^f_2\) denote the collection of all such histories. Then, firm B and the consumer move sequentially as follows:

\(^{10}\)However, we assume that firms know the prior distribution of consumer types when setting prices. We do not explicitly include this as part of the information available when setting prices, since the prior distribution is later incorporated when we define firms’ beliefs over consumer types.
Appendix A.3. Interpreting negative prices as pricing below cost. This choice of action spaces results does not have a material effect are common for all histories that result in firms observing information. Before moving on, we establish some additional notation. Given history \( h = (s, \theta, p_1^A, p_1^B, a_1) \in H_2^f \), the information available for firm B at the beginning of period 2 is

\[
\mathcal{J}(h) = \begin{cases} 
    h & \text{if } a_1 = 1, \\
    (p_1^A, p_1^B, a_1) & \text{if } a_1 = 0.
\end{cases}
\tag{2.5}
\]

Then, given a history \( h \in H_2^f \), firm B observes \( \mathcal{J}(h) \) and sets the price of product 2.

(ii) Consumer. Finally, the consumer decides whether to buy product 2 after firm B sets its price \( p_2^B \), i.e., after a history of the form \( h = (s, \theta, p_1^A, p_1^B, a_1, p_2^B) \); we denote the collection of all such histories by \( H_2^f \). After the consumer’s purchasing decision for product 2, payoffs are realized and the game ends.

Before moving on, we establish some additional notation. Given history \( h = (s_0, \theta_0, p_1^A, p_1^B, a_1) \), we let \( s(h) = s_0, p_1^A(h) = p_1^A, a_1(h) = a_1 \), etc.\footnote{Similarly, if \( I = \mathcal{J}(h) \) for some history \( h \), we define \( p_1^A(I) \) and \( p_1^B(I) \) as the corresponding product 1 prices that are common for all histories that result in firms observing information \( I \).} In addition, given a history \( h \) and an action \( a \), we denote the concatenation of \( h \) and \( a \) by \( \langle h, a \rangle \). Finally, we denote the set of possible information vectors observed by firms when setting prices for product \( t \) by \( \mathcal{I}_t \). It follows that \( \mathcal{I}_t \) can be written as the image of \( H_t^f \) under \( \mathcal{J} \); more precisely, we define \( \mathcal{I}_t = \{ \mathcal{J}(h) : h \in H_t^f \} \), for \( t = 1, 2 \). Note that for the first period, we simply have \( \mathcal{I}_1 = \{ \emptyset \} \). Moreover, we can write \( \mathcal{I}_2 \) as the union of two sets, which differ on whether the consumer’s type is known by firm B when setting the price of product 2. In turn, this depends on the consumer’s decision in the first period. That is, we write \( \mathcal{I}_2 = \mathcal{I}_2^0 \cup \mathcal{I}_2^1 \), where

\[
\mathcal{I}_2^j = \{ \mathcal{J}(h) : h \in H_2^f, a_1(h) = j \}, \text{ for } j = 0, 1.
\tag{2.6}
\]

**Strategies and beliefs** We define an assessment as a tuple \( (\gamma, \sigma^A_1, \sigma^B, \mu) \) of strategies for the consumer, firms A and B, and a belief system \( \mu \). In more detail, a strategy for the consumer consists of a pair of functions \( \gamma = (\gamma_1, \gamma_2) \) with \( \gamma_t : H_1^f \to \Delta(\{0, 1\}) \) for \( t = 1, 2 \), where \( \gamma_t(h) \) denotes the probability of buying product \( t = 1, 2 \) from firm B after history \( h \).

A pricing strategy for firm A is then a function of the information available at the beginning of period 1 to firm A’s action set,\footnote{For the rest of the paper, we assume that the firms’ action spaces in period 1 are \( S^A = [0, 1] \) and \( S^B = [-1/2, 1] \), interpreting negative prices as pricing below cost. This choice of action spaces results does not have a material effect in our findings, i.e., it allows us to study all equilibria in which firms play undominated strategies (see Claim 19 in Appendix A.3).} \( \sigma^A_1 : \mathcal{I}_1 \to S^A \). A pricing strategy for firm B consists of a pair
of functions (one for each product) $\sigma^B = (\sigma_1^B, \sigma_2^B)$, where $\sigma_1^B : \mathcal{I}_1 \to S^B$ and $\sigma_2^B : \mathcal{I}_2 \to \mathbb{R}$. Note that, for simplicity, we focus on the case where both firms set prices according to pure strategies.\footnote{In proving our results, we allow firm $B$ to use a mixed strategy when pricing product 2. As we establish, at equilibrium, firm $B$’s pricing strategy for product 2 is indeed pure.}

A belief system consists of a pair of functions $\mu = (\mu_1, \mu_2)$ that map firms’ available information at a given time to probability distributions on the consumer type space, i.e., $\mu_t : \mathcal{I}_t \to \Delta(T)$ for $t = 1, 2$. In particular, $\mu_1(\emptyset)$ represents the firms’ common beliefs on the consumer types when choosing their product 1 prices. Since the only information available to both firms when setting product 1 prices is the prior distribution of consumer types, we define these beliefs to be equal to this prior distribution, i.e., we define $\mu_1(\emptyset) = \mu_0$, where $\mu_0$ is the uniform distribution on the unit square. For the second period, given firm $B$’s available information $I \in \mathcal{I}_2$, $\mu_2(I)$ represents firm $B$’s information on the type $(s, \theta)$ when setting product 2’s price.

**Payoffs** First, we define the continuation utility for the consumer at time periods $t = 1, 2$. Given a history $h \in H_2^t$, we slightly abuse notation to write the consumer’s utility as a function of her action as $u_2(a_2; h) = u_2(a_2; s(h), \theta(h), p_2^B(h))$ (note that we omit the explicit dependence on the histories whenever it is clear from the context). We let $U_2$ denote the consumer’s continuation utility in period 2 given an assessment $(\gamma, \sigma_1^A, \sigma_2^B, \mu)$ and a history $h \in H_2^t$. Then, we have

$$U_2(\gamma, \sigma_1^A, \sigma_2^B, \mu | h) = \mathbb{E}[u_2(a_2; h) | h],$$

(2.7)

where the expectation is taken with respect to the consumer’s purchasing decision in period 2, i.e., $a_2 \sim \gamma_2(h)$. Similarly, for every history where the consumer moves in period 1, $h \in H_1^t$, we write $u_1(a_1; h) = u_1(a_1; s(h), \theta(h), p_1^A(h), p_1^B(h))$, and define the total expected utility given the assessment and $h$ as

$$U_1(\gamma, \sigma_1^A, \sigma_2^B, \mu | h) = \mathbb{E}[u_1(a_1; h) + u_2(a_2; \langle h, a_1, p_2^B \rangle)],$$

(2.8)

where the expectation is taken with respect to consumer purchase decisions $a_1, a_2$, and firm $B$’s product 2 price $p_2^B$. Similarly, we define firm $A$’s total payoff as

$$\Pi_1^A(\gamma, \sigma_1^A, \sigma_2^B, \mu | \emptyset) = \mathbb{E}[\pi_1^A(p_1^A, p_1^B, a_1)],$$

where $p_i^A = \sigma_i^A(\emptyset)$ for $i = 1, 2$, and the expectation is taken with respect to firms’ beliefs on consumer types for the first period, $\mu_1(\emptyset)$, and the consumer’s decision $a_1$. Proceeding similarly for firm B, given an information vector $I \in \mathcal{I}_2$ and an assessment, firm $B$’s continuation payoff at time 2 is

$$\Pi_2^B(\gamma, \sigma_1^A, \sigma_2^B, \mu | I) = \mathbb{E}[\pi_2^B(p_2^B, a_2) | I],$$

where the expectation is taken assuming that consumer types are distributed according to $\mu_2(I)$, taking firm $B$’s product 2 price as $p_2^B = \sigma_2^B(I)$, and with respect to the consumer’s decision $a_2$.\footnote{In proving our results, we allow firm $B$ to use a mixed strategy when pricing product 2. As we establish, at equilibrium, firm $B$’s pricing strategy for product 2 is indeed pure.}
Finally, firm B’s total expected profit in the game is

$$\Pi^B_1(\gamma, \sigma^A_1, \sigma^B, \mu | \emptyset) = E \left[ \pi^B_1(p^A_1, p^B_1, a_1) + \pi^B_2(p^B_2, a_2) \right],$$

where $p_i^A = \sigma^A_i(\emptyset)$ for $i = 1, 2$. The expectation is taken with respect to the consumer’s decisions $a_1, a_2$ and firm B’s price for product 2, $p^B_2$, assuming that consumer types are distributed according to $\mu_1(\emptyset)$.

Equilibrium definition We consider Perfect Bayesian Equilibria (PBE) in the game we described above. An assessment $(\gamma, \sigma^A_1, \sigma^B, \mu)$ is a PBE if and only if the strategy profile $(\gamma, \sigma^A_1, \sigma^B)$ is sequentially rational for all players given the belief system $\mu$, and $\mu$ is consistent with the strategy profile. In particular, a PBE $(\gamma, \sigma^A_1, \sigma^B, \mu)$ satisfies the following conditions:

(i) For any period $t = 1, 2$, any history $h \in H^c_t$, and any consumer strategy $\gamma'$ we have that

$$U_t(\gamma, \sigma^A_1, \sigma^B, \mu | h) \geq U_t(\gamma', \sigma^A_1, \sigma^B, \mu | h).$$

(ii) For any period $t = 1, 2$, any information vector $I \in I_t$, and any firm B’s pricing strategy $\sigma^B'$ we have that

$$\Pi^B_t(\gamma, \sigma^A_1, \sigma^B, \mu | I) \geq \Pi^B_t(\gamma, \sigma^A_1, \sigma^B', \mu | I).$$

(iii) For any firm A’s pricing strategy $\sigma^A'$, we have that

$$\Pi^A_t(\gamma, \sigma^A_1, \sigma^B, \mu | \emptyset) \geq \Pi^A_t(\gamma, \sigma^A', \sigma^B, \mu | \emptyset).$$

(iv) The belief system $\mu$ is consistent with the strategy profile $(\gamma, \sigma^A_1, \sigma^B)$. That is, given any information vector $I \in I_2$ we have that:

(a) If $I \in I^1_2$ (as defined in (2.6)), then $\mu_2(\cdot | I)$ assigns probability 1 to the true consumer type $(s(I), \theta(I))$. That is, for any Borel set $A \subseteq T$,\n
$$\mu_2(A | I) = 1_A(s(I), \theta(I)), \quad (2.9)$$

where $1_A$ denotes the indicator function of set $A$.

(b) If $I \in I^0_2$ (as defined in (2.6)), then for any Borel set $A \subseteq T$,

$$\mu_2(A | I) = \mathbb{P}_{(s, \theta) \sim \mu_1(\emptyset)}[(s, \theta) \in A | I, \gamma, \sigma],$$

and moreover, this expression satisfies Bayes’ rule whenever the event of reaching a history that results in firm B observing information vector $I$ has positive probability (according to $\mu_1(\emptyset)$), if players follow the strategy profile $(\gamma, \sigma^A_1, \sigma^B)$.  

12
As mentioned above, our main focus is a setting where firm B employs data tracking and the consumer is forward-looking, i.e., she takes into account the firm’s data tracking ability when making her purchase decisions. This is captured by condition (i) in our equilibrium definition given that the consumer makes her purchasing decision in period 1 to maximize her total expected utility over both time periods. To best illustrate the implications of data tracking, we also consider the following two settings:

**Restricted.** In the restricted setting, firm B does not have any data tracking capabilities, which in turn implies that the two markets in which firm B operates are entirely decoupled. Formally, firm B’s information in period 2 given history \( h \) is given by \( \mathcal{J}(h) = (p_1^A, p_1^B) \), irrespective of the consumer’s action in period 1 (contrast this with Expression (2.5)).

**Myopic.** Firm B employs data tracking but the consumer acts myopically. In particular, when deciding from which firm to buy product 1, she does not internalize the implications of her action to firm B’s pricing for product 2. In other words, a myopic consumer determines her action \( a_1 \) to maximize her utility in period 1 (given by Expression (2.1)), whereas a forward-looking consumer determines \( a_1 \) to maximize her aggregate utility over both product markets, as given by Expression (2.8). Thus, in this setting, condition (i) in the equilibrium definition for period 1 changes to

\[
\gamma_1(h) \in \arg \max_{\gamma_1(h) \in \Delta \{0,1\}} E \left[ u_1(a_1; h) \mid h \right],
\]

for all \( h \in H^c_1 \), where the expectation above is taken with respect to \( a_1 \sim \gamma_1(h) \).

**Model discussion** As a preface to the discussion below, we note that our objective is to illustrate the implications of data tracking, the endogenous privacy concerns it may generate, and how competition affects the associated equilibrium outcomes. To this end, we develop a two-period model that captures the basic elements of the setting of our interest. We now discuss the main assumptions of the model and their implications.

**Data tracking and personalization.** In the model, if the consumer buys product 1 from firm B, she reveals her type and firm B can use this information to offer a personalized price for product 2. With this modeling choice, we are implicitly assuming that (i) firm B has access to data tracking and personalization technologies, and (ii) it is able to infer the consumer’s type after transacting with her.

Assumption (i) is reasonable in the context of e-commerce applications. Tracking technologies are widespread in practice – customers have accounts for online stores which keep track of various aspects of their behavior, such as purchases and browsing history. In addition, personalization is increasingly used and can take the form of personalized pricing, targeted coupons or promotions, or tailored product assortments.\(^{15}\) In line with the spirit of our model, such practices entail leveraging

\(^{14}\)Note that, in the restricted setting, we assume that firm B cannot distinguish between histories where the consumer bought product 1 from it or not, given fixed product 1 prices. This is in line with our interpretation for the restricted setting, i.e., firm B cannot use any information that results from product 1 transactions to make decisions related to product 2.

\(^{15}\)For a detailed discussion, refer to Acquisti et al. (2016) and Dubé and Misra (2019).
information about customers for personalization.

We make assumption (ii) to capture the fact that firms can learn information about their customers from previous interactions beyond just observing that the customer chose to buy some products at certain prices. We incorporate this feature in our model by positing that firms infer customers’ types in the event of a transaction. This assumption may capture well certain contexts, e.g., when the population of customers can be clustered in a small number of types that can be easily identified by historical data (see, for example, Moon, Bimpikis, and Mendelson (2018)). In addition, it provides the intuition we aim to capture in a clean and transparent way, i.e., firms may learn substantial information about their consumers based on data from previous transactions.

Endogenous privacy concerns. We study a setting where the consumer does not have an intrinsic value for privacy, but would like to preserve it only to the extent that doing so prevents a future economic loss. Therefore, in the context of our model “privacy concerns” arise solely due to the possibility that revealing one’s data may be associated with economic consequences and they do not result from the consumer deriving a direct utility from maintaining her privacy.¹⁶

Competitive structure. We assume that both firms compete in the market for product 1, but firm B is a monopolist in the market for product 2. With this, we look to capture two features. First, that learning consumer information is valuable for firm B. Giving firm B monopoly power in the second market and the possibility of increasing her profits by leveraging its knowledge about the consumer achieves this. Second, we want to understand how the consumer’s choice depends on her option to buy product 1 while preserving her privacy, which is the role that firm A plays in the first period. A direct extension of our model is to assume that firm B is a monopolist in both markets. Proposition 3 establishes that the welfare implications of these two models can be strikingly different.¹⁷

Timeline of consumer purchases. We assume that the order in which the consumer makes her purchase decisions is fixed, i.e., first for product 1 and then for product 2. Indeed, one could formulate a model with a more flexible timeline, where the order of purchases may be stochastic, or where the consumer may consider to buy only one of the products, or even where she strategically chooses the sequence of her transactions. In addition to aiming for simplicity, we avoid a more flexible timeline for three main reasons. First, the current model setup provides firm B with a clear path to obtain and, then, potentially exploit consumer data. Also, we are interested in isolating the effect of data tracking in the model and introducing a less stringent timeline would complicate this considerably. Finally, several papers that study related questions assume a similar timeline and maintaining it allows us to contrast our results with theirs.¹⁸

¹⁶We refer the reader to Lin (2020), which differentiates between consumers’ intrinsic versus instrumental motives for protecting their privacy. In her terminology, our setting involves instrumental motives.
¹⁷We analyze the extension of our model to the monopoly context in Section EC 8 of the Electronic Companion.
¹⁸For example, Taylor (2004), Acquisti and Varian (2005), and Conitzer et al. (2012).
3 Equilibrium Outcomes

As a first step in our analysis, we provide a characterization of the equilibrium outcomes in the model economy of Section 2. First, we describe the equilibrium in the environment where firm B employs data tracking and consumers are forward-looking (Theorem 1). Then, we also describe the equilibrium outcomes associated with the two benchmarks we consider, i.e., (i) no data tracking, and (ii) consumers who are myopic in their decision making (Propositions 1 and 2). Armed with this characterization, we proceed in Section 4 to discuss the implications of adopting data tracking for both firms and the consumers. We provide the proof of Theorem 1 in Appendix A, while the proofs of Propositions 1 and 2 can be found in the Electronic Companion.

Equilibrium characterization In the presence of data tracking, a consumer anticipates that buying product 1 from firm B (the firm that operates in both markets), would result in zero net utility in period 2 as firm B would use the information generated in the transaction to price product 2 at the consumer’s valuation. Thus, all else equal, the consumer views firm A as a more attractive option than B given that firm A allows the consumer to maintain her “privacy,” i.e., avoid disclosing her type to firm B. In turn, firms determine their prices for product 1 accordingly. Theorem 1 below formalizes this intuition, states conditions that ensure existence, and provides a characterization of the equilibrium outcome.

![Equilibrium with Forward-looking Consumers](image)

Figure 2: Equilibrium outcome for a forward-looking consumer as a function of her type \((s, \theta)\) (we depict only interior equilibria, i.e., we assume that \(m < m_L\)). In addition, we let \(\bar{X}^*\) denote the following quantity \(\bar{X}^* = 1/2 + p_{1B}^* - p_{1A}^*\).
Theorem 1. There exist constants $m_L, m_H$ with $0 < m_L < m_H$ such that if $m < m_L$ or $m > m_H$, there exists an equilibrium in the forward-looking setting. Moreover, the equilibrium path can be characterized as follows:

(a) If $0 < m < m_L$, there exist unique prices $p_{1A}^* = p_{1A}^*(m) > 0$, $p_{1B}^* = p_{1B}^*(m)$, with $p_{1A}^* > p_{1B}^* > p_{1A}^* - 1/2$ such that, in any equilibrium, firms’ prices for product 1 are equal to $p_{1A}^*$ and $p_{1B}^*$, respectively. In addition, the expected product 1 demand for both firms is positive. Moreover, there exists a constant $\bar{\theta}^* = \bar{\theta}^*(m) \in (1/2, 1)$, and a function $g^* : [0, 1] \to \mathbb{R}$ defined by

$$g^*(t) = p_{1B}^* - p_{1A}^* + 1/2 + m(t - \bar{\theta})^+,$$

such that

(i) If the consumer’s type $(s, \theta)$ is such that $s > g^*(\theta)$, then, with probability one, the consumer buys product 1 from firm B; the firm perfectly observes the type of the consumer and, in the second period, sets a price equal to $m\theta$ for product 2, which the consumer also buys.

(ii) If the consumer’s type $(s, \theta)$ is such that $s < g^*(\theta)$, then, with probability one, the consumer buys product 1 from firm A; firm B sets a price equal to $p_{2B}^2 = m\bar{\theta}^*$ for product 2, which the consumer buys if $\theta > \bar{\theta}^*$.

(b) If $m > m_H$, in any equilibrium, firms’ prices for product 1 are equal to $p_{1A}^* = 0$ and $p_{1B}^* = -1/2$, respectively, and the consumer buys product 1 from firm B with probability one. In addition, firm B perfectly observes the consumer’s type and, in the second period, sets a price equal to her valuation for product 2, which the consumer buys.

The proof of Theorem 1 involves a backwards induction argument and proceeds in three steps: first, for any pair of product 1 prices, $p_{1A}^*$ and $p_{1B}^*$, we characterize the equilibria that arise in the subgame that follows the firms’ choices of such prices. Then, we leverage this characterization to define a simultaneous-move pricing game for firms A and B, where their corresponding profit functions represent the “on-the-equilibrium-path” profits that each firm obtains when prices are set to be equal to $p_{1A}^*$ and $p_{1B}^*$, respectively. We establish a formal mapping between the equilibrium strategies of the original game and those in the simultaneous-move pricing game. Finally, in the third step, we identify the range of values of $m$ for which an equilibrium exists in the simultaneous-move game and characterize its general structure. On the technical level, the proof and, in particular its third step, presents challenges given that essentially the two firms engage in an asymmetric pricing game, where firm B’s profit function is not quasi-concave in its price. This prevents us from using well-known methods to establish equilibrium existence. Instead, we directly obtain the firms’ best response correspondences and establish conditions that ensure the existence of a pure-strategy Nash equilibrium as well as its uniqueness (when it exists).\(^{19}\)

\(^{19}\)See Lemmas 1, 2 and 3 in Appendix A.

\(^{20}\)It is important to emphasize that Theorem 1 only ensures that an equilibrium exists when $m < m_L$ or $m > m_H$. In the Appendix, we establish that $m_L \approx 3.98$ and $m_H \approx 4.02$. This implies that an equilibrium may not exist only
As we already mentioned above, firm B has an incentive to offer a discount relative to firm A’s product 1 price in order to induce consumers to buy from it and generate information about their valuations for product 2. This results in firm A’s price being higher at equilibrium than firm B’s, i.e., \( p^*_A > p^*_B \). Moreover, Theorem 1 implies that two types of equilibria may arise: (i) \textit{interior}, which involve firm A setting a positive price and transacting with the consumer with strictly positive probability, and (ii) \textit{corner}, where firm A sets its price for product 1 to zero and does not transact with the consumer. Corner equilibria arise only when the potential profits that firm B can generate in product market 2 are relatively large (i.e., when the parameter \( m \) takes relatively high values). Then, firm B finds it optimal to set a sufficiently low price to capture the entire market for product 1.\(^{21}\) Figure 2 provides an illustration of the equilibrium outcome as a function of the consumer’s type.

Next, we describe the equilibria that arise in two alternative settings: (i) with no data tracking, and (ii) when firm B uses data tracking but consumers act myopically (Figure 3 provides an illustration). In particular, in the absence of data tracking, firms do not collect any consumer information that they can use in product market 2. Thus, the two product markets are entirely decoupled for both firms and the consumer. Proposition 1 summarizes the equilibrium outcome in this setting.

**Proposition 1.** For any \( m > 0 \), there exists an equilibrium in the restricted setting. In addition, the equilibrium path is essentially unique and takes the following form:\(^{22}\)

(i) Both firms set a price of 1/2 for product 1.

(ii) The consumer buys product 1 from firm A if her type satisfies \( s < 1/2 \), and she buys from firm B if \( s > 1/2 \). In particular, the expected demand for product 1 is 1/2 for both firms.

(iii) Firm B sets a price of \( p^{B,R}_2 = m/2 \) for product 2, which the consumer buys if \( \theta > 1/2 \).

In a nutshell, given that firm B does not collect any consumer information, it sets its price for product 2 based on the prior distribution for the consumer’s type. Then, in product market 1, firms are symmetric since there is no information flowing from the first to the second period. Thus, firms compete in a standard Hotelling spatial competition model with fixed locations, which results in both firms setting the same price and splitting the market evenly (see e.g., d’Aspremont, Gabszewicz, and Thisse (1979) and Osborne and Pitchik (1987)).

Finally, we consider the setting when firm B employs data tracking but consumers act myopically, i.e., they do not take into account the implications of their actions in the first period for firm B’s pricing in the second. Proposition 2 summarizes the equilibrium outcome in this setting.

\(^{21}\)Although we present this type of equilibria for completeness in this section, the analysis in Section 4 focuses on interior equilibria.

\(^{22}\)The consumer’s strategy is not uniquely pinned down when she is indifferent between buying product 1 from either firm, or between buying product 2 or not, but these events occur with probability zero.
Figure 3: Equilibrium outcome in a restricted economy (left) and when the consumer behaves myopically (right) as a function of her type \((s, \theta)\)

**Proposition 2.** For any \(m > 0\), there exists an equilibrium in the myopic setting. In addition, the equilibrium path is essentially unique and takes the following form:\(^{23}\)

(i) Firms’ prices for product 1 are, respectively,

\[
p_{A,M}^1 = \max \left\{ \frac{1}{2} - \frac{m}{12}, 0 \right\}, \quad p_{B,M}^1 = \max \left\{ \frac{1}{2} - \frac{m}{6}, -\frac{1}{2} \right\}.
\] (3.1)

(ii) The consumer buys product 1 from firm B if her type \((s, \theta)\) satisfies \(s > p_{B,M}^1 - p_{A,M}^1 + 1/2\). In that case, firm B perfectly observes the type of the consumer and, in the second period, sets a price equal to \(m\theta\) for product 2, which the consumer also buys.

(iii) The consumer buys product 1 from firm A if her type \((s, \theta)\) satisfies \(s < p_{B,M}^1 - p_{A,M}^1 + 1/2\). In that case, firm B’s sets a price of \(p_{2,B}^M = m/2\) for product 2, which the consumer buys if \(\theta > 1/2\).

(iv) Firm B faces a higher expected demand for product 1 than firm A. In particular, firm A’s expected demand for product 1 is equal to \(p_{1,A}^M < 1/2\).

The equilibrium outcome when consumers act myopically shares a number of features with the corresponding outcome in the forward-looking setting. In both cases, firm B finds it optimal to offer a discount for product 1 relative to firm A. In addition, as in the forward-looking setting, depending on the value of \(m\), two types of equilibria may arise: (i) interior, in which both firms

\(^{23}\)As in Proposition 1, the consumer’s strategy is not uniquely pinned in some histories that involve indifferences, but these events occur with probability zero.
transact with the consumer with strictly positive probability and, (iii) corner, in which firm B’s incentive to generate information for product 2 is sufficiently high for it to capture the entire product 1 market by pricing low.

However, there are also key differences between the structure of the equilibrium paths in the forward-looking and myopic settings. First, note that when the equilibrium is interior for the forward-looking setting (i.e., $0 < m < m_L$), the consumer strictly prefers to buy product 1 from firm A if her type satisfies $s < g^*(\theta)$, where $g^*$ is a (weakly) increasing function. In other words, consumers with high values of $\theta$ are relatively more likely to buy product 1 from firm A (see Figure 2). This captures the fact that the consumers with high valuations for product 2 have relatively more to gain by hiding their type from firm B and buy product 1 from firm A. In contrast to the myopic setting, a forward-looking consumer considers this trade-off when making her purchase decision for product 1.

A second and related key difference is that the price firm B sets for product 2 if the consumer bought product 1 from firm A is higher for the forward-looking setting than the corresponding one in the myopic setting, as $p^B_2 = m\bar{\theta}^*(m) > m/2$. In particular, a myopic consumer’s purchase decision for product 1 is independent of her valuation for product 2, i.e., the value of $\theta$. Thus, when the consumer buys product 1 from firm A, firm B sets its price for product 2 based on the prior distribution for $\theta$. In contrast, for a forward-looking consumer, her decision to purchase product 1 from firm A is informative about her valuation for product 2, as consumers with high valuations are relatively more likely to avoid buying product 1 from firm B (for an illustration refer to Figure 2). In turn, this is internalized by firm B when it updates its beliefs about the consumer’s type after period 1, resulting in a distribution that is skewed towards high values of $\theta$ (when the consumer buys product 1 from firm A). This leads firm B to set a higher price for product 2 than it would if such beliefs were uniform as in the myopic setting.

This dynamic highlights an important feature captured by our model: the actions of consumers that have a low opportunity cost for revealing their type (low-$\theta$) impact the equilibrium outcome for consumers that face higher costs (high-$\theta$). The fact that low valuation consumers are more likely to buy product 1 from firm B induces an externality on high valuation consumers, since their choices result in firm B’s equilibrium beliefs on $\theta$ to be skewed towards higher values, leading the price for product 2 to be higher than it would under the uniform prior as beliefs.

4 Implications of Data Tracking for Consumers and Firms

In this section, we build on the equilibrium characterization results of Section 3 to study the implications of data tracking on consumers and firms in our model economy. We begin our analysis by discussing consumer surplus (Theorem 2). Then, we consider firms’ pricing decisions and profits (Propositions 4, 5 and 6). Throughout this section, we focus on the case where the forward-looking setting admits an interior equilibrium, i.e., we assume that $m \in (0, m_L)$. In our view, this regime

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24In particular, the function $g^*(\theta)$ is proportional to the density function associated to firm B’s equilibrium beliefs over $\theta$ following a history where the consumer buys product 1 from firm A.
represents the most plausible and interesting setting to study, as otherwise, i.e., when \( m > m_H \), firm A does not engage in any transactions and makes zero expected profits. For the sake of completeness, we discuss the implications of data tracking for corner equilibria in Appendix C. The proof of Theorem 2 is provided in Appendix B and the proofs of Propositions 4, 5, and 6 can be found in the Electronic Companion.

**Consumer Surplus** Our first result establishes that data tracking leads to higher aggregate consumer surplus. Interestingly, this finding holds even if consumers act myopically, i.e., if they do not take into account the implications of their actions in product market 1 for pricing in product market 2.

**Theorem 2.** Suppose that \( m \in (0, m_L) \). Then, aggregate consumer surplus is higher in the presence of data tracking for both myopic and forward-looking consumers.

Theorem 2 highlights one of the main qualitative insights of our analysis: on aggregate, consumers are better off when firm B employs data tracking. The intuition behind this result can be best explained as follows: the presence of data tracking creates the incentive for firm B to lower its price for product 1 as a way of increasing its share of transactions and generating consumer information. In turn, this also results in lower prices from firm A due to competition. On the other hand, firm B can use the data it generates to extract additional surplus from consumers in product market 2. Our analysis establishes that the first effect, i.e., lower prices in product market 1, dominates the potentially adverse effects of price discrimination in product market 2. The left panel of Figure 4 depicts graphically the findings of Theorem 2, i.e., the increase of consumer surplus relative to the restricted economy, as a function of the value of consumer data (captured by parameter \( m \)).

Although consumers are on aggregate better off in the presence of data tracking, this does not hold across all consumer types. In particular, consumers with low valuations for product 2 (low \( \theta \)) tend to gain most from data tracking as they benefit from lower prices for product 1, while not experiencing a significant loss in net utility in product market 2 relative to a restricted economy. In contrast, high valuation consumers would be better off if the two product markets were decoupled for firm B, i.e., there was no information flow between them. The right panel of Figure 4 illustrates this point and adds to the discussion on the implications of data tracking by highlighting that they may differ considerably depending on a consumer’s idiosyncratic features.

Importantly, Theorem 2 is largely driven by competition: firm B’s incentive to lower its prices for product 1 results in its competitor, firm A, doing the same, which in turn benefits a fraction of consumers. To best illustrate the role of competition, we consider an alternative formulation where firm B is a monopolist in both markets. Section EC 8 of the Electronic Companion outlines the monopoly variant of our model economy and provides a characterization of the equilibria for the settings we consider.\(^{25}\) In turn, this allows us to obtain the following result for consumer surplus.

\(^{25}\)In particular, the consumer decides sequentially whether to buy product 1 and then product 2 from firm B. Then, her utility associated with product 1 is simply given as \( a_1 (\bar{u} - (1 - s) / 2 - p_1) \).
Figure 4: Percentage change in the equilibrium aggregate consumer surplus relative to the restricted setting, as a function of the value of consumer data $m$, for $\bar{u} = 1$ (left), and comparison of the consumer’s utility as a function of her type $(s, \theta)$ in the forward-looking and restricted settings (right) when $m = 2$. Forward-looking consumers with relatively high values of $\theta$, i.e., whose types fall into the shaded region, are better off in the absence of data tracking.

**Proposition 3.** Consider an economy with $\bar{u} \geq 1/2$ and $m > 0$. Then, when firm $B$ is a monopoly in both markets, we obtain that

(i) Aggregate consumer surplus is lower in the presence of data tracking when consumers behave myopically;

(ii) In addition, if $m \geq 4(1-\bar{u})$, aggregate consumer surplus is also lower in the presence of data tracking when consumers are forward-looking.

The juxtaposition of Proposition 3 with Theorem 2 establishes that competition plays a fundamental role in determining whether data tracking technologies may lead to an increase in consumer surplus. In particular, when consumers fail to internalize the impact of their actions in firm B’s pricing decisions in market 2, they are on aggregate worse off if firm B faces no competition and can employ data tracking. Thus, the findings of Theorem 2 are driven by the combination of the incentive that data tracking provides to firm B to lower its price for product 1 as well as the competitive response from firm A, which increases the size of the price discount. As statement (ii) of the proposition highlights, a similar insight holds for an economy with forward-looking consumers: the effects of data tracking on aggregate consumer surplus are largely dependent on competition. In fact, when consumers value product 1 sufficiently high, i.e., $\bar{u} \geq 1$, and firm B is a monopoly in both markets, they are better off if it does not have access to data tracking, regardless of the value of $m$. 

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The above findings about consumer surplus may help inform the discussion around policy design relative to data tracking. Although prior work has emphasized the adverse effects of tracking for myopic consumers (Taylor (2004), Acquisti and Varian (2005), and Bonatti and Cisternas (2020)), they have mainly focused on a monopoly context. We illustrate that the implications may be more nuanced and largely dependent not only on consumers’ level of sophistication, i.e., whether they internalize the effect of their purchase decisions on (future) firms’ actions, but also on the structure of competition in the market.

**Prices**  
As we discussed above, data tracking creates the incentive for firm B to set lower prices in the first product market as a way to increase its market share and obtain additional consumer information. The effect of firm B’s data tracking capabilities on firm A’s price is driven by the following trade-off. On the one hand, firm A faces pressure to decrease its price for product 1 relative to an economy where no such capabilities exist given firm B’s lower prices. On the other hand though, when consumers are forward-looking, they anticipate the implications of revealing their type to firm B when transacting with it. In other words, a fraction of consumers find firm A’s product relatively more attractive as transacting with A allows them to preserve their privacy regarding their type.

Data tracking results in price dispersion, i.e., firm A charging more for its product than firm B, relative to the economy where data tracking capabilities are not present and both firms set the same price. What is more, the extent of price dispersion is higher when consumers are forward-looking, since in this case the aforementioned privacy effect is also present in the economy. These observations are formalized in Proposition 4 below and illustrated in Figure 5.

**Proposition 4.** Suppose that \( m \in (0, m_L) \). Then, there exists dispersion in product 1 prices at equilibrium when firm B uses data tracking. Furthermore, the level of price dispersion is highest when consumers are forward-looking. In other words,

\[
\Delta^* = p_{1A}^* - p_{1B}^* > \Delta^M = p_{1A,M}^* - p_{1B,M}^* > 0,
\]

where \( \Delta^* \) and \( \Delta^M \) denote the level of product 1 dispersion in equilibrium, in an economy with forward-looking and myopic consumers, respectively.

**Profits**  
In the final part of this section, we turn our attention to the profits firms generate. When firm B employs data tracking, it finds it optimal to “subsidize” product 1 (set a lower price than in an economy without tracking) and generate higher profits from product 2. Clearly, when consumers behave myopically, tracking yields the highest benefits for firm B. Proposition 5 below establishes that firm B is better off with data tracking even when consumers are forward-looking and internalize the implications of the firm’s pricing decisions. Interestingly, this finding is somewhat at odds with prior work which argues that firms may find it optimal to commit not to use consumer information for personalized pricing when facing forward-looking consumers, albeit
Figure 5: Dispersion in the equilibrium prices for product 1 as a function of the value of consumer
data $m$

in models that differ from ours in a number of ways, e.g., they feature two customer types and
different tracking and personalization technologies (Taylor (2004), Villas-Boas (2004), and Acquisti
and Varian (2005)).

**Proposition 5.** Suppose that $m \in (0,m_L)$. Then, firm B always benefits from using data tracking
technologies. In addition, its equilibrium profits are higher when consumers behave myopically than
when they engage in forward-looking behavior.

As one would perhaps expect, the implications of data tracking for firm A are considerably
different and are summarized in Proposition 6 below.

**Proposition 6.** Suppose that $m \in (0,m_L)$. Then, firm A’s expected equilibrium profits are highest
in a restricted economy. On the other hand, suppose that firm B uses data tracking. Then, there
exists $\bar{m} \in (0,m_L)$ such that firm A’s equilibrium profits are higher with forward-looking than with
myopic consumers if and only if $m < \bar{m}$.

The first part of the proposition establishes that firm A is better off in an economy where its
competitor does not use data tracking. In other words, even though consumers attach a “privacy”
premium to firm A’s product, this does not compensate for the potential loss in its profits due
to firm B’s lower prices. The findings are more nuanced when we focus on consumer behavior.
Assuming that firm B tracks consumer data, firm A is better off in an economy with forward-
looking consumers only when the value of their data as captured by $m$ is relatively low. In other
words, even though forward-looking consumers attach additional value to firm A’s product, firms’/endogenous pricing decisions may result in lower profits for A than in an economy where consumers
act myopically, as firm B’s incentive to lower the price of product 1 is intensified when consumers
are forward-looking. Figure 6 provides a graphical illustration of our findings related to the firms' equilibrium profits.

As a final remark, note that the latter findings may point to a strategic opportunity for firm A. In particular, Proposition 6 provides conditions under which the firm would be better off in an economy with forward-looking consumers relative to one where they behave myopically. To the extent that highlighting the implications of data tracking, e.g., through firm A advertising its “privacy” features, would raise awareness among consumers and, in the context of our model, induce forward-looking behavior, the comparison of equilibrium profits in the proposition establishes when such an endeavor would benefit firm A. Interestingly, this depends on the value of consumer data as firm A’s efforts to educate consumers may be countered by firm B’s endogenous pricing response.

![Firm A’s Expected Profits](image1)

![Firm B’s Expected Profits](image2)

Figure 6: Percentage change in total expected profits in equilibrium for firm A (left) and B (right), relative to the restricted setting, as a function of the value of consumer data m

### 5 Concluding Remarks

In this paper we study the implications of data tracking technologies on firms’ pricing decisions, their profits, and, most importantly, consumer surplus. The economy we consider, which features two markets (a duopoly and a monopoly), allows us to focus on the role of competition and the consumers’ level of sophistication. In particular, our results establish that the adoption of data tracking may benefit consumers in the presence of competition. This result stems from the fact that the potential gains from acquiring consumer information through transactions provide the incentive for a firm to lower its prices. In turn, this increases the intensity of competition and results in lower prices for all consumers in the first market, the duopoly. Interestingly, given that
the firm can use the consumer data it collects for price discrimination in the second market, the gains in consumer surplus are not uniform among different consumer types: those with high valuations for the product in the second market are adversely affected from data tracking in contrast to those with low valuations. Thus, our results may contribute to the ongoing discussion around regulation aimed at protecting consumers’ data privacy by illustrating that the debate on whether to regulate data tracking practices should take into account not only the degree of consumers’ understanding of how their data may be used (captured in our model by whether consumers are forward-looking or myopic), but also the competitive structure of the market.

We make a number of simplifying assumptions to allow for tractable and transparent analysis. For example, we assume that consumers are either all taking into account the implications of their actions to firms’ pricing decisions or they are all acting myopically. In addition, we assume that data tracking technologies enable firm B to perfectly infer the type of a consumer after a transaction. Extending our model to the setting where only a fraction of consumers are forward-looking and/or where data tracking technologies provide only imperfect learning of consumers’ types are interesting avenues for further research. In particular, it would be instructive to study the incentives of firms to invest in adopting and improving the accuracy of such technologies.

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Appendix

The appendix is organized as follows: Appendix A provides the proof of Theorem 1, which characterizes the equilibrium outcomes in the setting with data tracking and forward-looking consumers. Appendix B provides the proof of Theorem 2, which results from comparing the aggregate consumer surplus for the three settings we consider. Finally, to complement the analysis of Section 4, we compare the equilibrium outcomes across three settings in the corner equilibrium regime (i.e., when \( m > m_H \)) in Appendix C.

In the interest of space, we have relegated the proofs of the rest of the results stated in the paper (Propositions 1 – 6) to the Electronic Companion.

A Proof of Theorem 1

Overview of the Proof  As discussed in Section 3, the proof of Theorem 1 proceeds in three steps, each of which we will state as an independent lemma. Informally, Lemma 1 characterizes the equilibria that arise in the continuation game that follows firms choices for produce 1 prices, \( p^A_1 \) and \( p^B_1 \), i.e., the subgame that follows histories of the form \( h = (s, \theta, p^A_1, p^B_1) \). Then, based on the characterization established in Lemma 1, we define a simultaneous-move pricing game for firms A and B, where their corresponding profit functions in terms of product 1 prices represent the “on-the-equilibrium-path” profits that each firm will obtain when prices are \( p^A_1 \) and \( p^B_1 \), respectively. Lemma 2 shows that any equilibria in the original game is associated to a Nash equilibrium in this simultaneous-move game and vice versa. Finally, in Lemma 3, we identify the range of values of \( m \) for which an equilibrium exists in this simultaneous-move game and characterize its general structure (i.e., the interior and corner regimes) and establish the uniqueness of such equilibrium.

To formally introduce each of these lemmas, we first define some auxiliary notation that we use to characterize the equilibrium strategies in the continuation game that follows firms’ choices for product 1 prices. First, given product 1 prices \( p^A_1 \) and \( p^B_1 \), let us define

\[
\bar{X}(p^A_1, p^B_1) = \max \left\{ 0, \min \left\{ p^B_1 - p^A_1 + 1/2, 1 \right\} \right\}.
\]

Note that \( \bar{X}(p^A_1, p^B_1) \) is a measure of the price dispersion for product 1. Let us also define the following auxiliary function that will allow us to characterize firm B’s pricing strategy for product 2, as we will see in Lemma 1. We define \( \tilde{\theta} : [0, 1] \times \mathbb{R}^+ \to \mathbb{R} \) as

\[
\tilde{\theta}(x, m) = \begin{cases} 
\frac{1}{m} \left( 2x + m - \sqrt{2x(2x + m)} \right), & \text{if } x \leq \tilde{x}(m), \\
\frac{1}{1+x} \left[ 1 - \frac{1}{2m} (1 - x)^2 \right], & \text{if } x > \tilde{x}(m),
\end{cases}
\]

where the cutoff value for the two cases above, \( \tilde{x} : \mathbb{R}^+ \to \mathbb{R} \), is given by:

\[
\tilde{x}(m) = \frac{1}{3} \left( \sqrt{(m-1)^2 + 3} - (m-1) \right).
\]
Finally, given any product 1 prices \( p_1^A \) and \( p_1^B \), and \( t \in [0,1] \), we define
\[
 g(t \mid p_1^A, p_1^B) = p_1^B - p_1^A + 1/2 + m \left( t - \hat{\theta} \left( \bar{X} \left( p_1^A, p_1^B \right), m \right) \right)^+ . \tag{A.4}
\]

As we will see, this quantity will help us characterize the type of the consumer that is indifferent between buying product 1 from either firm. Now that we have defined these functions, we can characterize the equilibrium strategies after firms set product 1 prices as follows.

**Lemma 1.** Fix an assessment \((\gamma, \sigma_1^A, \sigma_1^B, \mu)\) and let \( p_1^A = \sigma_1^A(\theta) \) and \( p_1^B = \sigma_1^B(\theta) \). Then, in any equilibrium, we must have that:26

(a) If \( p_1^B - p_1^A \leq -1/2 \), the consumer buys product 1 from firm B with probability one; the firm perfectly observes the type of the consumer and, in the second period, sets a price equal to the consumer’s valuation for product 2, that the consumer buys.

(b) If \( p_1^B - p_1^A > -1/2 \), we have that:

(i) If the consumer’s type \((s, \theta)\) is such that \( s > g(\theta \mid p_1^A, p_1^B) \), then, the consumer buys product 1 from firm B with probability one; the firm perfectly observes the consumer’s type and, in the second period, sets a price equal to \( m\theta \) for product 2, which the consumer buys.

(ii) If the consumer’s type \((s, \theta)\) is such that \( s < g(\theta \mid p_1^A, p_1^B) \), then, the consumer buys product 1 from firm A with probability one; firm B sets the price for product 2 equal to
\[
 p_2^B = m\bar{\theta}(\bar{X} \left( p_1^A, p_1^B \right), m),
\]
which the consumer buys if \( \theta > \bar{\theta}(\bar{X} \left( p_1^A, p_1^B \right), m) \),

where \( \bar{X}, \bar{\theta} \) and \( g \) are defined in (A.1), (A.2) and (A.4), respectively.

The analysis of the continuation game yields interesting managerial insights. First, note that a (forward-looking) consumer is indifferent between buying product 1 from either firm if and only if her type satisfies \( s = g(\theta \mid p_1^A, p_1^B) \). This condition contrasts with the corresponding one for the restricted and myopic settings, which is given by \( s = p_1^B - p_1^A + 1/2 \). In particular, if \( p_1^B > p_1^A - 1/2 \), the indifference condition in the forward-looking setting satisfies \( s = g(\theta \mid p_1^A, p_1^B) \geq p_1^B - p_1^A + 1/2 \), with the inequality being strict with positive probability.27 This inequality captures the fact that given product 1 prices that satisfy \( p_1^B > p_1^A - 1/2 \), there are consumers that would choose to buy product 2 from firm B in the myopic setting but will prefer to buy from firm A instead in the forward-looking setting, since, when making their purchase decision for product 1, they consider the cost of revealing their type to firm B by buying from it.

In addition, we establish that, in equilibrium, firm A captures a positive share of the product 1 market if and only if \( p_1^B - p_1^A > -1/2 \). That is, by offering a price that is lower than firm A’s

\[\text{We do not explicitly characterize the equilibrium outcomes when the consumer types satisfy } s = g(\theta \mid p_1^A, p_1^B) \text{ since the consumer is indifferent between buying product 1 from either firm. However, this occurs with probability zero and has no impact in subsequent results. The same reasoning applies for the case where } \theta = \bar{\theta}(\bar{X} \left( p_1^A, p_1^B \right), m).\]

\[\text{The inequality holds strictly if the consumer’s type satisfies } \theta > \bar{\theta}(\bar{X} \left( p_1^A, p_1^B \right), m).\]
by 1/2, firm B has the ability to capture all the market of product 1 transactions and, as a result, learn the consumer’s type with probability 1. Observe that this condition is exactly the same as in the setting with myopic consumers.\footnote{See Proposition 2.} Given that in the forward-looking setting firm B has data-tracking ability and consumers are aware of it, one might expect that firm B would need to offer a larger discount to ensure that all consumers buy from it; Lemma 1 shows that this is not the case.

The force driving this result is that as the price gap between $p_A^1$ and $p_B^1$ increases (with price $p_B^1$ being the lowest), more consumers are “persuaded” to buy product 1 from firm B even though they reveal their type. Moreover, the consumers that still prefer to buy from firm A (and avoid revealing their type) are those with relatively high valuations for product 2 (i.e., high $\theta$). However, this information is captured by firm B when updating its beliefs – as the price gap increases, firm B’s beliefs assign relatively more probability to high values of $\theta$, which leads to firm B setting a higher price for product 2 when the consumer buys from firm A in the first period. This can be seen by noting that this price is $p_B^2 = m\bar{\theta}(X(p_A^1,p_B^1),m)$, which is decreasing in $p_B^1$. In fact, when $p_B^1 \leq p_A^1 - 1/2$, we have that $\bar{\theta}(0,m) = 1$, i.e., firm B sets the price of product 2 for consumers that bought product 1 from firm A as the largest possible valuation, which induces all consumers to buy product 1 from firm B given the discount it offers relative to firm A.

Finally, observe that the equilibrium in the continuation game only depends on product 1 prices $p_A^1$ and $p_B^1$ through the quantity $X(p_A^1,p_B^1)$, that essentially measures the gap between these prices.\footnote{See Section EC 2 in the Electronic Companion.} This property is helpful to characterize the equilibria of the game, since, the problem of finding product 1 equilibrium prices can be reduced to finding the equilibrium value of $X$. Indeed, this property allows us to reduce a two-dimensional problem to a single dimension, which simplifies our analysis considerably.\footnote{Our choice of action spaces requires some justification. Note that setting a negative price is a dominated strategy for firm A (as it is better off by setting a price of zero), so no equilibria in undominated strategies are lost by requiring firm A prices to be non-negative. Once we take this into account, any price below $-1/2$ is strictly dominated for firm B (as it is better off with $p_B^1 = -1/2$), so no equilibria are lost by bounding firm B’s action space below by $-1/2$. Claim 19 in Appendix A.3 shows that restricting firms’ action spaces to $S^A$ and $S^B$ results in no loss of equilibria in undominated strategies for $G(m)$.}

Having characterized the equilibrium in the continuation game, we now focus on the second step of the proof, that consists of deriving firms’ profit functions in terms of product 1 prices, incorporating the subsequent equilibrium path that we characterized in Lemma 1, and defining a simultaneous-move pricing game with these functions. We define this game as follows.

\textbf{Definition 1.} For $m > 0$, let $G(m)$ be the two-player normal-form game with action spaces\footnote{See Section EC 2 in the Electronic Companion.}
Lemma 3. There exist constants \(m_L, m_H\) with \(0 < m_L \leq m_H\) such that:

1. If \(m < m_L\) (Interior equilibrium regime): \(G(m)\) admits a unique pure-strategy Nash equilibrium \((p_1^A, p_1^B)\). Moreover, these prices satisfy \(p_1^A > 0\) and \(p_1^B > p_1^A - 1/2\).

\(S^A = [0, 1]\) and \(S^B = [-1/2, 1]\), and profit functions \(\pi^A\) and \(\pi^B\) that we define as\(^{32}\)

\[
\pi^A(p_1^A, p_1^B, m) = p_1^A \psi(X(p_1^A, p_1^B), m), \tag{A.5}
\]
\[
\pi^B(p_1^A, p_1^B, m) = (1 - \psi(X(p_1^A, p_1^B), m)) p_1^B + m \phi(X(p_1^A, p_1^B), m), \tag{A.6}
\]

where \(\psi, \xi, \phi : [0, 1] \times \mathbb{R}^{++} \to \mathbb{R}\) are defined by

\[
\psi(x, m) = 2x\bar{\theta}(x, m), \tag{A.7}
\]
\[
\xi(x, m) = (1 - \psi(x, m)) \mathbb{E}_{\mu_0}[\theta | \xi \geq x + m (\theta - \bar{\theta}(x, m))^+], \tag{A.8}
\]
\[
\phi(x, m) = \xi(x, m) + \frac{1}{2} \psi(x, m) \bar{\theta}(x, m). \tag{A.9}
\]

While these expressions are formally derived in the proof of Claim 8, it is useful to (informally) preview their interpretation. Based on the equilibrium path described in Lemma 1, \(\psi(X(p_1^A, p_1^B), m)\) represents the expected demand for buying product 1 from firm A, given prices \(p_1^A\) and \(p_1^B\). Since firm A only participates in the market for product 1, its profit function is simply the product of the price it sets, \(p_1^A\), and its expected demand, \(\psi(X(p_1^A, p_1^B), m)\).

Similarly, firm B’s expected profit associated to product 1 is \(p_1^B (1 - \psi(X(p_1^A, p_1^B), m))\). However, firm B also sells product 2 in the second period, which results in a profit that is represented by the term \(m \phi(X(p_1^A, p_1^B), m)\) in (A.6). This last term is further split in two components: first, the term \(m \xi(X(p_1^A, p_1^B), m)\) represents the profit firm B makes by selling product 2 to a consumer that previously bought product 1 from it, and therefore revealed her valuation for product 2,\(^{33}\) while the remaining term in (A.9) represents the expected product 2 profit in the event that the consumer buys product 1 from firm A instead.

The second step of our proof concludes by establishing the following relationship between the equilibria of our original game and those in game \(G(m)\). Formally,

**Lemma 2.** If \(G(m)\) admits a pure strategy Nash equilibrium \((p_1^{A*}, p_1^{B*})\), then there exists an equilibrium in the forward-looking setting with \((\sigma_1^{A}(\emptyset), \sigma_1^{B}(\emptyset)) = (p_1^{A*}, p_1^{B*})\).

Conversely, if \((\gamma, \sigma_1^{A}, \sigma_1^{B}, \mu)\) is an equilibrium in the forward-looking setting, then \((p_1^{A*}, p_1^{B*}) = (\sigma_1^{A}(\emptyset), \sigma_1^{B}(\emptyset))\) is a pure strategy Nash equilibrium in \(G(m)\).

In the third and final step, we characterize the pure-strategy Nash equilibria of the game \(G(m)\). Formally, we show that

**Lemma 3.** There exist constants \(m_L, m_H\) with \(0 < m_L \leq m_H\) such that:

1. If \(m < m_L\) (Interior equilibrium regime): \(G(m)\) admits a unique pure-strategy Nash equilibrium \((p_1^A, p_1^B)\). Moreover, these prices satisfy \(p_1^A > 0\) and \(p_1^B > p_1^A - 1/2\).

\(^{32}\)We include the dependency of profit functions on the parameter \(m\) explicitly.

\(^{33}\)To see this, note that \(\xi(X(p_1^A, p_1^B), m)\) is defined as the expected value of \(\theta\), conditional on the consumer preferring to buy product 1 from firm B, multiplied by the probability that this event occurs.
2. If \( m > m_H \) (Corner equilibrium regime): \((p_A^1, p_B^1) = (0, -1/2)\) is the unique pure-strategy Nash equilibrium in \( G(m) \).

Note that \( G(m) \) is an asymmetric pricing game where, crucially, firm B’s profit function is not quasi-concave in \( p_B^1 \) in general (see Appendix A.3). In particular, firm B’s best-response correspondence is not convex-valued everywhere, as illustrated in Figure 7. Indeed, due to this fact, the game \( G(m) \) admits no pure-strategy Nash equilibria for some values of \( m \). Therefore, to prove Lemma 3, we directly characterize firms’ best-response correspondences for \( G(m) \) and establish conditions that ensure existence of a pure-strategy Nash equilibrium, as well as its uniqueness (when it exists). We provide an overview of the proof of Lemma 3 in Appendix A.3, which consists of a series of claims. In the interest of space, we have relegated the proofs of these claims to Section EC 1 of the Electronic Companion as they are, for the most part, algebraic exercises.

Figure 7: Illustration of the three possible shapes of firm B’s best-response correspondence in \( G(m) \), \( BR^B \), as we establish in Claim 16. The three charts above display the shape of \( BR^B(p_A^1, m) \) for \( m \in \{1/2, 1, 5\} \), from left to right. As can be seen in the second panel, \( BR^B(p_A^1, m) \) need not be convex-valued for all \( p_A^1 \in S^A \) when \( M_0 \leq m \leq M_1 \).

Moreover, in Section EC 2 of the Electronic Companion, we establish additional results to fully characterize the set of values of \( m \) for which \( G(m) \) admits a pure-strategy Nash equilibrium, and leverage these results to approximately compute the values of \( m_L \) and \( m_H \), and find that \( m_L \approx 3.98 \) and \( m_H \approx 4.02 \). This implies that an equilibrium may not exist only in a small interval of the parameter space (i.e., \( m \in (3.98, 4.02) \)). In fact, for some of these values, we indeed can show that no equilibrium exists\(^{34} \) (for instance, for \( m = 4 \)).

In what follows, Appendices A.1, A.2 and A.3 provide the proofs of Lemmas 1, 2 and 3, respectively.

\(^{34}\)We note that we require the firms’ pricing strategies to be pure strategies. However, Nash equilibria would exist if we allowed for mixed strategies. However, this may not be a particularly illuminating exercise given that pure-strategy equilibria exist for most of the parameter space.
A.1 Proof of Lemma 1

To prove Lemma 1, we establish some necessary conditions that any equilibrium \((\gamma, \sigma^A_1, \sigma^B, \mu)\) must satisfy. Later on, when proving Lemma 2, we show that these conditions are indeed sufficient.

To characterize the equilibrium conditions, we proceed by backwards induction in a series of claims as follows. Claim 1 characterizes the consumer’s purchasing strategy for product 2. Claim 2 then pins down firm B’s pricing strategy for product 2, in the subset of histories where the consumer buys product 1 from firm B, and therefore the firm perfectly observes the consumer’s type. Then, Claim 3 provides the form of the consumer’s purchasing strategy for product 1. Next, in Claim 4, we show that the probability of the consumer being indifferent between buying product 1 from either firm is zero, which simplifies the argument later on. Claim 5 then characterizes firm B’s beliefs on the consumer’s type, following a history where the consumer does not buy product 1 from firm B, assuming that this occurs with positive probability. Then, given these beliefs, we obtain the closed-form expression for firm B’s pricing strategy for product 2 following such histories in Claim 6. Finally Claim 7 provides a closed-form expression for the probability that the consumer buys product 1 from firm A as a function of product 1 prices, and in particular, establishes that this probability is zero when product 1 prices satisfy \(p^B_1 \leq p^A_1 - 1/2\). In what follows, we state and prove these claims, and explain the relationships amongst them in more detail. Then, we prove Lemma 1.

We start by characterizing the consumer’s purchasing strategy in the second period, which follows a simple form: buying when the price is lower than her valuation for product 2, not buying in the opposite case, and mixing between both actions when indifferent.

**Claim 1.** In any equilibrium, given any history \(h \in H^c_2\), we must have:

\[
\gamma_2(h) = \begin{cases} 
1, & \text{if } m\theta(h) > p^B_2(h) \\
q(h) \in [0,1], & \text{if } m\theta(h) = p^B_2(h) \\
0, & \text{if } m\theta(h) < p^B_2(h) 
\end{cases}
\]  

(A.10)

**Proof of Claim 1.** Given a history \(h \in H^c_2\), by sequential rationality in period 2, the consumer chooses \(a_2\) to maximize \(a_2(m\theta(h) - p^B_2(h))\). It follows that she is only indifferent when \(m\theta(h) = p^B_2(h)\). \(\square\)

Next, we show that when firm B observes the consumer’s type, it sets the price for product 2 as the consumer’s valuation and the consumer buys the product with probability 1.

**Claim 2.** Suppose that strategy \(\gamma\) satisfies the condition in Claim 1. Fix any history \(h \in H^c_2\) where the consumer bought product 1 from firm B (i.e., \(a_1(h) = 1\)). Then, in any equilibrium, we have that \(\sigma^B_2(h) = m\theta(h)\) with probability 1. Moreover, we must have that the mixing probability for \(\gamma_2\) at the resulting subsequent history of \(h\) is \(q(h, m\theta(h)) = 1\) (see (A.10)).

**Proof of Claim 2.** Since \(a_1(h) = 1\), we have that \(J(h) = h\). Then, by consistency of the beliefs for firm B, we have that, in equilibrium, \(\mu(\cdot | h)\) assigns probability 1 to the true type \((s(h), \theta(h))\).
Moreover, notice that it is always suboptimal for firm B to choose $p_2^B$ outside of $[0,m]$ (since this is the support of consumer valuations for product 2). Thus, sequential rationality for firm B implies

$$
\sigma_2^B(h) \in \arg \max_{p_2^B \in [0,m]} p_2^B \left( 1_{\{x: x < m\theta(h)\}}(p_2^B) + q \left( \langle h, p_2^B \rangle \right) 1_{\{x: x = m\theta(h)\}}(p_2^B) \right).
$$

Notice that an equilibrium can only exist if $q(\langle h, p_2^B \rangle) = 1$, in which case $\sigma_2^B(h) = m\theta(h)$. Otherwise, if $q(\langle h, p_2^B \rangle) < 1$, then all prices of the form $m\theta(h) - \varepsilon$ for $\varepsilon > 0$ small enough result in higher profit than $m\theta(h)$.

Now, we derive the consumer’s equilibrium strategy for period 1, given any strategies $\gamma_2$ and $\sigma_2^B$ that satisfy the conditions given in the previous two claims.

**Claim 3.** Suppose that strategies $\gamma$ and $\sigma^B$ satisfy the conditions in Claims 1 and 2 respectively. Given any product 1 prices $p_1^A$, $p_1^B$, and any $t \in [0,1]$, define

$$
\bar{g}(t \mid p_1^A, p_1^B) = p_1^B - p_1^A + 1/2 + \mathbb{E} \left[ (mt - p_2^B)^+ \right],
$$

where the expectation is taken w.r.t. $p_2^B$, and $p_2^B \sim \sigma_2^B(p_1^A, p_1^B, 0)$. Then, in any equilibrium, given a consumer history $h \in H_1^1$, $\gamma_1$ must take the following form:

$$
\gamma_1(h) = \begin{cases} 
1, & \text{if } s(h) > \bar{g}(\theta(h) \mid p_1^A(h), p_1^B(h)) \\
\beta(h) \in [0,1], & \text{if } s(h) = \bar{g}(\theta(h) \mid p_1^A(h), p_1^B(h)) \\
0, & \text{if } s(h) < \bar{g}(\theta(h) \mid p_1^A(h), p_1^B(h)).
\end{cases}
$$

Proof of Claim 3. Given history $h = (s, \theta, p_1^A, p_1^B) \in H_1^1$, the consumer’s total utility when buying product 1 from firm B ($a_1 = 1$) is

$$
U_1 = u_1(1; h) = \left( \bar{u} - (1 - s)/2 - p_1^B \right),
$$

as, by assumption, $\sigma_2^B$ and $\gamma_2$ satisfy the conditions in Claims 1 and 2 respectively. That is, firm B learns the consumer’s type and sets a price equal to the consumer’s valuation for product 2. On the other hand, choosing not to buy from firm B ($a_1 = 0$) yields

$$
U_1 = u_1(0; h) + \mathbb{E} \left[ u_2(a_2; s, \theta, p_2^B) \mid s, \theta \right] = \left( \bar{u} - s/2 - p_1^A \right) + \mathbb{E} \left[ (m\theta - p_2^B)^+ \mid \theta \right],
$$

where the equality follows since $p_2^B$ and $a_2$ are chosen according to $\sigma_2^B$ and $\gamma_2$, respectively. In addition, the last expectation is taken w.r.t. $p_2^B \sim \sigma_2^B(p_1^A, p_1^B, 0)$.

The consumer is only indifferent between either action if the previous two expressions are equal, which is equivalent to $s = \bar{g}(\theta \mid p_1^A, p_1^B)$. \qed

---

35Note that $\bar{g}$ is defined given a strategy $\sigma_2^B$ that satisfies the condition of Claim 2. In contrast, $g$, as defined in (A.4) results from plugging in the form of $\sigma_2^B(p_1^A, p_1^B, 0)$ that we derive in Claim 6. Thus, in equilibrium we will have that $g = \bar{g}$. 35
Given any choice for product 1 prices $p_1^A, p_1^B$, and any pair of strategies $\gamma, \sigma^B$ that satisfy the previous claims, we split the consumer type space according to the behavior induced by $\gamma$. We define $A_1(p_1^A, p_1^B)$ as the set of consumer types that strictly prefer buying from firm A given those prices. Analogously, let $A_0(p_1^A, p_1^B)$ and $A_I(p_1^A, p_1^B)$ be the set of consumer types that strictly prefer to buy from firm A and are indifferent between both firms, respectively. From Claim 3, we have that

$$A_1(p_1^A, p_1^B) = \{(s, \theta) \in [0, 1] \times [0, 1] : s > \bar{g}(\theta \mid p_1^A, p_1^B)\},$$
$$A_0(p_1^A, p_1^B) = \{(s, \theta) \in [0, 1] \times [0, 1] : s < \bar{g}(\theta \mid p_1^A, p_1^B)\},$$
$$A_I(p_1^A, p_1^B) = \{(s, \theta) \in [0, 1] \times [0, 1] : s = \bar{g}(\theta \mid p_1^A, p_1^B)\}. \quad (A.13)$$

We now show that any history in which the consumer is indifferent between buying product 1 from either firm occurs with probability zero in an equilibrium.

**Claim 4.** Suppose that strategies $\gamma$ and $\sigma^B$ satisfy the conditions in Claim 3. Then, given product 1 prices $p_1^A$ and $p_1^B$, in any equilibrium we must have $\mu_0(A_I(p_1^A, p_1^B)) = 0$.

**Proof of Claim 4.** The set of consumer types $s$ that satisfy $s = \bar{g}(\theta \mid p_1^A, p_1^B)$ has Lebesgue measure zero, and therefore has measure zero (relative to $\mu_0$).

The next result characterizes the equilibrium beliefs following a history where the consumer buys product 1 from A, assuming that this event occurs with positive probability, i.e., assuming that the event $A_0(p_1^A, p_1^B)$ occurs with positive probability if players follow $(\gamma, \sigma)$. In what follows, we denote by $\mu_2^\theta(\cdot \mid I)$ the CDF of the marginal distribution of $\theta$ induced by $\mu_2$, given $I \in T_2$, i.e., for $t \in [0, 1],

$$\mu_2^\theta(t \mid I) = \mu_2([0, 1] \times [0, t] \mid I) = \int_{[0,1] \times [0,t]} d\mu_2(s, \theta \mid I).$$

**Claim 5.** Fix prices $p_1^A, p_1^B$ and suppose that strategies $\gamma$ and $\sigma^B$ satisfy the conditions in Claim 3. Let $I = (p_1^A, p_1^B, 0) \in T_2^2$, and suppose that $\mu_0(A_0(p_1^A, p_1^B)) > 0$. Then, $\mu$ is consistent with $(\gamma, \sigma)$ at $I$ if and only if $\mu_2^\theta(\cdot \mid I)$ is the uniform probability measure on $A_0(p_1^A, p_1^B)$. In addition, $\mu_2^\theta(\cdot \mid I)$, admits a density that is proportional to

$$\bar{g}(z \mid p_1^A, p_1^B) = \max \{0, \min \{\bar{g}(z \mid p_1^A, p_1^B), 1\}\}. \quad (A.14)$$

**Proof of Claim 5.** Let $B$ be a Borel set in the unit square. By Bayes’ rule, and since $\mu_1(\emptyset) = \mu_0$ by definition, we have that

$$\mu_2(B \mid I) = \mathbb{P}_{\mu_0}[(s, \theta) \in B \mid I, \gamma, \sigma] = \frac{\mu_0(B \cap A_0(p_1^A, p_1^B))}{\mu_0(A_0(p_1^A, p_1^B))} = \frac{\int_B I_{A_0(p_1^A, p_1^B)}(s, \theta) d(s, \theta)}{\mu_0(A_0(p_1^A, p_1^B))}. \quad (A.15)$$

By Fubini’s theorem and the definition of $A_0(p_1^A, p_1^B)$, the marginal CDF of $\theta$ induced by $\mu_2(\cdot \mid I)$
satisfies

\[
\mu^\theta(t \mid I) = \frac{1}{\mu_0(\mathcal{A}_0(p^A_1, p^B_1))} \int_0^t \tilde{g}(\theta \mid p^A_1, p^B_1) d\theta = \frac{1}{\mu_0(\mathcal{A}_0(p^A_1, p^B_1))} \int_0^t \bar{g}(\theta \mid p^A_1, p^B_1) d\theta,
\]
as desired \qedhere

The next claim characterizes the form of firm B’s pricing strategy for product 2 in an equilibrium, following histories in which the consumer does not buy product 1 from firm B, assuming that this occurs with positive probability.

**Claim 6.** Fix prices \(p^A_1, p^B_1\) and suppose that strategies \(\gamma\) and \(\sigma^B\) satisfy the conditions in Claim 3. Let \(I = (p^A_1, p^B_1, 0) \in \mathcal{I}_2^0\), and suppose that \(\mu_0(\mathcal{A}_0(p^A_1, p^B_1)) > 0\). Then, in any equilibrium, we must have that \(\sigma^B_2(I) = m\theta(\bar{X}(p^A_1, p^B_1), m)\) with probability 1, where \(\bar{\theta}\) is defined as in (A.2).

**Proof of Claim 6.** Fix a history \(h = (s, \theta, p^A_1, p^B_1, a_1) \in \mathcal{H}_2^f\) such that the consumer does not buy product 1 from firm B, i.e., such that \(a_1 = 0\). Let \(I = \mathcal{J}(h) = (p^A_1, p^B_1, 0)\) be the information vector observed by firm B following such history. The proof consists of two parts. First, we show that in any equilibrium, there exists \(t^*(I) \in [0, 1]\) such that \(\sigma^B_2(I) = mt^*(I)\) with probability 1. In the second part, we show that \(t^*(I) = \bar{\theta}(\bar{X}(p^A_1, p^B_1), m)\).

Consider firm B’s pricing decision for product 2 given information \(I\). Since \(\gamma_2\) satisfies Claim 1 and the consumer’s valuation for product 2 is in \([0, m]\), it is suboptimal for firm B to choose \(p^B_2\) outside of \([0, m]\). Then, by sequential rationality, \(\sigma^B_2(I)\) takes values in \([0, m]\), in any equilibrium. We will now show that there exists \(t^*(I) \in [0, 1]\) such that \(\sigma^B_2(I) = mt^*(I)\) with probability 1.

To do so, we first write firm B’s continuation profit when setting the price of product 2, given \(I\). First, given the beliefs that \((s, \theta) \sim \mu_2(\cdot \mid I)\), Claims 4 and 5 imply that in any equilibrium we have that \(\mu_2(\mathcal{A}_I(p^A_1, p^B_1) \mid I) = 0\) and, moreover, by Claim 5 we have that \(\mu^\theta(\cdot \mid I)\) is atomless. Then, given \(\gamma_2\) and information \(I\), firm B’s expected continuation profit at price \(p^B_2 \in [0, m]\) is

\[
p^B_2 \left(1 - \mu^\theta(p^B_2/m \mid I)\right).
\]

We change variables by letting \(p^B_2 = mt\) with \(t \in [0, 1]\) and use the definition of \(\mu^\theta_2\) to rewrite the previous expression as \(mF(t)\), with

\[
F(t) = t \left(1 - \mu^\theta_2(t \mid I)\right) = \frac{t}{\mu_0(\mathcal{A}_0(p^A_1, p^B_1))} \int_t^1 \tilde{g}(z \mid p^A_1, p^B_1) dz, \quad (A.15)
\]

Consider the problem of maximizing \(F(t)\) for \(t \in [0, 1]\). A solution to this problem exists as \(F\) is a continuous function and \([0, 1]\) is compact. In addition, \(F\) is differentiable in \((0, 1)\) and

\[
F'(t) = 1 - \mu^\theta_2(t \mid I) - \frac{t}{\mu_0(\mathcal{A}_0(p^A_1, p^B_1))} \tilde{g}(t \mid p^A_1, p^B_1). \quad (A.16)
\]

Note that \(F(0) = F(1) = 0\), and also that \(F'(0) = 1\). Thus, there exist values of \(t \in (0, 1)\) such
that $F(t) > 0$. Hence, any solution $t^* \in \operatorname{arg\,max}_{t \in [0,1]} F(t)$ lies in $(0,1)$, and satisfies $F'(t^*) = 0$.

We claim that this optimization problem has a unique solution. To see this, let $t_0 = \inf \{ t \in [0,1] : \mu_0^B(t \mid I) > 0 \}$ and observe that $\bar{g}(t \mid p_1^A, p_1^B)$ is non-decreasing in $t$ (from (A.11) and (A.14)). Then, by (A.16), we have that $F'(t) = 1$ for $t < t_0$, and that $F'(t)$ is strictly decreasing for $t_0 < t \leq 1$. Therefore, there exist a unique $t^* \in [0,1]$ that maximizes $F(t)$. By sequential rationality, we must have that $\sigma_2^B(I) = mt^*$ with probability 1, which concludes the first step of the proof.

We now show that $t^* = \bar{\theta}([X(p_1^A, p_1^B), m])$, where $\bar{\theta}$ is defined as in (A.2). To derive this expression, first notice that as $\sigma(p_1^A, p_1^B)$, where we abbreviate $\bar{\theta}(t^*)$ occurs.

On the other hand, by Claim 5 we have

$$
\mu_0 \left( \mathcal{A}_0(p_1^A, p_1^B) \right) = \mu_0 \left( \{ s < \bar{g}(\theta \mid p_1^A, p_1^B) \} \right) = \int_0^1 \bar{g}(t \mid p_1^A, p_1^B) dt = \int_0^1 \min \left\{ X + m(t - t^*)^+, 1 \right\} dt,
$$

where we abbreviate $\bar{X} = \bar{X}(p_1^A, p_1^B)$. We then solve the system given by equations (A.17) and (A.18) to obtain the expression for $t^*$, which, as we argued before, exists and is uniquely defined in $[0,1]$. By computing the integral on (A.18), we obtain:

$$
\mu_0 \left( \mathcal{A}_0(p_1^A, p_1^B) \right) = \begin{cases} 
\bar{X} + \frac{1}{2} m \left(1 - t^*\right)^2, & \text{if } \bar{X} + m \left(1 - t^*\right) \leq 1. \\
\bar{X} + \left(1 - \bar{X}\right) \left(1 - t^*\right) - \frac{1}{2m} \left(1 - \bar{X}\right)^2, & \text{if } \bar{X} + m \left(1 - t^*\right) > 1.
\end{cases}
$$

Plugging in (A.19) to (A.17), simple algebra yields $t^* = \bar{\theta}(\bar{X}, m)$, as desired. \hfill \square

The next claim provides an expression for the probability that the consumer buys product 1 from firm A, given product 1 prices $p_1^A$ and $p_1^B$, i.e., the probability (under $\mu_0$) that the event $\mathcal{A}_0(p_1^A, p_1^B)$ occurs.

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Claim 7. Fix strategies \( \gamma \) and \( \sigma^B \) that satisfy the conditions in Claims 3 and 6, respectively. Then, for any prices \( p^A_1 \in S^A \) and \( p^B_1 \in S^B \), we have that

\[
\mu_0 (A_0(p^A_1, p^B_1)) = \psi (X(p^A_1, p^B_1), m),
\]

where \( \psi \) is defined in (A.7). In particular, \( A_0(p^A_1, p^B_1) \) occurs with positive probability under \( \mu_0 \) if and only if \( p^B_1 - p^A_1 > -1/2 \).

Proof of Claim 7. First assume that \( \mu_0 (A_0(p^A_1, p^B_1)) > 0 \). It follows from (A.17) in the proof of Claim 6 that we can write this probability as

\[
\psi (X(p^A_1, p^B_1), m) = 2X(p^A_1, p^B_1)\bar{\theta} (X(p^A_1, p^B_1), m) > 0,
\]

which implies that \( \bar{X}(p^A_1, p^B_1) > 0 \), and therefore \( p^B_1 - p^A_1 > -1/2 \).

Now assume that \( \mu_0 (A_0(p^A_1, p^B_1)) = 0 \). We claim that \( \bar{X}(p^A_1, p^B_1) = 0 \) (i.e., that \( p^B_1 - p^A_1 \leq -1/2 \)). To show this, assume towards a contradiction that \( p^B_1 - p^A_1 > -1/2 \). Then, there exists \( \varepsilon > 0 \) such that \( p^B_1 - p^A_1 + 1/2 > \varepsilon \), which implies that \( \bar{g}(t | p^A_1, p^B_1) > \varepsilon \) for all \( t \in [0, 1] \). It follows that

\[
\mu_0 (A_0(p^A_1, p^B_1)) = \mu_0 \left( \{ s < \bar{g}(\theta | p^A_1, p^B_1) \} \right) > \mu_0 \left( \{ s < \varepsilon \} \right) = \min \{ \varepsilon, 1 \} > 0,
\]

a contradiction. Therefore, \( \bar{X}(p^A_1, p^B_1) = 0 \). The result follows since \( \psi(0, m) = 0 \) for all \( m > 0 \). \( \square \)

Finally, we leverage the previous claims to prove Lemma 1.

Proof of Lemma 1. First suppose that \( p^B_1 - p^A_1 \leq -1/2 \). By Claims 4 and 7, we have that \( \mu_0 (A_1(p^A_1, p^B_1)) = 1 \), i.e., the consumer buys product 1 from firm B with probability 1. Since firm B observes the consumer’s type if she buys product 1 from it, we have that firm B indeed observes the type with probability 1, and, by Claim 2, sets the price of product 2 as the consumer’s valuation for that product, which the consumer buys with probability 1.

Now suppose that \( p^B_1 - p^A_1 > -1/2 \). By Claim 6, we have that \( \bar{g}(t | p^A_1, p^B_1) = g(t | p^A_1, p^B_1), \) for all \( t \in [0, 1] \), where these expressions are given in (A.11) and (A.4), respectively.

Then, by Claim 3, we have that if the consumer’s type satisfies \( s < g(\theta | p^A_1, p^B_1) \), she buys product 1 from firm A with probability 1. If that is the case, then, by Claim 6, firm B sets the price of product 2 as \( p^B_2 = m\bar{\theta}(X(p^A_1, p^B_1), m) \). By Claim 1, if the consumer’s type is such that \( \theta > \bar{\theta}(X(p^A_1, p^B_1), m) \), she buys product 2 with probability 1.

Similarly, Claim 3 implies that if the consumer’s type satisfies \( s > g(\theta | p^A_1, p^B_1) \), she buys product 1 from firm B with probability 1. By Claim 2, firm B then sets the price of product 2 as the consumer’s valuation (i.e., \( m\theta \)), and the consumer buys it with probability 1.

\( \square \)

A.2 Proof of Lemma 2

The proof of Lemma 2 consists of two steps. First, in Claim 8, we formally derive firms’ profit functions in terms of product 1 prices, assuming that the consumer and firm B play equilibrium
strategies following firms’ choices for product 1 prices. We show that these profit functions are indeed the profit functions of the auxiliary game $G(m)$. In the second step (Claim 9), we show that given a pure-strategy Nash equilibrium for $G(m)$, we can construct an equilibrium for the original game.

**Claim 8.** Fix an assessment $(\gamma, \sigma_A^1, \sigma_B, \mu)$ that satisfies the conditions of Lemma 1, and let $p_1^A = \sigma_A^1(\emptyset)$ and $p_1^B = \sigma_B^1(\emptyset)$. Then,

$$
\Pi_1^A (\gamma, \sigma_A^1, \sigma_B, \mu \mid \emptyset) = \pi^A (p_1^A, p_1^B, m), \quad \text{and} \quad \Pi_1^B (\gamma, \sigma_A^1, \sigma_B, \mu \mid \emptyset) = \pi^B (p_1^A, p_1^B, m)
$$

where $\pi^A$ and $\pi^B$ are defined as in (A.5) and (A.6), respectively.

**Proof of Claim 8.** The proof consists of computing the firms’ profit functions taking into account the necessary conditions for equilibrium given in Lemma 1. Note that these necessary conditions fully characterize the equilibrium after product 1 prices are set, except for events that occur with probability zero.\(^{36}\)

First, consider firm A. By definition of A’s total payoff and as $\mu_1(\emptyset) = \mu_0$, we have

$$
\Pi_1^A (\gamma, \sigma_A^1, \sigma_B, \mu \mid \emptyset) = E_{\mu_0} \left[ \pi^A (p_1^A, p_1^B, a_1) \right],
$$

where the expectation is taken with respect to the consumer type and $a_1$, which is chosen according to strategy $\gamma_1$, that, by assumption, satisfies Claim 3. Recall that firm A only participates in period 1, so its total payoff is $\pi^A (p_1^A, p_1^B, a_1) = p_1^A (1 - a_1)$. Thus,

$$
\Pi_1^A (\gamma, \sigma_A^1, \sigma_B, \mu \mid \emptyset) = E_{\mu_0} \left[ \pi^A (p_1^A, p_1^B, a_1) \right] = E_{\mu_0} \left[ p_1^A (1 - \gamma_1(s, \theta, p_1^A, p_1^B)) \right] = p_1^A \mu_0 (\{(s, \theta) \in A_0(p_1^A, p_1^B)\}) = p_1^A \psi (X(p_1^A, p_1^B, m)) = \pi^A (p_1^A, p_1^B, m),
$$

where the third and fourth equalities follow from Claims 4 and 7, and $A_0(p_1^A, p_1^B)$ and $\psi$ are defined in (A.13) and (A.7), respectively.

Now consider firm B. Again, since $\mu_1(\emptyset) = \mu_0$, we have

$$
\Pi_1^B (\gamma, \sigma_A^1, \sigma_B, \mu \mid \emptyset) = E_{\mu_0} \left[ \pi^B (p_2^B, a_2) \right],
$$

where the expectation is taken with respect to the consumer’s type and the consumer’s actions $a_1, a_2$, that are chosen according to $\gamma_1$ and $\gamma_2$ respectively, and $p_2^B$ is chosen according to $\sigma_2^B$. Following the same argument as with firm A, we have that

$$
E_{\mu_0} [\pi^B (p_1^A, p_1^B, a_1)] = p_1^A (1 - \psi (X(p_1^A, p_1^B), m)). \quad \text{(A.20)}
$$

Moreover, since the probability that the consumer is indifferent between buying product 1 from either firm is zero (by Claim 4), we have that

$$
E_{\mu_0} [\pi^B (p_2^B, a_2)] = E_{\mu_0} \left[ p_2^B a_2 1_{A_0(p_1^A, p_1^B)}(s, \theta) \right] + E_{\mu_0} \left[ p_2^B a_2 1_{A_1(p_1^A, p_1^B)}(s, \theta) \right] \quad \text{(A.21)}
$$

\(^{36}\)Namely, the event that the consumer’s type satisfies either $s = g(\theta \mid p_1^A, p_1^B)$ or $\theta = \bar{\theta} (p_1^A, p_1^B, m)$. 40
We now analyze these two components separately. For the event where the consumer buys product 1 from firm B, by Claim 7 we have (abbreviating $\tilde{X} = X(p^A_1, p^B_1)$):

$$
\mathbb{E}_{\mu_0} \left[ p^B_2 a_2 1_{A_1(p^A_1,p^B_1)}(s, \theta) \right] = \mu_0 \left( A_1(p^A_1,p^B_1) \right) \mathbb{E}_{\mu_0} \left[ m\theta \mid (s, \theta) \in A_1(p^A_1,p^B_1) \right] \\
= m \left( 1 - \psi(\tilde{X}, m) \right) \mathbb{E}_{\mu_0} \left[ \theta \mid s > \tilde{X} + m \left( \theta - \bar{\theta}(\tilde{X}, m) \right) \right] \\
= m \xi (\tilde{X}, m),
$$

where the last step simply uses the definition of $\xi$ in (A.8). On the other hand, for the event where the consumer buys product 1 from firm A we have, by Claims 1 and 6

$$
\mathbb{E}_{\mu_0} \left[ p^B_2 a_2 1_{A_0(p^A_1,p^B_1)}(s, \theta) \right] = \mathbb{E}_{\mu_0} \left[ m\bar{\theta}(\tilde{X}, m) g_2(s, \theta, p^A_1, p^B_1, 0) 1_{A_0(p^A_1,p^B_1)}(s, \theta) \right] \\
= \mathbb{E}_{\mu_0} \left[ m\bar{\theta}(\tilde{X}, m) 1_{A_0(p^A_1,p^B_1) \cup \{ \theta > \bar{\theta}(\tilde{X}, m) \}}(s, \theta) \right] \\
= m \bar{\theta}(\tilde{X}, m) \mu_0 \left( A_0(p^A_1,p^B_1) \cap \{ \theta > \bar{\theta}(\tilde{X}, m) \} \right). \tag{A.23}
$$

We now compute the probability $\mu_0 \left( A_0(p^A_1,p^B_1) \cap \{ \theta > \bar{\theta}(\tilde{X}, m) \} \right)$. We do this in two steps, first notice that

$$
\mu_0 \left( A_0(p^A_1,p^B_1) \cap \{ \theta > \bar{\theta}(\tilde{X}, m) \} \right) = \mu_0 \left( A_0(p^A_1,p^B_1) \right) - \mu_0 \left( A_0(p^A_1,p^B_1) \cap \{ \theta \leq \bar{\theta}(\tilde{X}, m) \} \right) \\
= \psi(\tilde{X}, m) - \mu_0 \left( \{ \theta \leq \bar{\theta}(\tilde{X}, m), s < g(\theta \mid p^A_1, p^B_1) \} \right), \tag{A.24}
$$

where we use the definition of $A_0(p^A_1,p^B_1)$ given in (A.13), and Claim 7 to obtain the last equality. Now, we compute $\mu_0 \left( \{ \theta \leq \bar{\theta}(\tilde{X}, m), s < g(\theta \mid p^A_1, p^B_1) \} \right)$:

$$
\mu_0 \left( \{ \theta \leq \bar{\theta}(\tilde{X}, m), s < g(\theta \mid p^A_1, p^B_1) \} \right) = \int_0^{\bar{\theta}(\tilde{X}, m)} \int_0^{g(\theta \mid p^A_1, p^B_1)} ds d\theta = \int_0^{\bar{\theta}(\tilde{X}, m)} g(\theta \mid p^A_1, p^B_1) d\theta \\
\int_0^{\bar{\theta}(\tilde{X}, m)} \tilde{X} d\theta = \tilde{X} \bar{\theta}(\tilde{X}, m), \tag{A.25}
$$

where the final steps follows since $g(\theta \mid p^A_1, p^B_1) = \tilde{X}$ for $\theta \leq \bar{\theta}(\tilde{X}, m)$. Plugging back to (A.24) and using the fact that $\psi(\tilde{X}, m) = 2\tilde{X} \bar{\theta}(\tilde{X}, m)$ results in

$$
\mu_0 \left( A_0(p^A_1,p^B_1) \cap \{ \theta > \bar{\theta}(\tilde{X}, m) \} \right) = \psi(\tilde{X}, m) - \tilde{X} \bar{\theta}(\tilde{X}, m) = \psi(\tilde{X}, m)/2. \tag{A.26}
$$

Thus, we can rewrite (A.23) as

$$
\mathbb{E}_{\mu_0} \left[ p^B_2 a_2 1_{A_0(p^A_1,p^B_1)}(s, \theta) \right] = m \psi(\tilde{X}, m) \bar{\theta}(\tilde{X}, m)/2.
$$

Plugging back this expression and (A.22) into (A.21) results in

$$
\mathbb{E}_{\mu_0} \left[ p^B_2 (p^B_2, a_2) \right] = m \left( \xi (\tilde{X}, m) + \psi(\tilde{X}, m) \bar{\theta}(\tilde{X}, m)/2 \right) = m \phi (\tilde{X}, m),
$$
where $\phi$ is defined as in (A.9). Finally, it follows from (A.20) that
\[
\Pi^B_1 (\gamma, \sigma^A_1, \sigma^B, \mu \mid \emptyset) = p^A_1 (1 - \psi (X(p^A_1, p^B_1), m)) + m\phi (X(p^A_1, p^B_1), m) = \pi^B (p^A_1, p^B_1, m),
\]
as desired.

Next, we establish that given a pure-strategy Nash equilibrium in $G(m)$, we can construct an equilibrium for the forward-looking setting.

**Claim 9.** If $G(m)$ admits a pure strategy Nash equilibrium $(p^A_1^\ast, p^B_1^\ast)$, then there exists an equilibrium in the forward-looking setting with $(\sigma^A_1(\emptyset), \sigma^B_1(\emptyset)) = (p^A_1^\ast, p^B_1^\ast)$.

**Proof of Claim 9.** Suppose that there exists a PSNE $(p^A_1^\ast, p^B_1^\ast)$ in $G(m)$. We will construct an equilibrium $(\gamma, \sigma^A_1, \sigma^B, \mu)$ for the forward-looking setting. To do so, we rely on the necessary conditions for equilibrium established in Lemma 1, adding some refinements to fully define the strategies and beliefs at histories that occur with probability zero.

Let us first define consumer strategies. For period 1, given a history $h \in H^*_1$, define $\gamma_1$ by
\[
\gamma_1(h) = \begin{cases} 
1, & \text{if } s(h) \geq g (\theta(h) \mid p^A_1(h), p^B_1(h)), \\
0, & \text{if } s(h) < g (\theta(h) \mid p^A_1(h), p^B_1(h)),
\end{cases}
\]
where $g$ is defined in (A.4). For the second period, given a history $h \in H^*_2$, define $\gamma_2$ by
\[
\gamma_2(h) = \begin{cases} 
1, & \text{if } m\theta(h) \geq p^B_2(h), \\
0, & \text{if } m\theta(h) < p^B_2(h).
\end{cases}
\]
For firm A, simply take $\sigma^A_1(\emptyset) = p^A_1^\ast$. For firm B, let the pricing strategy for product 1 be $\sigma^B_1(\emptyset) = p^B_1^\ast$. For product 2, given $I \in I_2$ define
\[
\sigma^B_2(I) = \begin{cases} 
m\theta(I), & \text{if } I \in I^1_2, \\
m\theta(X(p^A_1(I), p^B_1(I)), m), & \text{if } I \in I^0_2,
\end{cases}
\]
where $I^0_2$ and $I^1_2$ are defined in (2.6). Finally, we complete the definition of our proposed assessment by defining a belief system as follows. For period 1, simply let $\mu_1(\emptyset) = \mu_0$. For the second period, given $I \in I_2$, define $\mu_2(\cdot \mid I)$ as follows:

1. If $I \in I^1_2$, let $\mu_2(\cdot \mid I)$ be the probability distribution that assigns probability 1 to the true consumer type $(s(I), \theta(I))$. That is, for any Borel set $B \subseteq T$, let $\mu(B \mid I) = 1_B(s(I), \theta(I))$.
2. If $I \in I^0_2$, we have two cases.
   (a) If $p^B_1(I) - p^A_1(I) + 1/2 > 0$, let $\mu_2(\cdot \mid I)$ be the uniform probability distribution in the
set $\tilde{A}_0(I)$, which we define as

$$\tilde{A}_0(I) = A_0(p^A_1(I), p^B_1(I)) = \{(s, \theta) \in T : s < g(\theta | p^A_1(I), p^B_1(I))\},$$

that is, for any Borel set $B \subseteq T$, we have\(^{37}\)

$$\mu_2(B | I) = \frac{1}{\mu_0(\tilde{A}_0(I))} \int_B 1_{\tilde{A}_0(I)}(s, \theta) d(s, \theta). \quad (A.27)$$

(b) If $p^B_1(I) - p^A_1(I) + 1/2 = 0$, let $\mu_2(\cdot | I)$ be the uniform probability distribution in $[0, 1] \times \{1\}$. We claim that $(\gamma, \sigma^A_1, \sigma^B, \mu)$ is an equilibrium. To show this, we first show that this assessment satisfies sequential rationality for each player, and then proceed to show that the belief system $\mu$ is consistent given the strategies.

**Consumer.** Consider period 2 and let $h \in H^c_2$. It follows that for any strategy $\gamma'$ we have

$$U_2(\gamma, \sigma^A_1, \sigma^B, \mu | h) = (m\theta(h) - p^B_2(h))^{+} \geq (m\theta(h) - p^B_2(h)) \gamma'_2(h) = U_2(\gamma', \sigma^A_1, \sigma^B, \mu | h).$$

Whereas for period 1 and $h \in H^c_1$, with some algebra we can write

$$U_1(\gamma, \sigma^A_1, \sigma^B, \mu | h) = \bar{u} - (1 - s(h))/2 - p^B_1(h) + (g(\theta(h) | p^A_1(h), p^B_1(h)) - s(h))^{+} \geq \bar{u} - (1 - s(h))/2 - p^B_1(h) + (1 - \gamma'_1(h)) (g(\theta(h) | p^A_1(h), p^B_1(h)) - s(h)) = U_1(\gamma', \sigma^A_1, \sigma^B, \mu | h).$$

Thus, $\gamma$ is sequentially rational for the consumer given $\sigma$ and $\mu$.

**Firm B.** Let $\sigma^{B'}$ be a strategy for firm B and take $I \in T^2_2$. Given the belief system $\mu$ described above, we have three cases. First, consider the case where $I \in T^2_1$. It follows that for any $\sigma^B_2(I) \in \mathbb{R}$,

$$\Pi^B_2(\gamma, \sigma^A_1, \sigma^B, \mu | I) = m\theta(I) \geq \sigma^{B'}_2(I) \mathbf{1}_{\{\sigma^{B'}_2(I) \leq m\theta(I)\}}(\sigma^B_2(I)) = \sigma^{B'}_2(I) \gamma_2((I, \sigma^B_2(I))) = \Pi^B_2(\gamma, \sigma^A_1, \sigma^{B'}, \mu | I)$$

Now consider the case with $I \in T^0_2$ and $p^B_1(I) - p^A_1(I) + 1/2 > 0$. Given $\gamma$ and $\mu$, we can write firm B’s continuation of profit as a function of the price it sets for product 2 as

$$p^B_2 \mu_2([0, 1] \times [p^B_2/m, 1] | I).$$

\(^{37}\)It is easy to see, following the same argument as in the proof of Claim 7, that $\mu_0(\tilde{A}_0(I)) > 0.$
Following the same argument as in the proof of Claim 6, we can rewrite this expression as
\[
\frac{p_2^B}{\mu_0(\mathcal{A}_0(h))} \int_{\min\{p_2^B/m,1\}^+}^1 \min\{g(t \mid p_1^A(I), p_1^B(I)) , 1\} dt.
\]
In the proof of Claim 6, we have shown that this expression has a unique maximizer, which is \(p_2^B = \sigma_2^B(I) = m\theta(X(p_1^A(I), p_1^B(I)), m)\). Thus, \(\sigma^B\) satisfies sequential rationality in the second period for firm B given information vector \(I, \gamma, \sigma^A_1\) and \(\mu\).

Finally, if \(I \in \mathcal{I}_2^0\) and \(p_1^B(I) - p_1^A(I) + 1/2 \leq 0, \mu_2(\cdot \mid I)\) assigns probability 1 to the event that \(\{\theta = 1\}\) and we can therefore rewrite B’s continuation of profit as a function of the price it sets for product 2 (given \(\gamma\) and \(\mu\)) as
\[
p_2^B 1_{\{p_2^B \leq m\}}(p_2^B),
\]
which is clearly maximized with \(p_2^B = \sigma_2^B(I) = m\theta(0, m) = m\). It follows that \(\sigma^B\) satisfies sequential rationality for firm B in the second period, given any information vector \(I \in \mathcal{I}_2\), and given \(\gamma, \sigma^A_1\), and \(\mu\).

To show that \(\sigma^B\) satisfies sequential rationality for firm B in period 1, notice that \(\gamma\) and \(\sigma^B_2\) that satisfy the conditions in Claims 1, 2, 3 and 6. Then, by Claim 8, and since prices \((p_1^A, p_1^B)\) form a Nash equilibrium in \(G(m)\) it follows that, for any strategy \(\sigma^B\)
\[
\Pi_1^B (\gamma, \sigma^A_1, \sigma^B, \mu \mid \emptyset) = \pi^B(p_1^A, p_1^B, m) \geq \pi^B(p_1^A, \sigma^B, (\emptyset), m) = \Pi_1^B (\gamma, \sigma^A_1, \sigma^B, \mu \mid \emptyset).
\]

**Firm A.** Following the same argument as for firm B, we have that \(\sigma^A_1\) satisfies sequential rationality for firm A given \(\sigma^B, \gamma\) and \(\mu\).

**Consistency of beliefs.** It remains to show that the belief system \(\mu\) is consistent given the strategy profile \((\gamma, \sigma)\). First, notice that if \(I \in \mathcal{I}_2^0\), \(\mu_2(\cdot \mid I)\) satisfies our definition of consistency as it assigns probability 1 to the event associated with the true consumer type observed in information vector \(I\).

Now consider \(I = (p_1^A, p_1^B, 0) \in \mathcal{I}_2^0\). By definition of \(\gamma_1\), it follows that \(I\) is reached given prices \(p_1^A\) and \(p_1^B\) if and only if the event \(\mathcal{A}_0(I)\) occurs. By following the same argument as in the proof of Claim 7, it follows that
\[
\mu_0(\mathcal{A}_0(I)) = \mu_0(\mathcal{A}_0(p_1^A(I), p_1^B(I))) = \psi(X(p_1^A(I), p_1^B(I)), m).
\]
Therefore, if \(X(p_1^A(I), p_1^B(I)) > 0\), we have that \(\mu_0(\mathcal{A}_0(I)) > 0\). In such cases, notice from (A.27) that we have defined \(\mu_2(\cdot \mid I)\) by applying Bayes’ rule to \(\mu_0 = \mu_1(\emptyset)\), so \(\mu\) satisfies consistency with the strategy profile at such information vectors.

Finally, if \(X(p_1^A(I), p_1^B(I)) = 0\), we have that \(\mu_0(\mathcal{A}_0(I)) = 0\). Thus, we can define beliefs arbitrarily at these information sets while maintaining consistency. We conclude that \(\mu\) is consistent with \((\gamma, \sigma)\). Thus, \((\gamma, \sigma^A_1, \sigma^B, \mu)\) is an equilibrium in the forward-looking setting, as desired. \(\square\)
Finally, given that we can construct an equilibrium in the forward-looking setting given a PSNE for $G(m)$, we can establish the relationship between the equilibria on both games as stated in Lemma 2.

**Proof of Lemma 2.** First suppose that $(\gamma, \sigma_A^1, \sigma_B, \mu)$ is an equilibrium in the forward-looking setting, and define $(p_A^1, p_B^1) = (\sigma_A^1(\emptyset), \sigma_B(\emptyset))$. By sequential rationality for firms in period 1, and Claim 8, it follows that $(p_A^1, p_B^1)$ is a PSNE in $G(m)$.

Conversely, if $(p_A^1, p_B^1)$ is a PSNE in $G(m)$, it follows from Claim 9 that there exists an equilibrium in the forward-looking setting with $(\sigma_A^1(\emptyset), \sigma_B(\emptyset)) = (p_A^1, p_B^1)$.

**A.3 Proof of Lemma 3**

In what follows, we prove Lemma 3, which states that $G(m)$ admits pure-strategy Nash equilibria (henceforth, PSNE) when the parameter $m$ satisfies $m < m_L$ or $m > m_H$, where $m_L$ and $m_H$ are constants that we will characterize. In addition, we show that when $G(m)$ admits a PSNE, it is the unique PSNE of the game.

To do so, we establish a series of claims that characterize firms’ best-response correspondences in $G(m)$. We start by providing some technical conditions about the auxiliary functions $\bar{\theta}, \psi, \xi$ and $\phi$ (defined in (A.2), (A.7), (A.8) and (A.9)) in Claim 10, and then show that the firms’ profit functions $\pi_A$ and $\pi_B$ are continuous in prices in Claim 11. Then, we characterize firm A’s best response-correspondence in Claim 12. In particular, we show that it is always convex-valued.

We then turn our focus to firm B’s best-response correspondence, which need not be convex-valued in general, as illustrated on Figure 7. Claims 13 and 14 provide some monotonicity properties for firm B’s best-response correspondence. In particular, they provide conditions under which firm B has an incentive to choose $p_B^1 \leq p_A^1 - 1/2$, which induces the consumer to buy product 1 from it regardless of her type, as established in Lemma 1. These two claims allow us to show that when the value of $m$ is large enough, the only PSNE of $G(m)$ is for firm A to set a price of zero, and for firm B to set a price of $-1/2$, which we do in Claim 15. We refer to this equilibrium as the *corner equilibrium*.\(^{38}\)

Claim 16 then provides a characterization of firm B’s best-response correspondence, and in particular it shows that it is single-valued everywhere for all small enough values of $m$. Combined with the fact that firm A’s best-response correspondence is always convex-valued, this allows us to establish in Claim 17 that $G(m)$ admits a PSNE for all small enough values of $m$. Moreover, we show that is this case, the equilibrium prices satisfy $p_A^1 > 0$ and $p_B^1 > p_A^1 - 1/2$; we refer to such equilibria as *interior equilibria*. Finally, Claim 18 establishes that $G(m)$ admits at most one interior PSNE. We then prove Lemma 3 based on all these claims, and we formally prove Theorem 1.

To conclude this section, we establish in Claim 19 that restricting the firms’ action spaces in $G(m)$ to be $S_A = [0, 1]$ and $S_B = [-1/2, 1]$ (as given in Definition 1) is without loss, in the sense

\(^{38}\)To make this definition complete, we say a PSNE $(p_A^1, p_B^1)$ of $G(m)$ is a corner equilibrium if $p_B^1 \leq p_A^1 - 1/2$.\)
that the set of PSNE in undominated strategies of the game remains the same if we make no restrictions on the firms’ action spaces.

In order to keep our exposition as brief as possible, and since the proofs of all these claims are primarily algebraic exercises, we have relegated the proofs of Claims 10 – 19 to Section EC 1 of the Electronic Companion.

The following claim provides various properties of the auxiliary functions we have previously defined. These properties are useful to prove our results later on.

**Claim 10.** Let \( \bar{\theta}, \psi, \xi \) and \( \phi \) be as defined in (A.2), (A.7), (A.8) and (A.9), respectively. Then, for every fixed \( m > 0 \), the functions \( \bar{\theta}(x,m) \), \( \psi(x,m) \), \( \xi(x,m) \) and \( \phi(x,m) \) are all continuous and bounded functions of \( x \in [0,1] \), and continuously differentiable for \( x \in (0,1] \). Moreover, we have that:

1. \( \bar{\theta}(x,m) \) is strictly decreasing in \( x \).
2. \( \psi(x,m) \) is strictly increasing and strictly concave in \( x \).
3. \( \xi(x,m) \) is strictly decreasing in \( x \).
4. \( \phi(x,m) \) is strictly decreasing and strictly convex in \( x \).

The following claim establishes that the firms’ profit functions in \( G(m) \) are continuous in prices. This implies that firms’ best-response correspondences are upper hemicontinuous in the competitor’s price, which enables us to use fixed point theorems later on. The proof follows directly from the properties given in Claim 10.

**Claim 11.** For fixed \( m > 0 \), \( \pi^A \) and \( \pi^B \) are continuous functions of \( (p^A_1, p^B_1) \).

Next, we characterize the form of firm A’s best-response correspondence. In particular, note that this correspondence is always convex-valued.

**Claim 12.** For \( m > 0 \), let \( BR^A(p^B_1, m) \) be firm A’s best-response correspondence in \( G(m) \), given firm B’s price \( p^B_1 \in S^B \). Then, \( BR^A(-1/2, m) = S^A \). Moreover, \( BR^A(p^B_1, m) \) is single-valued for any \( p^B_1 > -1/2 \) and, in addition, \( p^A_1 = BR^A(p^B_1, m) \) is the unique solution to

\[
p^A_1 = Z \left( \bar{X} \left( p^A_1, p^B_1 \right), m \right),
\]

where we define \( Z : [0,1] \times \mathbb{R}^{++} \rightarrow \mathbb{R} \) as

\[
Z(x,m) = \frac{\psi(x,m)}{\psi_x(x,m)}.
\]

\[\overset{39}{\text{We denote the partial derivative of } \psi \text{ w.r.t } x \text{ by } \psi_x \text{ and similarly for other functions. Note that, by Claim 10, } \psi_x(x,m) \text{ is formally defined only for } x \in (0,1]. \text{ Slightly abusing notation, we define } \psi_x(0, m) = \lim_{t \rightarrow 0^+} \psi_x(t, m) = 2, \text{ so that } Z(x,m) \text{ is well defined for all } x \in [0,1].}\]
We now study firm B’s best-response correspondence. We denote firm B’s best-response correspondence in \( G(m) \) given firm A’s price \( p_1^A \) by \( BR^B(p_1^A, m) \). In contrast to the case for firm A, a challenge that arises is that firm B’s profit function is not always quasiconcave in its own price. To see why quasiconcavity may fail, recall from equation (A.6) that firm B’s profit function can be written as \( \pi^B(p_1^A, p_1^B) = (1 - \psi(X(p_1^A, p_1^B), m)) p_1^B + m \phi(X(p_1^A, p_1^B), m) \). The first term is quasiconcave in \( p_1^B \), whereas the second term is in fact convex in \( p_1^B \); adding these two terms produces instances where \( \pi^B \) is not quasiconcave in its own price. This results in cases where firm B’s best-response correspondence is not convex-valued (see Figure 7). Moreover, this leads to the game \( G(m) \) having no PSNE for some values of \( m \) (see Remark 1 at the end of this section). The following two claims establish a series of properties that help us characterize firm B’s best-response correspondence.

The next result establishes two properties. First, that given firm A’s price \( p_1^A \), firm B will have an incentive to capture the entire market for product 1 transactions by setting a price of \( p_1^A - 1/2 \) provided that \( m \) is large enough. Second, that if for a fixed value of \( m \) and some firm A’s price \( p_1^A \in S^A \), firm B has an incentive to capture all the product 1 market by setting a price of \( p_1^A - 1/2 \), then this is also the case for any \( m' > m \).

**Claim 13.** Fix \( p_1^A \in S^A \). Then, the following two properties hold:

1. There exists \( m_0 = m_0(p_1^A) \) such that for all \( m > m_0 \) we have that \( BR^B(p_1^A, m) = p_1^A - 1/2 \).

2. If \( BR^B(p_1^A, m) = p_1^A - 1/2 \) for some \( m > 0 \), then \( BR^B(p_1^A, m') = p_1^A - 1/2 \) for any \( m' > m \).

Next, we establish that if firm B has an incentive to capture the entire market for product 1 when firm A chooses some price \( p_1^A \), that will also be the case if firm A sets any larger price.

**Claim 14.** Fix \( m > 0 \), and let \( p_1^A \in S^A \) be such that \( p_1^A - 1/2 \in BR^B(p_1^A, m) \). Then, for any \( y > p_1^A \), we have that \( BR^B(y, m) = y - 1/2 \).

The previous two claims allow us to show that \( (p_1^A, p_1^B) = (0, -1/2) \) is the only PSNE in \( G(m) \) for all \( m \) large enough. Recall that we refer to this type of equilibrium as the corner case.

**Claim 15.** There exists \( m_H > 0 \) such that for every \( m > m_H \), \( (p_1^{A*}, p_1^{B*}) = (0, -1/2) \) is the only pure-strategy Nash equilibrium of \( G(m) \). In addition, there are no corner equilibria when \( m < m_H \).

Claim 15 characterizes the unique PSNE of \( G(m) \) for \( m > m_H \). We now proceed to do so for small values of \( m \), and to prove that these equilibria are interior, i.e., that prices satisfy \( p_1^A > 0 \), \( p_1^B > -1/2 \) and \( p_1^B > p_1^A - 1/2 \). The first step towards this goal is to characterize the shape of firm B’s best-response correspondence, which is addressed by the following result. As we can see in the second panel of Figure 7, there is a range of values of \( m \) for which firm B’s best-response correspondence need not be convex-valued everywhere (i.e., when \( M_0 \leq m \leq M_1 \)). However, Claim 16 shows that \( BR^B(p_1^A, m) \) will indeed be single-valued everywhere except for at most one price \( p_1^A \), and in particular, that it is single-valued for all \( p_1^A \in S^A \), provided that \( m \) is small enough.
Claim 16. There exist constants $0 < M_0 < M_1$ that define the shape of $BR^B$ as follows:

(i) If $0 < m < M_0$, $BR^B(p_A^1, m)$ is single-valued for all $p_A^1 \in S^A$. Moreover, $p_B^1 = BR^B(p_A^1, m)$ satisfies $p_B^1 > p_A^1 - 1/2$ and:

$$p_B^1 = V(X(p_A^1, p_B^1), m),$$

where we define $V : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}$ as

$$V(x, m) = \frac{1 - \psi(x, m) + m\phi(x, m)}{\psi_x(x, m)}.$$  \hspace{1cm} (A.30)

(ii) If $M_0 \leq m \leq M_1$, there exists $d(m) \in S^A$ such that

(a) If $p_A^1 < d(m)$, $BR^B(p_A^1, m)$ is single-valued and satisfies equation (A.30).

(b) If $p_A^1 > d(m)$, $BR^B(p_A^1, m) = p_A^1 - 1/2$.

(iii) If $m > M_1$, we have that $BR^B(p_A^1, m) = p_A^1 - 1/2$ for all $p_A^1 \in S^A$.

This last result shows that even though $\pi^B$ is not quasiconcave in $p_B^1$ in general, firm B’s best-response correspondence $BR^B(p_A^1, m)$ is single-valued for all $p_A^1 \in S^A$ and all small enough $m$. This allows us to use standard results from Game Theory to prove the existence of a PSNE in $G(m)$ for all small enough $m$.

Claim 17. There exists $M_0 > 0$ such that if $m < M_0$, $G(m)$ admits a pure-strategy Nash equilibrium. Moreover, such equilibria must be interior.

The next claim establishes that $G(m)$ admits at most one interior PSNE.

Claim 18. If $G(m)$ admits an interior PSNE, it is the unique interior PSNE of the game.

Finally, we prove Lemma 3 based on the previous results.

Proof of Lemma 3. Let $m_H$ be as in Claim 15, so that $(p_A^{1*}, p_B^{1*}) = (0, -1/2)$ is the unique PSNE of $G(m)$ when $m > m_H$. Now define $E$ as the set of positive values of $m$ for which $G(m)$ admits an interior PSNE, and let

$$m_L = \sup\{m' > 0 : [0, m'] \subseteq E\}.$$  \hspace{1cm} (A.32)

Take $M_0$ as in Claim 17, so that $m_L \geq M_0 > 0$. In addition, by Claim 15, we have that $m_L \leq m_H$. By definition of $m_L$, $G(m)$ admits an interior PSNE for $m \in (0, m_L)$. Moreover, in this case, such equilibrium is the only PSNE of $G(m)$ by Claims 15 and 18.

We now prove Theorem 1 based on Lemmas 1, 2 and 3.

\footnote{Note that $V(x, m)$ is well defined since $\psi_x(x, m) > 0$ for $x \in [0, 1]$ and $m > 0$.}
Proof of Theorem 1. Let $m \in (0, m_L) \cup (m_H, \infty)$, where $m_L$ and $m_H$ are as in Lemma 3. By Lemma 3, $G(m)$ admits a unique PSNE $(p_1^A, p_1^B)$. By Lemma 2, there exists an equilibrium $(\gamma, \sigma_1^A, \sigma_1^B, \mu)$ in the forward-looking setting with $(\sigma_1^A(\emptyset), \sigma_1^B(\emptyset)) = (p_1^A, p_1^B)$. We now consider the two cases for the value of $m$ separately.

If $m > m_H$, we have that $(p_1^A, p_1^B) = (0, -1/2)$ by Lemma 3. The description of the resulting equilibrium path follows from Lemma 1.

Now consider $m < m_L$. By Lemma 3, $(p_1^A, p_1^B)$ is an interior PSNE, i.e., we have that $p_1^A > 0$ and $p_1^B > p_1^A - 1/2$. In particular, this implies that $\bar{X}(p_1^A, p_1^B) > 0$, and thus, by Claim 7, the expected product 1 demand for both firms is positive. Moreover, let us define $\bar{\theta}(m) = \bar{\theta}(\bar{X}(p_1^A, p_1^B), m)$. The description of the resulting equilibrium path follows from Lemma 1. Finally, the fact that $p_1^A > p_1^B$ follows from Claim 22 in Appendix B. \qed

Note that Theorem 1 only ensures that an equilibrium exists when either $m < m_L$ or $m > m_H$. Although the constant $m_H$ is fully characterized in the proof of Claim 15 as the smallest value of $m$ such that $BR^B(0, m) = -1/2$, we have not yet provided any tool to compute the value of $m_L$. However, we do so in Section EC 2 of the Electronic Companion (see Proposition 8), where we characterize the set of values of $m$ for which $G(m)$ admits an interior PSNE. In particular, this characterization allows us to show that:

Remark 1. The constants $m_L$ and $m_H$ satisfy $m_L < 4 < m_H$. In addition, $G(4)$ admits no PSNE. Moreover, we can numerically approximate $m_L \approx 3.98$ and $m_H \approx 4.02$.

We refer the reader to Section EC 2 of the Electronic Companion for the proof of Remark 1.

To conclude this section, we establish that the restriction of the action spaces we have imposed (i.e., choosing $S^A = [0, 1]$ and $S^B = [-1/2, 1]$) leads to no loss of PSNE in undominated strategies. Formally, the following claim shows that the PSNE of $G(m)$ remains unchanged, if we define action spaces as $\bar{S}^A = \bar{S}^B = \mathbb{R}$ instead and consider only equilibria in which firm A plays no dominated strategies, i.e., such that firm A sets non-negative prices.\footnote{Setting a negative price need not be a dominated strategy for firm B, as it has an incentive to learn the type of the consumer by setting low prices for product 1.}

Claim 19. Let $\bar{G}(m)$ be the two-player normal-form game with action spaces $\bar{S}^A = \bar{S}^B = \mathbb{R}$, and profit functions $\pi^A$ and $\pi^B$ as defined in (A.5) and (A.6). Then, for any $m > 0$, $\bar{G}(m)$ and $G(m)$ have the same pure-strategy Nash equilibria in undominated strategies.

B Proof of Theorem 2

In this appendix, we establish a series of claims that allow us to compare the equilibrium expected consumer surplus for our three settings. In what follows, we refer to the equilibrium expected consumer surplus on each setting simply as “consumer surplus” (CS). These comparisons result in the proof Theorem 2, that states that in the interior equilibrium regime ($m < m_L$), consumer surplus...
surplus is higher with data tracking in the economy, and that this holds both with myopic and with forward-looking consumers.

The results we prove are as follows. Claim 20 provides the expressions for consumer surplus in the myopic and restricted settings and establishes that consumer surplus is larger in the myopic setting when \(0 < m < m_L\). Then, Claim 21 provides the expression for consumer surplus in the forward-looking setting, and Claim 22 provides some auxiliary properties about the product 1 price dispersion in the forward-looking setting. Finally, Claim 23 shows that consumer surplus is higher in the forward-looking than in the myopic setting, for \(0 < m < m_L\). We conclude by proving Theorem 2, which follows directly from Claims 20 and 23.

Claim 20. For any \(m > 0\), the equilibrium consumer surplus in the restricted and myopic settings is given, respectively, by:

\[
\begin{align*}
\text{CS}_R(m) &= \bar{u} - \frac{5}{8} + \frac{m}{8}, \\
\text{CS}_M(m) &= \begin{cases} \\
\bar{u} - \frac{5}{8} + \frac{m}{144} (27 - m), & \text{if } m \leq 6, \\
\bar{u} + \frac{1}{4}, & \text{otherwise.}
\end{cases}
\end{align*}
\]

(B.1)

In particular, for \(m \in (0, m_L)\), consumer surplus is higher in the myopic than in the restricted setting.

Proof of Claim 20. First consider the restricted setting, and assume firms set prices \(p^A_1, p^B_1\) for product 1, and let \(\text{CS}_1(p^A_1, p^B_1)\) be the expected utility associated to product 1 that the consumer obtains if firms set such prices. It is easy to show that, in equilibrium, the consumer strictly prefer to buy product 1 from firm A if and only if her type \((s, \theta)\) satisfies \(s < \bar{X}(p^A_1, p^B_1)\). We then compute \(\text{CS}_1(p^A_1, p^B_1)\) as (abbreviating \(\bar{X}(p^A_1, p^B_1)\) as \(\bar{X}\)):

\[
\begin{align*}
\text{CS}_1(p^A_1, p^B_1) &= \mathbb{E}_{\mu_0} \left[ u_1 \left( 1_{(s > \bar{X})}(s, \theta); s, \theta, p^A_1, p^B_1 \right) | p^A_1, p^B_1 \right] \\
&= \bar{u} - (\bar{X} p^A_1 + (1 - \bar{X}) p^B_1) - \frac{1}{4} (\bar{X}^2 + (1 - \bar{X})^2) \\
&= \bar{u} - p^B_1 - \frac{1}{4} + \frac{1}{2} \bar{X}^2,
\end{align*}
\]

(B.2)

where, as before, \(\mu_0\) is the uniform distribution on the unit square.

By Proposition 1, both firms set a price of 1/2 for product 1 in equilibrium, so we have that the equilibrium consumer surplus associated to product 1 is \(\text{CS}_R(m) \equiv \text{CS}_1(1/2, 1/2) = \bar{u} - 5/8\).

Now consider product 2. When firm B sets a price of \(p^B_2\) for product 2 (regardless of the consumer’s type, since we consider the restricted setting), the consumer surplus associated to product 2 is

\[
\begin{align*}
\text{CS}_2(p^B_2) &= \mathbb{E}_{\mu_0} \left[ u_2 \left( 1_{(p^B_2 / m < \theta)}(\theta); s, \theta, p^B_2 \right) | p^B_2 \right] \\
&= \mathbb{E}_{\mu_0} \left[ (m \theta - p^B_2)^+ \right].
\end{align*}
\]

In equilibrium, firm B sets a price of \(p^B_{2,R} = m/2\) for product 2, so we have that, the consumer

\[\text{See Claim 31 in Section EC 3.}\]
surplus associated to product 2 is \(CS^R_2(m) \equiv CS_2(m/2) = m/8\). By adding the surplus across both products, we have that the total consumer surplus is \(CS^R(m) = \bar{u} - 5/8 + m/8\).

Now consider the myopic setting, and recall from Proposition 2 that in this case, the equilibrium prices for product 1 are

\[
p^1_{A,M}(m) = \max \{0, 1/2 - m/12\}, \quad p^1_{B,M}(m) = \max \{-1/2, 1/2 - m/6\}.
\]

The surplus derived from product 1 can be obtained by plugging these prices into equation (B.2) (and taking \(\bar{u}\)) is defined as in

\[
\begin{align*}
CS^M_1(m) &\equiv CS_1\left(p^1_{A,M}(m), p^1_{B,M}(m)\right) = \begin{cases} \bar{u} - \frac{5}{8} + \frac{m}{288}(m + 36), & \text{if } m \leq 6, \\ \bar{u} + \frac{1}{4}, & \text{if } m > 6. \end{cases}
\end{align*}
\]

Regarding product 2, by Proposition 2, the consumer receives zero surplus with probability \(1 - \bar{X}^M(m)\) (since, if \(s > \bar{X}^M(m)\), she buys product 1 from firm B and is subsequently offered a personalized price equal to her valuation for product 2). With the remaining probability, \(\bar{X}^M(m)\), the consumer is offered a price of \(m/2\), as in the restricted setting. It follows that the consumer surplus associated to product 2 in the myopic setting is (by independence of \(s\) and \(\theta\)):

\[
CS^M_2(m) = \bar{X}^M(m)CS^R_2(m) = \frac{m}{8}\max\left\{0, \frac{1}{2} - \frac{m}{12}\right\} = \max\left\{0, \frac{m(6 - m)}{96}\right\}.
\]

By computing \(CS^M(m) = CS^M_1(m) + CS^M_2(m)\), we get the expression given in (B.1).

Finally, comparing the expressions for \(CS^R(m)\) and \(CS^M(m)\) given in (B.1), we have that \(CS^R(m) < CS^M(m)\) if and only if \(0 < m < 7\). In particular, since \(m_L < 4\), this holds for all \(0 < m < m_L\).

We now consider the consumer surplus comparisons involving the forward-looking setting. The following claim provides an expression for consumer surplus in this setting.

**Claim 21.** For \(0 < m < m_L\), let \(p^1_{A,s}(m)\) and \(p^1_{B,s}(m)\) the unique product 1 equilibrium prices in the forward-looking setting. Let \(\bar{X}^s(m) = p^1_{B,s}(m) - p^1_{A,s}(m) + 1/2\), and \(\bar{\theta}^s(m) = \bar{\theta}(\bar{X}^s(m), m)\), where \(\bar{\theta}\) is defined as in (A.2). Then, consumer surplus in the forward-looking setting is given by

\[
CS^{FL}(m) = \bar{u} - \frac{1}{4} - p^1_{B,s}(m) + \frac{1}{2}(\bar{X}^s(m))^2 + \frac{1}{2}m\bar{X}^s(m)(1 - \bar{\theta}^s(m))^2 + \frac{1}{6}m^2(1 - \bar{\theta}^s(m))^3. \quad (B.3)
\]

**Proof of Claim 21.** Let \(0 < m < m_L\), so that by Theorem 1, there exists an equilibrium in the forward-looking setting, say \((\gamma^*, \sigma^A, \sigma^B, \mu^*)\). Without loss of generality, we take the equilibrium in which the consumer buys product 1 from firm B if indifferent.\(^{43}\) Let \(U^s(s, \theta)\) be the total utility

\(^{43}\)Recall that the equilibrium outcome is unique, except for the zero probability event in which the consumer is
if the consumer’s type is \((s, \theta)\) in that equilibrium, i.e.,

\[
U^*(s, \theta) = U_1 \left( \gamma^*, \sigma_{1}^A, \sigma_{1}^B, \mu^* \mid (s, \theta, p_1^A(m), p_1^B(m), m) \right),
\]

where \((p_1^A(m), p_1^B(m))\) are the unique product 1 equilibrium prices from Theorem 1. By Theorem 1, the consumer strictly prefers to buy product 1 from firm B if her type \((s, \theta)\) satisfies \(s > g^*(\theta)\), where \(g^*(\theta) = X^*(m) + m (\theta - \bar{\theta}^*(m))\). It follows then that the equilibrium utility for the consumer as a function of her type \((s, \theta)\) is

\[
U^*(s, \theta) = \begin{cases} 
\bar{u} - \frac{1}{2} s - p_1^A(m) + m (\theta - \bar{\theta}^*(m))^+ & \text{if } s < g^*(\theta), \\
\bar{u} - \frac{1}{2} (1 - s) - p_1^B(m) & \text{if } s \geq g^*(\theta).
\end{cases}
\]

We now compute \(CS^{FL}(m) = \mathbb{E}_{\mu_0}[U^*(s, \theta)]\), where \(\mu_0\) is the uniform distribution on the unit square, as before. For a fixed \(\theta\), let \(h(\theta) = \int_0^1 U^*(s, \theta) ds\), so that \(CS^{FL}(m) = \int_0^1 h(\theta) d\theta\). First consider \(\theta \leq \bar{\theta}^*(m)\) so that \(g^*(\theta) = X^*(m) \in (0, 1)\) in this case. Thus, we have (omitting the dependency on \(m\) in the notation):

\[
h(\theta) = \mathbb{E}[U^*(s, \theta); s < g^*(\theta)] + \mathbb{E}[U^*(s, \theta); s \geq g^*(\theta)]
\]

\[
= \bar{u} - \int_0^{X^*} \left( \frac{1}{2} s + p_1^A \right) ds - \int_{X^*}^1 \left( \frac{1}{2} (1 - s) + p_1^B \right) ds
\]

\[
= \bar{u} - \bar{X}^* p_1^A - (1 - \bar{X}^*) p_1^B - \frac{1}{4}(\bar{X}^*)^2 - \frac{1}{4}(1 - \bar{X}^*)^2
\]

\[
= \bar{u} - p_1^B + \bar{X}^* (p_1^B \bar{p}_1^A - \bar{p}_1^A) - \frac{1}{4} (2(\bar{X}^*)^2 - 2\bar{X}^* + 1)
\]

\[
= \bar{u} - p_1^B + \frac{1}{2}(\bar{X}^*)^2 - \frac{1}{4}
\]

Similarly, for \(\theta > \bar{\theta}^*(m)\) we have\(^{44}\)

\[
h(\theta) = \bar{u} - \int_0^{g^*(\theta)} \left( \frac{1}{2} s + p_1^A - m (\theta - \bar{\theta}^*) \right) ds - \int_{g^*(\theta)}^1 \left( \frac{1}{2} (1 - s) + p_1^B \right) ds
\]

\[
= \bar{u} + mg^*(\theta) (\theta - \bar{\theta}^*) - g^*(\theta) p_1^A - (1 - g^*(\theta)) p_1^B - \frac{1}{4} (g^*(\theta))^2 - \frac{1}{4}(1 - g^*(\theta))^2
\]

\[
= \bar{u} + mg^*(\theta) (\theta - \bar{\theta}^*) - p_1^B + g^*(\theta) (p_1^B - p_1^A) - \frac{1}{4} (2(g^*(\theta))^2 - 2g^*(\theta) + 1)
\]

\[
= \bar{u} + mg^*(\theta) (\theta - \bar{\theta}^*) - p_1^B + g^*(\theta) X^* - \frac{1}{2}(g^*(\theta))^2 - \frac{1}{4}
\]

\[
= \bar{u} - p_1^B + g^*(\theta) (X^* + m (\theta - \bar{\theta}^*)) - \frac{1}{2}(g^*(\theta))^2 - \frac{1}{4}
\]

\[
= \bar{u} - p_1^B + \frac{1}{2}(g^*(\theta))^2 - \frac{1}{4}
\]

indifferent between buying product 1 or not, which does not impact the expected utility computation once we take the expectation over consumer types.

\(^{44}\)Claim 27 establishes that \(X^*(m) < \bar{\theta}(m)\), which implies that \(g^*(1) < 1\), so integrating from 0 to \(\min\{1, g^*(\theta)\}\) gives the same result as integrating from 0 to \(g^*(\theta)\).
Finally, we compute consumer surplus by integrating $h(\theta)$:

$$CS^{FL}(m) = \int_0^1 h(\theta) d\theta$$

$$= \bar{u} - p_1^{B*} + \frac{1}{2}(\bar{X}^*)^2 - \frac{1}{4} + \frac{1}{2} \int_{\bar{\theta}^*}^1 \left(2m\bar{X}^*(\theta - \bar{\theta}^*) + m^2(\theta - \bar{\theta}^*)^2\right) d\theta$$

$$= \bar{u} - p_1^{B*} + \frac{1}{2}(\bar{X}^*)^2 - \frac{1}{4} + \frac{1}{2}m\bar{X}^*(1 - \bar{\theta}^*)^2 + \frac{1}{6}m^2(1 - \bar{\theta}^*)^3$$

\[\Box\]

Before comparing consumer surplus in the forward-looking and myopic settings, we establish the following technical conditions on the quantity $\bar{X}^*(m) = p_1^{B*}(m) - p_1^{A*}(m) + 1/2$.

**Claim 22.** For $0 < m < m_L$, $\bar{X}^*(m)$ is continuous in $m$ and $0 < \bar{X}^*(m) < 1/2$.

*Proof of Claim 22.* From Step 3 in the proof of Claim 27 (see Section EC 1.1), we have that $x = \bar{X}^*(m)$ solves $N(x, m) = 0$ given fixed $m$ for $x \in [0, 1]$, where

$$N(x, m) = m^2 - 4m + 20x^2 + 12mx - 10x - \sqrt{2x(2x + m)}(1 - 2x). \quad (B.4)$$

Note that $N(0, m) = m(m - 4) < 0$ for $m \in (0, 4)$. In particular, this holds for $m \in (0, m_L)$ since $m_L < 4$ by Remark 1. In addition, $N(1/2, m) = m^2 + 2m > 0$.

Moreover, since $N(x, m)$ is strictly convex in $x \in [0, 1]$ for fixed $m > 0$, and since $N(0, m) < 0 < N(1/2, m)$, $x = \bar{X}^*(m)$ is the unique solution to $N(x, m) = 0$ with $x \in [0, 1]$, and in particular, we have that $0 < \bar{X}^*(m) < 1/2$ for all $m \in (0, m_L)$.

Finally, we have that $N_x(\bar{X}^*(m), m) > 0$ by strict convexity of $N(x, m)$, and therefore $\bar{X}^*(m)$ is a continuous function of $m \in (0, m_L)$ by the implicit function theorem. \[\Box\]

Now, we show that consumer surplus is higher in the forward-looking than in the myopic setting when $0 < m < m_L$.

**Claim 23.** For $0 < m < m_L$, we have that $CS^{FL}(m) > CS^M(m)$.

Let us first describe the approach we follow to prove the result, which consists of the following steps.

1. Find a function $F : (0, m_L) \rightarrow \mathbb{R}$ such that $CS^{FL}(m) - CS^M(m) \geq F(m)$ for $0 < m < m_L$.

2. Show that: (i) $F(m)$ is continuous in $m$, (ii) that it has no roots in $(0, m_L)$, and that (iii) $F(m) > 0$ for some $m \in (0, m_L)$. We do this as follows:

   (i) Show that we can write $F(m) = \Gamma(\bar{X}^*(m), m)$, for $m \in (0, m_L)$, where $\Gamma : [0, 1] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function. Continuity of $F$ will follow from the fact that $\bar{X}^*(m)$ is continuous in $m$. 

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(ii) To show that \( F(m) = \Gamma(\bar{X}^*(m), m) \) has no roots in \((0, m_L)\), we first note that given \( 0 < m < m_L \), \( x = \bar{X}^*(m) \) is the unique solution to \( N(x, m) = 0 \) with \( x \in (0, 1/2) \), where \( N(x, m) \) is defined in (B.4). Therefore, showing that \( F \) has no roots in \((0, m_L)\) is equivalent to showing that the system of equations given by \( \{ \Gamma(x, m) = 0, N(x, m) = 0 \} \) has no solutions with \( x \in (0, 1/2) \) and \( m > 0 \).

While obtaining the solutions of this system of equations is complicated, we achieve this by finding a change of variables that allow us to transform the system above to one with two polynomial equations with integer coefficients in two unknowns. We then use a well-established method to obtain the solutions of the transformed system, and find that none of them are feasible in the original system. Formally, the method relies on computing the Gröbner basis of the polynomial system, and obtaining an equivalent system in triangular form, for which the solutions can be approximated with high precision. We refer the reader to Cox, Little, and O’Shea (2015) and Sturmfels (2002) for an introduction to these tools.\(^{45}\)

(iii) Directly show that \( F(m_0) > 0 \) for some \( m_0 \in (0, m_L) \).

We now prove Claim 23 by following these steps.

**Proof of Claim 23.** Let \( m \in (0, m_L) \). We proceed according to the steps described above.

**Step 1.** From equations (B.1) and (B.3) we have that (omitting the dependence of \( \bar{X}^*(m) \), \( \bar{\theta}^*(m) \), \( p_1^{A*}(m) \) and \( p_1^{B*}(m) \) on \( m \)):

\[
CS^{FL}(m) - CS^{M}(m) = \frac{3}{8} - \frac{m}{144} (27 - m) - \frac{1}{2} (\bar{X}^*)^2 + \frac{m \bar{X}^* (1 - \bar{\theta}^*)^2}{2} + \frac{1}{6} m^2 (1 - \bar{\theta}^*)^3
\]

\[
> \frac{3}{8} - \frac{m}{144} (27 - m) - \frac{1}{2} (\bar{X}^*)^2 + \frac{m \bar{X}^* (1 - \bar{\theta}^*)^2}{2}
\]

\[
= \frac{7}{8} - \frac{m}{144} (27 - m) - \frac{1}{2} (\bar{X}^*)^2 + \frac{m \bar{X}^* (1 - \bar{\theta}^*)^2}{2},
\]

where the first inequality follows since \( \frac{1}{6} m^2 (1 - \bar{\theta}^*)^3 > 0 \) and the second step follows by plugging in \( \bar{X}^*(m) = p_1^{B*}(m) - p_1^{A*}(m) + 1/2 \).

**Step 2.** Recall that by Claim 12, \( p_1^{A*}(m) = Z(\bar{X}^*(m), m) \), where \( Z \) is defined as in (A.29). Therefore, we can write \( F(m) = \Gamma(\bar{X}^*(m), m) \) where

\[
\Gamma(x, m) = \frac{7}{8} - \frac{m}{144} (27 - m) - Z(x, m) - x + \frac{1}{2} x^2 + \frac{1}{2} m x (1 - \bar{\theta}(x, m))^2.
\]

(B.5)

Since both \( Z(x, m) \) and \( \bar{\theta}(x, m) \) are continuous functions of \( (x, m) \in [0, 1] \times \mathbb{R}^+ \), \( \Gamma(x, m) \) is also continuous.\(^{46}\) By Claim 22, we know that \( \bar{X}^*(m) \) is continuous for \( 0 < m < m_L \). Thus

\(^{45}\)Intuitively, this method performs a similar procedure to Gaussian elimination but with a system of polynomial rather than linear equations. This method is intractable by hand, but it is implemented in various software packages, such as Mathematica and Sage.

\(^{46}\)It is easy to establish continuity of \( Z \) and \( \bar{\theta} \) following a similar argument as in the proof of Claim 10.
\( F(m) = \Gamma(\bar{X}(m), m) \) is continuous for \( 0 < m < m_L \).

Moreover, by the proof of Claim 22, we know that \( \bar{X}(m) \) is the unique solution of \( N(x, m) = 0 \) with \( 0 < x < 1 \), where \( N(x, m) \) is defined in (B.4), and, by Claim 22, we know that \( 0 < \bar{X}(m) < 1/2 \) for all \( 0 < m < m_L \).

Then, by the preceding argument, showing that \( F(m) = \Gamma(\bar{X}(m), m) \) has no roots in \( (0, m_L) \) is equivalent to showing that the following system of equations admits no solution with \( 0 < x < 1/2 \) and \( 0 < m < m_L \).

\[
\begin{align*}
\Gamma(x, m) &= 0, \\ N(x, m) &= 0. 
\end{align*}
\tag{B.6}
\]

However, we know that \( \bar{X}(m) < \tilde{x}(m) \) (by Claim 27 in Section EC 1), so it suffices to show that this system of equations has no solution with \( 0 < x < \min\{1/2, \tilde{x}(m)\} \) and \( 0 < m < m_L \).

By plugging in the expressions for \( Z(x, m) = \psi(x, m)/\psi_2(x, m) \) and \( \tilde{\theta}(x, m) \) for \( x < \tilde{x}(m) \) into (B.5) (from equations (A.2), (A.7) and (EC 1.4)) and simplifying the resulting expression, we can write \( \Gamma(x, m) = J_1(x, m)/J_2(x, m) \) for \( 0 < x < \tilde{x}(m) \), where

\[
\begin{align*}
J_1(x, m) &= 2J_3(x, m) + \sqrt{2x(2x + m)}J_4(x, m), \\
J_2(x, m) &= 144m \left( 2m + 6x - \sqrt{2x(2x + m)} \right), \\
J_3(x, m) &= m^4 + 3m^3(x - 9) + 2304x^4 + 9m^2(24x^2 - 41x + 14) + 18mx(84x^2 - 40x + 21), \\
J_4(x, m) &= 27m^2 - m^3 - 2304x^3 - 18m(44x^2 - 8x + 7).
\end{align*}
\]

Notice that \( J_2(x, m) > 0 \) for all \( x, m > 0 \). Therefore, for \( 0 < x < \tilde{x}(m) \) and \( m > 0 \), system (B.6) can be written equivalently as \( \{J_1(x, m) = 0, N(x, m) = 0\} \). Thus, is suffices to show that this system has no solutions with \( 0 < x < 1/2 \) and \( 0 < m < m_L \).

To do so, we now change variables to \( w = \sqrt{2x}, z = \sqrt{2x + m} \), so that \( \sqrt{2x(2x + m)} = wz \). By plugging the change of variables to \( J_1 \) and \( N \) we can write

\[
\begin{align*}
J_1(w^2/2, z^2 - w^2) &= (w - z)Q_1(w, z), \\
N(w^2/2, z^2 - w^2) &= Q_2(w, z), 
\end{align*}
\tag{B.7}
\]

where,

\[
\begin{align*}
Q_1(w, z) &= 17w^7 - 72w^6z - 110w^4z^3 - z^2(z^2 - 21)(z^2 - 6)(w + 2z) + w^5(91z^2 + 45) \\
&\quad + w^3(z^4 + 216z^2 - 126) + w^2z^3(4z^2 + 234), \\
Q_2(w, z) &= -w^2 - wz + w^3z - 4z^2 + 4w^2z^2 + z^4.
\end{align*}
\]

Showing that the system \( \{J_1(x, m) = 0, N(x, m) = 0\} \) has no solutions with \( 0 < x < 1/2 \) and \( 0 < m < m_L \) is then equivalent to showing that the transformed system given by \( \{J_1(w^2/2, z^2 - w^2) = 0, N(w^2/2, z^2 - w^2) = 0\} \) has no solutions with \( 0 < w < 1 \) and \( 0 < z^2 - w^2 < m_L \). In
particular, it suffices to show that the system admits no solutions with $w > 0$ and $w > z$.

It follows from (B.7), that it suffices to analyze the solutions of the following system

$$
Q_1(w, z) = 0 \\
Q_2(w, z) = 0
$$

(B.8)

Since $Q_1$ and $Q_2$ are polynomials with integer coefficients, this system can be solved by computing the Gröbner basis of $Q_1, Q_2$, and solving the resulting triangular system (see Cox et al. (2015) and Sturmfels (2002) for a detailed introduction to these tools, or Sturmfels (2005) for a short overview). We find that system (B.8) has five real solutions,\footnote{We obtain the solutions by using the Solve routine in Mathematica. The real solutions to system (B.8) are $(w, z) = (0, 0), \pm(1, 1), \pm(1.32385, 0.589199)$.} but none of them satisfy that $0 < w < 1$ and $z > 0$. It follows that system (B.6) has no solution with $0 < x < 1/2$ and $0 < m < m_L$, and therefore, $F(m)$ has no roots in $(0, m_L)$.

To conclude, consider $m = 2$. By Remark 1, we have that $m_L > 2$. Direct computation shows that $\bar{x}^*(2) \approx 0.2423, \bar{\theta}^*(2) \approx 0.6937, p_A^A(2) \approx 0.2763$, and thus $F(2) \approx 0.06126 > 0$. \hfill \qed

Based on the previous results, we can now prove Theorem 2.

**Proof of Theorem 2.** Let $m \in (0, m_L)$. By Claims 20 and 23, we have that $CS^{FL}(m) > CS^M(m) > CS^R(m)$, as desired. \hfill \qed

## C Implications of Data Tracking in the Corner Equilibrium Regime

In this appendix, we consider the regime where $m > m_H$. In this regime, the value of data is high enough so that the forward-looking setting admits a corner equilibrium, i.e., firm B captures the entire market of product 1 transactions in equilibrium (see Theorem 1). Following a similar structure to that of the comparisons presented in Section 4, Proposition 7 compares the equilibrium aggregate consumer surplus, firms’ profits, and dispersion in product 1 prices when $m > m_H$.

It is important to remark that in contrast with the comparisons established in Theorem 2, Proposition 7 finds that consumers may be worse off when firm B may use data tracking if the value of their data (i.e, $m$) is high enough. However, this only occurs in the corner equilibrium regime, in which firm B has a strong enough incentive to set a low enough price for product 1 to induce the consumer to buy product 1 from it, regardless of her type. As we have argued, we find this to be the less interesting and plausible case of our model, as it implies that firm A receives no purchases.

**Proposition 7.** Let $m_H$ be as defined in Theorem 1. Then, for $m \in (m_H, \infty)$ we have that:

(i) The comparisons regarding consumer surplus depend on the value of $m$ as follows:
(a) If $m_H < m < 7$, the aggregate consumer surplus is highest in the forward-looking, and lowest in the restricted setting. That is, the comparisons established in Theorem 2 hold in this region.\footnote{However, for $m \geq 6$, both the forward-looking and the myopic setting result in the same corner equilibrium, which results in the same level of consumer surplus.}

(b) If $m > 7$, the aggregate consumer surplus is highest in the restricted setting, while the forward-looking and myopic settings result in the same consumer surplus. Thus, in contrast to Theorem 2, aggregate consumer surplus is lower in the presence of data tracking if the value of data is high enough.

(ii) As in Proposition 4, the dispersion in product 1 equilibrium prices is higher with forward-looking than with myopic consumers if $m < 6$. For $m \geq 6$, both the forward-looking and the myopic setting result in the same equilibrium prices.

(iii) As in Proposition 5, firm B benefits from using data tracking technologies, even if consumers are forward-looking.

(iv) As in Proposition 6, firm A’s expected profits are highest in the restricted setting, followed by the myopic and forward-looking settings. In particular, firm A’s expected profit is zero in the forward-looking setting for all $m > m_H$, while this occurs in the myopic setting when $m \geq 6$.

Proof of Proposition 7. Let $m > m_H$, and recall from Theorem 1 that in equilibrium, firms’ prices for product 1 are $p_A^*(m) = 0$, $p_B^*(m) = -1/2$, with the consumer buying product 1 from firm B, and firm B observing her type with probability 1.

Part (i). For $m \geq 6$, the myopic and forward-looking settings result in the same equilibrium outcomes (by Theorem 1 and Proposition 2), and therefore the same level of aggregate consumer surplus, which is $\bar{u} + 1/4$ (from Equation (B.1)). It follows from Equation (B.1) that consumer surplus is larger in the restricted than in these two settings if and only if

$$\bar{u} + \frac{1}{4} < \bar{u} - \frac{5}{8} + \frac{m}{8}.$$ 

This inequality reduces to $m > 7$, which proves point (b), and point (a) for $6 \leq m \leq 7$. Now consider the case with $m < 6$, for which $CS^M(m) = \bar{u} - 5/8 + m(27 - m)/144$. Straightforward calculus shows that this expression is increasing for $m \in [0, 27/2]$, so in particular $CS^M(m) < CS^M(6)$ for $m \in (m_H, 6)$. But since we have a corner equilibrium in the forward-looking setting for such values of $m$, it follows that $CS^M(6) = \bar{u} + 1/4 = CS^{FL}(m)$. Thus, $CS^{FL}(m) > CS^M(m)$. It remains to show that $CS^M(m) > CS^R(m)$ for $m_H < m < 6$, which is equivalent to

$$\bar{u} - \frac{5}{8} + \frac{m}{144} (27 - m) > \bar{u} - \frac{5}{8} + \frac{m}{8}.$$ 

This inequality holds for $m < 9$, and in particular for $m \in (m_H, 6)$, which concludes the proof of (i).
**Part (ii).** For \( m > m_H \), we have that

\[
p_1^{A*}(m) - p_1^{B*}(m) = 1/2 \geq \min \left\{ \frac{1}{2}, \frac{m}{12} \right\} = p_1^{A,M}(m) - p_1^{B,M}(m).
\]

**Part (iii).** For \( m > m_H \), it is easy to compute firm B’s equilibrium profit in the restricted and forward-looking settings which are \( \pi_B^R(m) = (m + 1)/4 \) and \( \pi_B^F(m) = (m - 1)/2 \), respectively. Since \( m > m_H > 4 \) (by Remark 1), we have that \( (m - 1)/2 > (m + 1)/4 \), so the profit is higher in the forward-looking than in the restricted setting. For \( m \geq 6 \), the myopic and the forward-looking settings result in the same outcome. For \( m < 6 \), firm B’s equilibrium profit in the myopic setting is \( \pi_B^M(m) = m/4 + (1/2 + m/12)^2 \) (see equation (EC 6.3) in Section EC 6). Then, for \( m_H < m < 6 \), profits are highest in the myopic setting since the following inequality holds for \( m_H < m < 6 \):

\[
\pi_B^M(m) = \frac{m}{4} + \left( \frac{1}{2} + \frac{m}{12} \right)^2 > \frac{m - 1}{2} = \pi_B^F(m).
\]

**Part (iv).** In the forward-looking setting, firm A’s profit is zero while this is also the case in the myopic setting when \( m \geq 6 \). In contrast, firm A’s profit in the restricted setting is \( 1/4 \) regardless of the value of \( m \). It remains to see that firm A’s profit in the restricted setting is larger than in the myopic one for \( m_H < m < 6 \), which indeed holds since \( \pi_A^R(m) = \frac{1}{4} > \max \left\{ \frac{1}{2} - \frac{m}{12}, 0 \right\}^2 = \pi_A^M(m) \). 

\( \square \)
Electronic Companion

This companion is organized as follows:

Section EC 1 provides the proofs of Claims 10–19, which are intermediate steps in the proof of Lemma 3.

Section EC 2 provides an additional result that characterize the set of values of $m$ for which the game $G(m)$ admits an interior pure-strategy Nash equilibrium (Proposition 8). This result allows us to provide numerical approximations for the constants $m_L$ and $m_H$.

Sections EC 3 and EC 4 prove Propositions 1 and 2, respectively, which characterize the equilibrium in the restricted and myopic settings. Then, Sections EC 5, EC 6, EC 7 provide the proofs of Propositions 4, 5, and 6, respectively, which compare equilibrium outcomes across the three settings we consider.

Finally, in Section EC 8, we extend our analysis to consider the case where firm B is a monopolist in both product markets. In particular, we provide the proof of Proposition 3.

EC 1  Additional proofs for Section A.3 (proofs of Claims 10–19)

In this section we prove Claims 10–19, which complete the proof of Lemma 3 as described in Section A.3. When proving these claims, we state some additional results whose proofs we defer to the end of this section (see Subsection EC 1.1).

Proof of Claim 10. Fix $m > 0$. The proof is organized as follows: first, we show all the stated properties for $\bar{\theta}$. Then, based on these results we prove the corresponding properties for $\psi$, $\xi$ and $\phi$ respectively.

Properties of $\bar{\theta}$. We first show that $\bar{\theta}(x, m)$ is continuous in $x$, at any $x \in [0, 1]$. Recall the definition of $\bar{\theta}$ from (A.2):

$$
\bar{\theta}(x, m) = \begin{cases} 
\frac{1}{m} \left(2x + m - \sqrt{2x(2x + m)}\right), & \text{if } x \leq \tilde{x}(m) \\
\frac{1}{1 + x} \left[1 - \frac{1}{2m}(1 - x)^2\right], & \text{if } x > \tilde{x}(m),
\end{cases}
$$

where $\tilde{x}(m) = \frac{1}{3} \left(\sqrt{(m-1)^2 + 3} - (m-1)\right)$. It is easy to verify continuity by inspection when $x \neq \tilde{x}(m)$, so we only need to prove continuity at $x = \tilde{x}(m)$. That is, we want to verify that the expression

$$
\frac{1}{m} \left(2x + m - \sqrt{2x(2x + m)}\right) = \frac{1}{1 + x} \left[1 - \frac{1}{2m}(1 - x)^2\right]
$$

is satisfied for $x = \tilde{x}(m)$. A few algebraic steps are necessary to verify this. First, one can verify that $x = \tilde{x}(m)$ is a solution for the quadratic equation given by $(1 + x)^2 = 2x(2x + m)$, which can also be written as $(1 - x)^2 = 4x^2 + 2(m-2)x$. Then, by plugging these two conditions above and
simplifying the resulting expression, we have that continuity will hold if and only if

\[ 3\tilde{x}(m)^2 + 2(m - 1)\tilde{x}(m) - 1 = 0. \]

This expression holds, since it can be rewritten as \((1 + \tilde{x}(m))^2 = 2\tilde{x}(m)(2\tilde{x}(m) + m)\). Thus, \(\bar{\theta}(x, m)\)
is continuous in \(x\) for fixed \(m > 0\).

To show that \(\bar{\theta}(x, m)\) is differentiable in \(x\), notice that this clearly holds by inspection at any \(x \neq \tilde{x}(m)\), and the partial derivative of \(\bar{\theta}\) w.r.t. \(x\) is\(^{49}\)

\[
\bar{\theta}_x(x, m) = \begin{cases} \frac{1}{m} \left[ 2 - \frac{4x + m}{\sqrt{4x^2 + 2mx}} \right], & \text{if } x \leq \tilde{x}(m). \\ \left[ \frac{2}{m} - 1 \right] \left[ \frac{1}{(1+x)^2} \right] - \frac{1}{2m}, & \text{if } x > \tilde{x}(m). \end{cases}
\]

(EC 1.1)

As before, we want to show that the expression for the two cases coincide when \(x = \tilde{x}(m)\). Simple algebra shows that equating these two expressions of reduces to

\[(1 + x)^2 = 2mx + 4x^2, \quad \text{(EC 1.2)}\]

which holds for \(x = \tilde{x}(m)\). Thus, \(\bar{\theta}_x(x, m)\) is a continuous function of \(x \in (0, 1]\), i.e, \(\bar{\theta}(x, m)\) is continuously differentiable in this region.\(^{50}\) Moreover, simple algebra shows that \(\bar{\theta}_x(x, m) < 0\) for all \(x \in (0, 1]\), which implies that \(\bar{\theta}(x, m)\) is decreasing in \(x\). Finally, by evaluating \(\bar{\theta}(1, m) = 1/2\) and \(\bar{\theta}(0, m) = 1\), we conclude that \(\bar{\theta}(x, m) \in [1/2, 1]\) for all \(x \in [0, 1]\).

**Properties of \(\psi\).** Recall from (A.7) that \(\psi\) is defined by \(\psi(x, m) = 2x\bar{\theta}(x, m)\), for \(x \in [0, 1]\). Since \(\bar{\theta}(x, m)\) is continuously differentiable in \(x\) and bounded, so is \(\psi(x, m)\). Thus, we only need to show that \(\psi\) is strictly concave and strictly increasing in \(x\). We will first prove strict concavity. Notice from (EC 1.1) that \(\bar{\theta}(x, m)\) is twice differentiable in \(x\) for all \(x \in (0, 1)\) with \(x \neq \tilde{x}(m)\), and its second partial derivative w.r.t. \(x\) is

\[
\bar{\theta}_{xx}(x, m) = \begin{cases} \frac{m}{(4x^2 + 2mx)^{3/2}}, & \text{if } x < \tilde{x}(m) \\ \frac{2(m-2)}{m(1+x)^4}, & \text{if } x > \tilde{x}(m). \end{cases}
\]

(EC 1.3)

It follows that \(\psi(x, m)\) is also twice differentiable in \(x\) for all \(x \in (0, 1), x \neq \tilde{x}(m)\). Since \(\psi_x(x, m)\) is continuous in \(x\), to show strict concavity of \(\psi\) in \(x\), it suffices to show that \(\psi_{xx}(x, m) < 0\) for all \(x \neq \tilde{x}(m)\).

To show this, we first obtain the expression for this second derivative. To derive a convenient expression, first note that by Claim 7 and equation (A.19) we have that

\[
\psi(x, m) = \begin{cases} x + \frac{1}{2}m \left( 1 - \bar{\theta}(x, m) \right)^2, & \text{if } x \leq \tilde{x}(m). \\ x + (1-x)(1-\bar{\theta}(x, m)) - \frac{1}{2m} (1-x)^2, & \text{if } x > \tilde{x}(m). \end{cases}
\]

\(^{49}\)Once we show \(\bar{\theta}_x(x, m)\) is continuous at \(x = \tilde{x}(m)\), we can define \(\bar{\theta}_x(\tilde{x}(m), m)\) as any of the two cases by continuity.

\(^{50}\)Note that \(\lim_{t \to 0^+} \bar{\theta}_x(t, m) = -\infty\), but \(\bar{\theta}_x(x, m)\) is finite for \(x \in (0, 1]\).
Thus, we can write the partial derivative of $\psi$ w.r.t. $x$ as

$$
\psi(x, m) = \begin{cases}
1 - m (1 - \bar{\theta}(x, m)) \bar{\theta}(x, m), & \text{if } x \leq \bar{x}(m), \\
\bar{\theta}(x, m) + \frac{1}{m}(1 - x) - (1 - x)\bar{\theta}(x, m), & \text{if } x > \bar{x}(m).
\end{cases}
$$

(EC 1.4)

And therefore,

$$
\psi_{xx}(x, m) = \begin{cases}
m \left[ \bar{\theta}(x, m) \right] ^2 - (1 - \bar{\theta}(x, m))\bar{\theta}(x, m), & \text{if } x < \bar{x}(m), \\
2\bar{\theta}(x, m) - \frac{1}{m} - (1 - x)\bar{\theta}(x, m), & \text{if } x > \bar{x}(m).
\end{cases}
$$

(EC 1.5)

We now show that this expression is negative for the two cases we have. First consider $0 < x < \bar{x}(m)$. After plugging in (EC 1.3) to (EC 1.5) and algebraic manipulation we have

$$
\psi_{xx}(x, m) = \frac{m}{4x^2 + 2mx} \left[ \frac{16x(2x + m) - 4(4x + m)\sqrt{4x^2 + 2mx}}{m^2} + \sqrt{\frac{2x}{2x + m}} \right].
$$

Then, we want to show that for all $x \in (0, \bar{x}(m))$,

$$
\frac{16x(2x + m) - 4(4x + m)\sqrt{4x^2 + 2mx}}{m^2} + \sqrt{\frac{2x}{2x + m}} < 0.
$$

By multiplying both sides by $m^2\sqrt{\frac{2x + m}{2x}}$, the inequality reduces to

$$
(2x + m) \left[ 8\sqrt{2x(2x + m)} - 4(4x + m) \right] < -m^2.
$$

(EC 1.6)

Notice that the LHS of this inequality is strictly increasing in $x$. Therefore, it suffices to show that (EC 1.6) holds, even if weakly for $x = \bar{x}(m)$. Plugging in equation (EC 1.2) to the LHS above results in (abbreviating $\bar{x} = \bar{x}(m)$):

$$
(2\bar{x} + m) \left[ 8\sqrt{2\bar{x}(2\bar{x} + m)} - 4(4\bar{x} + m) \right] = (2\bar{x} + m) \left[ 8(1 + \bar{x}) - 4(4\bar{x} + m) \right]
$$

$$
= 4(2\bar{x} + m) \left[ 2 - m - 2\bar{x} \right]
$$

$$
= 8\bar{x}^2 - 16(m - 1)\bar{x} - 4m(m - 2)
$$

where the last step follows since, by equation (EC 1.2), $2(m - 1)\bar{x} = 1 - 3\bar{x}^2$. Plugging back into inequality (EC 1.6), we want to show that $8(1 - \bar{x}^2) + 4m(m - 2) \geq m^2$, which is equivalent to

$$
\bar{x}(m)^2 \leq 1 - m + \frac{3}{8}m^2.
$$

(EC 1.7)

This inequality holds, in fact strictly,\(^{51}\) for all $m > 0$. Thus, $\psi_{xx}(x, m) < 0$ for $0 < x < \bar{x}(m)$.

\(^{51}\)To see this, let $f(m) = 1 - m + \frac{3}{8}m^2 - \bar{x}(m)^2$, and notice that $f(0) = 0$ and that $f(m)$ is increasing in $m$. 

EC – 3
Now consider $\tilde{x}(m) < x < 1$. Plugging in (EC 1.3) to (EC 1.5) results in

\[
\psi_{xx}(x, m) = 2 \left[ \frac{2 - m}{m(1 + x)^2} - \frac{1}{2m} \right] - \frac{1}{m} + 2(1 - x) \left( \frac{2 - m}{m(1 + x)^3} \right)
\]

\[
= 2 \left( \frac{2 - m}{m(1 + x)^2} \right) \left( \frac{2}{1 + x} \right) - \frac{2}{m}
\]

\[
= \frac{2}{m} \left( \frac{4 - 2m}{(1 + x)^3} - 1 \right)
\]

So it suffices to show that $4 - 2m < (1 + x)^3$ for $x > \tilde{x}(m)$. In particular, notice that it suffices to show that (abbreviating $\tilde{x} = \tilde{x}(m)$):

\[
4 - 2m \leq (1 + \tilde{x})^2 = 2m\tilde{x} + 4\tilde{x}^2.
\]  

(EC 1.8)

Since $\tilde{x} \in [0, 1]$, this inequality is equivalent to

\[
\tilde{x} = \frac{1}{3} \left( \sqrt{(m - 1)^2 + 3} - (m - 1) \right) \geq 1 - \frac{m}{2},
\]

which can be easily verified. Therefore, $\psi_{xx}(x, m) < 0$ for $\tilde{x}(m) < x < 1$.

We have shown that $\psi_{xx}(x, m) < 0$ for all $x \in (0, 1)$ with $x \neq \tilde{x}(m)$. Since $\psi_x(x, m)$ is continuous in $x$, it follows that $\psi_x(x, m)$ is strictly decreasing in $x$, i.e., that $\psi(x, m)$ is strictly concave in $x$.

Finally, by strict concavity we have that $\psi_x(x, m) > \psi_x(1, m) = 1/2$ for all $x \in [0, 1)$, which implies that $\psi(x, m)$ is increasing in $x$.

**Properties of $\xi$.** Recall from (A.8) that $\xi(x, m)$ is defined as

\[
\xi(x, m) = (1 - \psi(x, m)) \mathbb{E}_{\mu_0} \left[ \theta \mid s \geq x + m(\theta - \bar{\theta}(x, m))^+ \right].
\]

By conditioning the expectation of $\theta$ on whether the consumer buys from firm A or B in the first period, and by Claims 7, we have that

\[
\frac{1}{2} = \mathbb{E}_{\mu_0} [\theta] = \psi(x, m) \mathbb{E}_{\mu_0} \left[ \theta \mid s < x + m(\theta - \bar{\theta}(x, m))^+ \right] + \xi(x, m).
\]

Observe that $\xi(0, m) = 1/2$. Moreover, for $x > 0$, by Claim 5, we know that the conditional distribution of $\theta$ on $\{s < x + m(\theta - \bar{\theta}(x, m))^+\}$ admits a density function given by $\hat{g}(t \mid x, m)/\psi(x, m)$, where, for $t \in [0, 1]$ we have

\[
\hat{g}(t \mid x, m) = \min \left\{ x + m\left( t - \bar{\theta}(x, m) \right)^+, 1 \right\}.
\]

Then, for $x > 0$ we have that

\[
\xi(x, m) = \frac{1}{2} - \int_0^1 \hat{g}(t \mid x, m) dt = \int_0^1 t \left( 1 - \min \left\{ x + m\left( t - \bar{\theta}(x, m) \right)^+, 1 \right\} \right) dt.
\]  

(EC 1.9)
Since $\tilde{\theta}(x, m)$ is continuous in $x$, so is the above expression. Thus $\bar{\xi}(x, m)$ is continuous in $x$ for all $x > 0$. Moreover, to verify that $\xi(x, m)$ is also continuous at $x = 0$, notice by plugging into the previous expression that $\lim_{x \to 0^+} \xi(x, m) = 1/2 = \xi(0, m)$.

In addition, note that $\xi(1, m) = 0$, so by continuity, $\xi(x, m)$ is a bounded function of $x \in [0, 1]$. Finally, to verify that $\xi(x, m)$ is continuously differentiable in $x$, for $x \in (0, 1]$, we compute the integral in \eqref{EC:1.11} and obtain:

$$
\bar{\xi}(x, m) = \begin{cases} 
\frac{1-x}{2} - \frac{m}{6} (1 - \tilde{\theta}(x, m))^2(\tilde{\theta}(x, m) + 2), & \text{if } x \leq \tilde{x}(m) \\
\frac{1-x}{2} \tilde{\theta}(x, m)^2 + \frac{(1-x)^2}{2m} \tilde{\theta}(x, m) + \frac{(1-x)^3}{6m^2}, & \text{if } x > \tilde{x}(m).
\end{cases}
$$

\text{(EC 1.10)}

Since $\bar{\theta}(x, m)$ is differentiable in $x$ (for $x \in (0, 1]$), it can be seen by inspection that so is $\bar{\theta}(x, m)$ when $x \neq \tilde{x}(m)$. To verify that $\xi$ is also differentiable at $x = \tilde{x}(m)$, we compute the derivative of $\xi(x, m)$ w.r.t. $x$ from \eqref{EC:1.10} to obtain:

$$
\xi_x(x, m) = \begin{cases} 
-\frac{1}{2} + \frac{m}{2} \tilde{\theta}_x(x, m) (1 - \tilde{\theta}(x, m)^2) & \text{if } x \leq \tilde{x}(m) \\
-\frac{1}{2} \tilde{\theta}(x, m)^2 + (1 - x) \left( \tilde{\theta}_x(x, m) - \frac{1}{m} \right) \left( \tilde{\theta}(x, m) + \frac{1-x}{2m} \right) & \text{if } x > \tilde{x}(m).
\end{cases}
$$

\text{(EC 1.11)}

By continuity of $\tilde{\theta}(x, m)$ and $\tilde{\theta}_x(x, m)$, we have that $\xi_x(x, m)$ is continuous when $x \neq \tilde{x}(m)$. Performing an similar algebraic argument as in the proof for $\theta_x$ show that this is also the case when $x = \tilde{x}(m)$, therefore $\xi_x(x, m)$ is continuous in $x$. In addition, since $\tilde{\theta}_x(x, m)$ is finite for $x \in (0, 1]$, so is $\xi_x(x, m)$. Finally, notice that since $\tilde{\theta}_x(x, m) < 0$, it follows from \eqref{EC:1.11} that $\xi_x(x, m) < 0$ and therefore $\xi(x, m)$ is strictly decreasing in $x$.

**Properties of $\phi$.** Recall from \eqref{A.5} that $\phi$ is defined as $\phi(x, m) = \xi(x, m) + \frac{1}{2} \psi(x, m) \tilde{\theta}(x, m)$. By the previous arguments, $\xi(x, m)$, $\psi(x, m)$ and $\tilde{\theta}(x, m)$ are continuously differentiable functions of $x$. Therefore, so is $\phi(x, m)$.

We now show that $\phi(x, m)$ is strictly convex in $x$. To do so, by plugging in the closed-form expressions for $\tilde{\theta}(x, m)$ and $\xi(x, m)$ given in \eqref{A.2} and \eqref{EC:1.10} into $\phi(x, m) = \xi(x, m) + \frac{1}{2} \psi(x, m) \tilde{\theta}(x, m)$, and simplifying the resulting expression, we have that

$$
\phi(x, m) = \begin{cases} 
\frac{3m^2(1-x) + 16x^3 - 2x \sqrt{2x(2x+m)(4x-m)}}{6m^2}, & \text{if } x \leq \tilde{x}(m) \\
\frac{12m^2 + (1-x)^3(1+7x)}{24m^2(1+x)}, & \text{if } x > \tilde{x}(m).
\end{cases}
$$

\text{(EC 1.12)}

By taking the derivative w.r.t $x$, we have that

$$
\phi_x(x, m) = \begin{cases} 
\frac{2x(m^2 - 4mx - 16x^2) + \sqrt{2x(2x+m)}(16x^2 - m^2)}}{2m^2 \sqrt{2x(2x+m)}}, & \text{if } x \leq \tilde{x}(m) \\
\frac{2m^2 \sqrt{2x(2x+m)}}{[4m^2 + (1-x)^2(7x^2 + 10x - 1)]^{3/2}}, & \text{if } x > \tilde{x}(m).
\end{cases}
$$

\text{(EC 1.13)}

We know that this expression is continuous in $x$ by the previous argument. Moreover, note by inspection that $\phi(x, m)$ is twice differentiable at any $x \neq \tilde{x}(m)$ with $0 < x < 1$, and the second

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52Formally, we define $\xi_x(\tilde{x}(m), m) = \lim_{x \to \tilde{x}(m)^-} \xi_x(x, m)$, which is equal to the limit from the right by continuity.
partial derivative is

\[
\phi_{xx}(x, m) = \begin{cases} 
\frac{x(4x+m)(m^2-16mx-32x^2)+32x^2(2x+m)\sqrt{2x(2x+m)}}{m^2[2x(2x+m)]^{3/2}}, & \text{if } x < \bar{x}(m) \\
\frac{(9-7x)(1+x)^3+4(m^2-4)}{4m^2(1+x)^3}, & \text{if } x > \bar{x}(m).
\end{cases}
\] (EC 1.14)

One can show that \(\phi_{xx}(x, m) > 0\) for all \(x \in (0, 1)\) with \(x \neq \bar{x}(m)\) based on similar algebraic arguments to the ones we used before. Thus, \(\phi_x(x, m)\) is strictly increasing in \(x\), which implies that \(\phi(x, m)\) is strictly convex in \(x\).

Finally, by strict convexity we have that for all \(x \in (0, 1)\), \(\phi_x(x, m) \leq \phi_x(1, m) = -1/8\), which implies that \(\phi(x, m)\) is strictly decreasing in \(x\).

**Proof of Claim 11.** Recall from (A.5) and (A.6) that we can write firms’ profit functions as

\[
\pi^A(p^A_1, p^B_1, m) = p^A_1 \psi(X(p^A_1, p^B_1), m),
\]

\[
\pi^B(p^A_1, p^B_1, m) = (1 - \psi(X(p^A_1, p^B_1), m)) p^B_1 + m \phi(X(p^A_1, p^B_1), m).
\]

The result follows since \(X(p^A_1, p^B_1) = \max \{0, \min \{p^B_1 - p^A_1 + 1/2, 1\}\}\) is continuous in prices, and, by Claim 10, both \(\psi(x, m)\) and \(\phi(x, m)\) are continuous functions of \(x\).

Next, we prove Claim 12, that characterizes firm A’s best-response correspondence in \(G(m)\).

**Proof of Claim 12.** First note that if firm B sets \(p^B_1 = -1/2\), firm A obtains a profit of zero for any choice of \(p^A_1 \in S^A\), therefore any price in \(S^A\) is a best response to \(p^B_1 = -1/2\).

Now fix \(p^B_1 \in S^B\) such that \(p^B_1 > -1/2\). If firm A sets a price such that \(p^A_1 \geq p^B_1 + 1/2\), we have \(X(p^A_1, p^B_1) = 0\), which implies that \(\pi^A(p^A_1, p^B_1, m) = 0\). Moreover, for any \(p^A_1 \in (0, p^B_1 + 1/2)\), we have that \(X(p^A_1, p^B_1) > 0\), and therefore \(\pi^A(p^A_1, p^B_1, m) > 0\). Thus, any best response price lies in \([0, p^B_1 + 1/2) \cap S^A\).

Now consider \(p^A_1 < p^B_1 - 1/2\). For any such price, we have \(X(p^A_1, p^B_1) = 1\), and thus \(\pi^A(p^A_1, p^B_1, m) = p^A_1\). Thus, any price below \(p^B_1 - 1/2\) is dominated by taking \(p^A_1 = p^B_1 - 1/2\), and it follows that any best response price lies in \([\max\{0, p^B_1 - 1/2\}, p^B_1 + 1/2) \cap S^A\).

Since \(S^A = [0, 1]\), we can write firm A’s profit maximization problem given firm B’s price \(p^B_1\) as

\[
\max \{\pi^A(p^A_1, p^B_1, m) : p^A_1 \in [\max\{0, p^B_1 - 1/2\}, \min\{p^B_1 + 1/2, 1\}]\}.
\] (EC 1.15)

We claim that this problem has a unique solution given any choice of \(p^B_1 \in S^B\), and moreover, that this solution lies in the interior of the interval \([\max\{0, p^B_1 - 1/2\}, \min\{p^B_1 + 1/2, 1\}]\). In order to show this, first notice that for any \(p^A_1\) in this interval, we can write

\[
\pi^A(p^A_1, p^B_1, m) = p^A_1 \psi(p^B_1 - p^A_1 + 1/2, m).
\]

\(^{53}\)Since \(\psi_x(x, m)\) is continuous in \(x\) for all \(x \in [0, 1]\) and locally increasing at all \(x \neq \bar{x}(m)\), it follows that it is also increasing at \(x = \bar{x}(m)\).
By Claim 10, we have that \( \psi(x, m) \) is differentiable, strictly increasing and strictly concave in \( x \). Therefore, \( \pi^A(p^A_1, p^B, m) \) is strictly concave and differentiable in \( p^A_1 \), for \( p^A_1 \in [\max\{0, p^B_1 - 1/2\}, \min\{p^A_1 + 1/2, 1\}] \). By strict concavity, problem (EC 1.15) has a unique solution, so it only remains to show that this solution is interior. We consider two cases depending on the value of \( p^A_1 \) to show this.

**Case 1.** If \( p^B_1 \leq 1/2 \), it follows that the domain of problem (EC 1.15) is \([0, p^B_1 + 1/2]\). Setting either \( p^A_1 = 0 \) or \( p^A_1 = p^B_1 + 1/2 \) results in a profit of zero, and therefore is dominated by small enough positive prices. Thus, the solution to (EC 1.15) is interior.

**Case 2.** If \( 1/2 < p^B_1 \leq 1 \), the interval of interest is \([p^B_1 - 1/2, 1]\). To show that \( p^A_1 = p^B_1 - 1/2 \) is suboptimal, notice that

\[
\frac{\partial}{\partial p^A_1} \pi^A(p^A_1, p^B, m) = \psi\left(p^B_1 - p^A_1 + 1/2, m\right) - p^A_1 \psi_x\left(p^B_1 - p^A_1 + 1/2, m\right).
\]

(EC 1.16)

Plugging in \( p^A_1 = p^B_1 - 1/2 \) results in

\[
\frac{\partial}{\partial p^A_1} \pi^A\left(p^B_1 - 1/2, p^B, m\right) = \psi\left(1, m\right) - \left(p^B_1 - 1/2\right) \psi_x\left(1, m\right) = 1 - \frac{1}{2} \left(p^B_1 - \frac{1}{2}\right) = \frac{5}{4} - \frac{1}{2} p^B_1 > 0,
\]

since \( p^B_1 < 5/2 \). Thus, \( \pi^A(p^A_1, p^B, m) \) is increasing at \( p^A_1 = p^B_1 - 1/2 \). In particular, \( p^A_1 = p^B_1 - 1/2 \) is not a best-response price for firm A. It remains to show that \( p^A_1 = 1 \) does not solve problem (EC 1.15).

To do so, we will show that for any \( m > 0 \) and \( 1/2 < p^B_1 \leq 1 \), we have that \( \frac{\partial}{\partial p^A_1} \pi^A(1, p^B, m) < 0 \). In order to show this, first notice by taking the derivative of (EC 1.16) w.r.t. to \( p^B_1 \) we have

\[
\frac{\partial}{\partial p^B_1} \frac{\partial}{\partial p^A_1} \pi^A(p^A_1, p^B, m) = \psi_x\left(p^B_1 - p^A_1 + 1/2, m\right) - p^A_1 \psi_x\left(p^B_1 - p^A_1 + 1/2, m\right) > 0,
\]

where the inequality follows since \( \psi \) is strictly concave and increasing in \( x \). Thus, it suffices to show that \( \frac{\partial}{\partial p^A_1} \pi^A(1, 1, m) < 0 \), i.e., that \( \psi(1/2, m) - \psi_x(1/2, m) < 0 \), for all \( m > 0 \). By computing this expression we obtain

\[
\psi\left(1/2, m\right) - \psi_x\left(1/2, m\right) = \begin{cases} 
-\frac{(11+8m)}{36m} & \text{if } m > 5/4, \\
\frac{\sqrt{m+1}-m-2}{\sqrt{m+1}+m+1} & \text{if } m \leq 5/4,
\end{cases}
\]

which is negative for all \( m > 0 \). Thus, \( \pi^A(p^A_1, p^B_1, m) \) is decreasing at \( p^A_1 = 1 \), given a fixed \( p^B_1 \in [1/2, 1] \).

In any case, the unique solution to (EC 1.15) is interior, and therefore is such that that the partial derivative of \( \pi^A \) w.r.t. \( p^A_1 \) is zero. From (EC 1.16), we can rewrite this condition as \( p^A_1 = Z(p^B_1 - p^A_1 + 1/2, m) \), where \( Z \) is defined in (A.29).

That is, for any \( p^B_1 \in S^B \), there exists a unique \( p^A_1 \in [\max\{0, p^B_1 - 1/2\}, \min\{p^B_1 + 1/2, 1\}] \subset S^A \) that solves (EC 1.15). In addition such \( p^A_1 \) is the unique solution in \( S^A \) to \( p^A_1 = Z(p^B_1 - p^A_1 + 1/2, m) \),
We next prove the series of claims that characterize firm B’s best-response correspondence.

**Proof of Claim 13.** We start by proving the first property, i.e., we want to show that \( \pi^B (p^A_1, p^B, m) \) is (strictly) maximized by choosing \( p^B_1 = p^A_1 - 1/2 \) for all \( m \) large enough, i.e., that for any \( p^A_1 \in S^A \) there exists \( m_0 = m_0(p^A_1) \) such that, for all \( m > m_0 \) and any \( p^B_1 \in S^B \), we have

\[
\pi^B (p^A_1, p^A_1 - 1/2, m) > \pi^B (p^A_1, p^B_1, m). \tag{EC 1.17}
\]

First notice that choosing any price below \( p^B_1 < p^A_1 - 1/2 \) is suboptimal since for any such price, we have \( \bar{X} (p^A_1, p^B_1) = X (p^A_1, p^A_1 - 1/2) = 0 \), and therefore \( \pi^B (p^A_1, p^B_1, m) = p^B_1 + m/2 \). Thus, any price such that \( p^B_1 < p^A_1 - 1/2 \) is dominated by setting a price of \( p^A_1 - 1/2 \). Therefore, (EC 1.17) holds for any \( p^B_1 < p^A_1 - 1/2 \).

In addition, notice that setting a price \( p^B_1 > p^A_1 + 1/2 \) results in the same profits as choosing \( p^A_1 + 1/2 \) since \( \bar{X} (p^A_1, p^B_1) = \bar{X} (p^A_1, p^A_1 + 1/2) = 1 \). This implies that

\[
\max_{p^B_1 \in S^B} \pi^B (p^A_1, p^B_1, m) = \max_{p^B_1 \in [p^A_1 - 1/2, p^A_1 + 1/2] \cap S^B} \pi^B (p^A_1, p^B_1, m). \tag{EC 1.18}
\]

To establish the result, we now show that given \( p^A_1 \in S^A \), \( \pi^B (p^A_1, p^B_1, m) \) is strictly decreasing in \( p^B_1 \), for \( p^B_1 \in [p^A_1 - 1/2, p^A_1 + 1/2] \cap S^B \), for all \( m \) large enough. To see this, notice that in this region we can write

\[
\pi^B (p^A_1, p^B_1, m) = p^B_1 (1 - \psi (p^B_1 - p^A_1 + 1/2, m)) p^B_1 + m \phi (p^B_1 - p^A_1 + 1/2, m).
\]

Since, by Claim 10, \( \psi(x, m) \) and \( \phi(x, m) \) are differentiable for \( x \in (0, 1) \), we take the derivative of \( \pi^B \) w.r.t. \( p^B_1 \) to obtain (changing variables to \( x = p^B_1 - p^A_1 + 1/2 \in [0, 1] \)):

\[
\frac{\partial}{\partial p^B_1} \pi^B (p^A_1, x + p^A_1 - 1/2, m) = 1 - \psi (x, m) - (x + p^A_1 - 1/2) \psi_x (x, m) + m \phi_x (x, m). \tag{EC 1.19}
\]

We will show that this expression is negative for all \( x \in (0, 1) \) and all \( m \) large enough. To do so, notice that since \( \phi \) is strictly convex in \( x \) (by Claim 10), we have that for \( 0 < x < 1 \), \( \phi_x(x, m) < \phi_x(1, m) = -1/8 \). Combining this with (EC 1.19) results in

\[
\frac{\partial}{\partial p^B_1} \pi^B (p^A_1, x + p^A_1 - 1/2, m) < 1 - \left(x + p^A_1 - \frac{1}{2}\right) \psi_x (x, m) - \frac{m}{8}. \tag{EC 1.20}
\]

Since \( \psi_x > 0 \) (by Claim 10), the expression in the RHS above is negative when \( x \geq 1/2 - p^A_1 \) for all \( m \) large enough. Consider then \( 0 < x < 1/2 - p^A_1 \). By concavity of \( \psi \) in \( x \), we have that \( \psi_x(x, m) < \lim_{t \to 0^+} \psi_x(t, m) = 2 \). Therefore, for \( 0 < x < 1/2 - p^A_1 \) we have

\[
\frac{\partial}{\partial p^B_1} \pi^B (p^A_1, x + p^A_1 - 1/2, m) < 2 \left(1 - p^A_1\right) - \frac{m}{8}, \tag{EC 1.21}
\]
which implies that (EC 1.19) is strictly negative for all $x \in (0, 1)$, provided that $m > 16 \left(1-p_1^A\right)$. Thus, we define $m_0(p_1^A) = 16 \left(1-p_1^A\right)$. It follows that for any $m > m_0(p_1^A)$ and any $p_1^B \in S^B$, (EC 1.17) holds, as desired.

We now prove the second property. Suppose that $BR^B(p_1^A, m) = p_1^A - 1/2$ for some $m > 0$. Then, $\pi^B(p_1^A, p_1^A - 1/2, m) > \pi^B(p_1^A, p_1^B, m)$ for all $p_1^B \in S^B$. In particular, this implies that $\pi^B(p_1^A, p_1^B, m)$ is strictly decreasing at $p_1^B = p_1^A - 1/2$, i.e., that $\frac{\partial}{\partial p_1^B} \pi^B(p_1^A, p_1^A - 1/2, m) < 0$. Plugging in the expression for this derivative (from (EC 1.19)), we have\(^{54}\)

$$1 - \psi(0, m) - (p_1^A - 1/2) \psi_x(0, m) + m \phi_x(0, m) = 1 - 2 \left(p_1^A - 1/2\right) - m/2 < 0. \quad \text{(EC 1.22)}$$

By rearranging terms, it follows that $m > 4 \left(1-p_1^A\right)$. Given this condition, we rely on the following claim, which we prove in Section EC 1.1, to complete the proof.

**Claim 24.** Let $f : [0, 1] \times S^A \times \mathbb{R}^+ \to \mathbb{R}$ be given by

$$f(x, p_1^A, m) = \pi^B(p_1^A, p_1^A - 1/2, m) - \pi^B(p_1^A, x + p_1^A - 1/2, m).$$

Fix $p_1^A \in S^A$ and $x \in (0, 1]$. Then, $f(x, p_1^A, m)$ is strictly increasing in $m$ for $m > 4 \left(1-p_1^A\right)$.

Since $BR^B(p_1^A, m) = p_1^A - 1/2$, it follows that $f(x, p_1^A, m) > 0$ for all $0 < x \leq 1$. By Claim 24, and since $m > 4 \left(1-p_1^A\right)$, it follows that this also holds for any $m' > m$, i.e.,

$$\pi^B(p_1^A, p_1^A - 1/2, m') > \pi^B(p_1^A, p_1^B, m'),$$

for all $p_1^B \in S^B$ such that $p_1^A - 1/2 < p_1^B \leq p_1^A + 1/2$. Thus, by (EC 1.18), we have that $BR^B(p_1^A, m') = p_1^A - 1/2$. \(\square\)

**Proof of Claim 14.** Fix $m > 0$, we omit the dependency on $m$ in the rest of the proof to simplify the notation. By assumption, $\pi^B(p_1^A, p_1^A - 1/2) \geq \pi^B(p_1^A, p_1^B)$ for all $p_1^B \in S^B$, i.e.,

$$p_1^A - 1/2 + \frac{m}{2} \geq (1 - \psi(X(p_1^A, p_1^B))) p_1^B + m \phi(X(p_1^A, p_1^B)).$$

In particular, it follows from (EC 1.18) that for all\(^{55}\) $x \in (0, \min\{1, 3/2 - p_1^A\}]$,

$$p_1^A - 1/2 + \frac{m}{2} \geq (p_1^A - 1/2 + x)(1 - \psi(x)) + m \phi(x). \quad \text{(EC 1.23)}$$

Let $y > p_1^A$ such that $y \in S^A$, and assume towards a contradiction that there exists $p_1^{B'} \in S^B$ with $p_1^{B'} > y - 1/2$ such that $\pi^B(y, y - 1/2) \leq \pi^B(y, p_1^{B'})$. Without loss of generality, suppose that $p_1^{B'} \leq y + 1/2$, as otherwise setting a price of $y + 1/2$ yields the same profits as $p_1^{B'}$.

Define $x' = p_1^{B'} - y + 1/2$ and notice that $0 < x' \leq \min\{1, 3/2 - y\} < \min\{1, 3/2 - p_1^A\}$. We

\(^{54}\)We define $\psi_x(0, m) = \lim_{t \to 0^+} \psi_x(t, m) = 2$, and $\phi_x(0, m) = \lim_{t \to 0^+} \phi_x(t, m) = -1/2$.

\(^{55}\)We have this condition instead of $x \in (0, 1]$ to ensure that $p_1^B \in S^B$. 

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have that
\[
\pi^B(y, y - 1/2) = y - 1/2 + \frac{m}{2} \leq (y - 1/2 + x')(1 - \psi(x')) + m\phi(x')
\]
\[
= (p_1^A - 1/2 + x')(1 - \psi(x')) + m\phi(x') + (y - p_1^A)(1 - \psi(x'))
\]
\[
\leq p_1^A - 1/2 + \frac{m}{2} + (y - p_1^A)(1 - \psi(x'))
\]
where the last step follows from (EC 1.23). The last inequality implies that \(y - p_1^A \leq (y - p_1^A)(1 - \psi(x'))\), from where we have that \(\psi(x') \leq 0\), and therefore \(x' = 0\), a contradiction.

Finally, by (EC 1.18), any possible best response to \(y\) lies in \([y - 1/2, \min\{1, y + 1/2\}]\), from where it follows that \(BR(y, m) = y - 1/2\).

\[\square\]

Proof of Claim 15. Define \(m_H = \inf\{m > 0 : -1/2 \in BR^B(0, m)\}\). By Claim 13, we have that \(m_H < \infty\) and that \(BR^B(0, m) = -1/2\) for all \(m > m_H\).

Fix \(m > m_H\). For firm A, we have that \(BR^A(-1/2, m) = S^A\) for all \(m > 0\) by Claim 12. It follows that \((p_1^{A*}, p_2^{B*}) = (0, -1/2)\) is a PSNE of \(G(m)\).

We now show that it is the unique PSNE. Suppose that there is another PSNE \((p_1^A, p_1^B)\) in \(G(m)\), given \(m > m_H\). If \(p_1^A > 0\), then, by Claim 14, firm B’s best response to \(p_1^A\) is \(p_1^B = p_1^A - 1/2 > -1/2\). Thus, we have that \(X(p_1^A, p_1^B) = p_1^B - p_1^A + 1/2 = 0\), and therefore \(\pi^A(p_1^A, p_1^B, m) = 0\). Then, for all \(\varepsilon > 0\) small enough, we have \(X(p_1^A - \varepsilon, p_1^B) > 0\), and \(\pi^A(p_1^A - \varepsilon, p_1^B, m) > 0\). In particular, \(p_1^A \notin BR^A(p_1^B, m)\), contradicting that \((p_1^A, p_1^B)\) is a Nash equilibrium. Thus, \((0, -1/2)\) is the unique PSNE of \(G(m)\).

Finally, by definition of \(m_H\), if \(m < m_H\), then \((0, -1/2)\) is not a PSNE of \(G(m)\). Moreover, by the preceding argument, any pair of prices \((p_1^A, p_1^B)\) with \(p_1^B \leq p_1^A - 1/2\) cannot be a PSNE if \(m < m_H\).

\[\square\]

To prove Claim 16, we rely on the following result, which establishes that as we take \(m\) to zero, firm B’s profit function in \(G(m)\) converges to the corresponding profit function associated to product 1 in the restricted setting.\(^{56}\) We provide the proof of Claim 25 in Section EC 1.1.

Claim 25. Fix any \(p_1^A \in S^A\), \(p_1^B \in S^B\). Then,
\[
\lim_{m \to 0^+} \pi^B(p_1^A, p_1^B, m) = p_1^B \left(1 - X\left(p_1^A, p_1^B\right)\right).
\]

Proof of Claim 16. Fix \(m > 0\) and consider firm B’s profit maximization problem when firm A sets a price of \(p_1^A \in S^A\), which by (EC 1.18) in the proof of Claim 13 can be written as:
\[
\max \left\{ \pi^B(p_1^A, p_1^B, m) : p_1^B \in [p_1^A - 1/2, \min\{p_1^A + 1/2, 1\}] \right\}.
\]
Notice that this problem always has a solution for any \(p_1^A \in S^A\) since \(\pi^B\) is a continuous function, and \([p_1^A - \frac{1}{2}, \min\{1, p_1^A + \frac{1}{2}\}]\) is a compact set, i.e., \(BR^B(p_1^A, m)\) is non-empty for all \(p_1^A \in S^A\).

\(^{56}\)Note that this is the profit function in the Standard Hotelling competition model d’Aspremont et al. (1979).
Given $m > 0$, define $d(m)$ as the smallest value of $p_1^A$ such that this problem has a corner solution, i.e., that $p_1^B = p_1^A - 1/2$ solves the problem above:

$$d(m) = \inf \{ p_1^A \in S^A : p_1^A - 1/2 \in BR^B(p_1^A, m) \},$$

where $d(m) = \infty$ if the set on which the infimum is taken is empty. The rest of the proof consists of studying the properties of $d(m)$, and in particular, showing that $BR^B(p_1^A, m)$ is single-valued when $p_1^A < d(m)$. We proceed in several steps.

**Step 1.** We show that for any $m > 0$, and any $p_1^A \in S^A$, $\min\{p_1^A + 1/2, 1\} \notin BR^B(p_1^A, m)$. Therefore, $BR^B(p_1^A, m)$ consists only of interior solutions to (EC 1.24), or the corner solution $p_1^B = p_1^A - 1/2$.

If $p_1^A \leq 1/2$, we have that $\min\{p_1^A + 1/2, 1\} = p_1^A + 1/2$. Observe that $p_1^B = p_1^A + 1/2$ is suboptimal in (EC 1.24) as it results in a profit of zero from product 1 and minimizes the profit from product 2. Thus, $p_1^A + 1/2 \notin BR^B(p_1^A, m)$.

Now suppose that $p_1^A > 1/2$. We will show that $p_1^B = 1$ is suboptimal in (EC 1.24), by showing that choosing $p_1^B = p_1^A - 1/2$ results in higher profits for firm B. To see this, notice that given $p_1^A > 1/2$,

$$\pi^B (p_1^A, 1, m) = 1 - \psi(3/2 - p_1^A, m) + m\phi(3/2 - p_1^A, m).$$

On the other hand, $\pi^B (p_1^A, p_1^A - 1/2, m) = p_1^A - 1/2 + m/2$. We will now show that $\pi^B (p_1^A, 1, m) < \pi^B (p_1^A, p_1^A - 1/2, m)$.

Since $\phi(x, m) < \phi(0, m) = 1/2$ for any $x \in (0, 1]$ (by Claim 10), we have that $m\phi(3/2 - p_1^A, m) < m/2$, so it suffices to show that $1 - \psi(3/2 - p_1^A, m) \leq p_1^A - 1/2$. This inequality can be rewritten as $3/2 - p_1^A \leq \psi(3/2 - p_1^A, m)$, which indeed holds.\(^{57}\) Thus, $1 \notin BR^B(p_1^A, m)$.

**Step 2.** We show that if $p_1^A < d(m)$, there is only one interior solution to problem (EC 1.24). In particular, $BR^B(p_1^A, m)$ is single-valued and defined by (A.30).

To show this, first notice that by the definition of $d(m)$, and by Step 1, $BR^B(p_1^A, m)$ consists only of interior solutions to problem (EC 1.24) when $p_1^A < d(m)$. That is, $BR^B(p_1^A, m)$ contains only prices in $(p_1^A - 1/2, \min\{p_1^A + 1/2, 1\})$. We now show that in this case, there is only one interior solution to problem (EC 1.24).

Fix $p_1^A \geq 0$ and assume that $p_1^A < d(m)$. By changing variables to $x = p_1^B - p_1^A + 1/2 \in [0, 1]$, we can rewrite the first-order condition for profit maximization of problem (EC 1.24) as

$$H(x, m) - p_1^A + \frac{1}{2} = 0,$$

where we define $H : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}$ as

$$H(x, m) = V(x, m) - x.$$  \(^{(EC \ 1.26)}\)

\(^{57}\)Notice that since $\psi(0, m) = 0$, $\psi(1, m) = 1$ and $\psi$ is strictly concave in $x$, we have that $\psi(x, m) > x$ for all $x \in (0, 1)$ and all $m > 0$.
On Section EC 1.1, we prove that:

**Claim 26.** For any fixed \( m > 0 \), \( H(x, m) \) is a strictly quasiconcave function of \( x \).

It follows that there are at most two values of \( x \) that solve equation (EC 1.25) given \( p^A_1 \). Note that if there exist two solutions one of them must correspond to a local minimum and the other one to a local maximum of (EC 1.24), and only the one corresponding to the local maximum can be a best response. If there is only one solution, it must correspond to a maximizer (since it is the only candidate for an interior solution). Therefore, there exists at most one interior solution to (EC 1.24). In particular, if \( p^A_1 < d(m) \), we know that an interior solution to (EC 1.24) exists, and therefore it must be the unique solution to problem (EC 1.24). By reversing the change of variables from \( p^B_1 \) to \( x \), we have that the best-response price \( p^B_1 \) satisfies \( p^B_1 = V(p^B_1 - p^A_1 + 1/2, m) \), where \( V \) is defined in (A.31). In addition, such \( p^B_1 \) lies in \( S^B \) since the domain of problem (EC 1.24) is a subset of \( S^B \).

**Step 3.** If \( p^A_1 > d(m) \), then \( BR^B(p^A_1, m) = p^A_1 - 1/2 \).

This follows directly from Claim 14, and the definition of \( d(m) \).

**Step 4.** For all small enough \( m > 0 \), we have that \( d(m) = \infty \), i.e., \( p^A_1 - 1/2 \notin BR^B(p^A_1, m) \).

Recall that \( \pi^B(p^A_1, p^B_1, m) \) is only defined for \( m > 0 \). We now augment this definition by letting, for \( m = 0 \),

\[
\pi^B(p^A_1, p^B_1, 0) = p^B_1 (1 - \bar{X}(p^A_1, p^B_1)).
\]

By evaluating \( \pi^B(p^A_1, p^B_1, 0) \) at \( p^B_1 = p^A_1 / 2 + 1/4 \) and \( p^B_1 = p^A_1 - 1/2 \), we have that for any \( p^A_1 \in S^A \),

\[
\pi^B(p^A_1, p^A_1 / 2 + 1/4, 0) - \pi^B(p^A_1, p^A_1 - 1/2, 0) = (p^A_1 / 2 + 1/4)^2 - (p^A_1 - 1/2)^2.
\]

Given that \( p^A_1 \in S^A = [0, 1] \), it is easy to verify that this expression is minimized when \( p^A_1 = 1 \), and therefore we have that \( \pi^B(p^A_1, p^A_1 / 2 + 1/4, 0) - \pi^B(p^A_1, p^A_1 - 1/2, 0) \geq 1/16 \) for all \( p^A_1 \in S^A \).

It is easy to verify that for any fixed prices \( p^A_1 \in S^A \) and \( p^B_1 \in S^B \), \( \pi^B(p^A_1, p^B_1, m) \) is a continuous function\(^{58} \) of \( m \). Therefore, by Claim 25, we have that for all small enough \( m > 0 \),

\[
\pi^B(p^A_1, p^A_1 / 2 + 1/4, m) - \pi^B(p^A_1, p^A_1 - 1/2, m) > 0,
\]

which implies that \( p^B_1 = p^A_1 - 1/2 \) is suboptimal in problem (EC 1.18) for all small enough \( m > 0 \). Thus, \( d(m) = \infty \) for all small enough \( m > 0 \).

Let us then define \( M_0 = \sup\{m > 0 : d(m) = \infty \} \). By the previous argument, it follows that \( M_0 > 0 \). Moreover, if \( m < M_0 \), we have that \( p^A_1 < d(m) \) for all \( p^A_1 \in S^A \). Part (i) of the proposition then follows by Step 2.

**Step 5.** There exists \( M_1 > 0 \) such that if \( m > M_1 \), \( BR^B(p^A_1, m) = p^A_1 - 1/2 \) for all \( p^A_1 \in S^A \).

To verify this, let \( M_1 = m_H = \inf\{m > 0 : -1/2 \in BR^B(0, m)\} \), as in the proof of Claim 15. From the proof of Claim 15, it follows that if \( m > M_1 \), \( BR^B(p^A_1, m) = p^A_1 - 1/2 \) for all \( p^A_1 \in S^A \).

\(^{58}\)Following the same reasoning as in Claim 10, we can show that \( \theta, \psi, \xi \) and \( \phi \) are continuous functions of \( m \), when \( m > 0 \).
where we define \( W \).

By combining these two equations, it follows that \((EC\ 1.28)\), we can write

\[ X(p^A_1, p^B_1) = p^B_1 - p^A_1 + 1/2 \in (0, 1). \]

Moreover, any such solution satisfies

\[ X(p^A_1, p^B_1) = p^B_1 - p^A_1 + 1/2 \in (0, 1). \]

Step 6. \( d(m) \) is non-increasing in \( m \).

If \( d(m) < \infty \), then for any \( m' > m \), we have that \( d(m') < d(m) \) by Claim 13. The property trivially holds if \( d(m) = \infty \). Then, by steps 4 and 5, it follows that \( d(m) \in (0,1) \) if and only if \( M_0 < m < M_1 \). Part (ii) of the Claim then follows from Steps 2 and 3.

Proof of Claim 17. Take \( M_0 \) from Claim 16 and fix \( m < M_0 \). Then both firms best-response correspondences are single-valued (therefore compact and convex-valued). Moreover, these correspondences are also upper hemicontinuous by continuity of \( \pi^A \) and \( \pi^B \), as established in Claim 11. It follows from Kakutani’s fixed point theorem \( G(m) \) admits a PSNE. In addition, all such equilibria must be interior by Claims 12 and 16.

Proof of Claim 18. Fix \( m > 0 \) and let \((p^A_1, p^B_1)\) be an interior pure-strategy NE for \( G(m) \). Then, since \((p^A_1, p^B_1)\) is not a corner equilibrium, it follows that \( X(p^A_1, p^B_1) = p^B_1 - p^A_1 + 1/2 \in (0, 1) \). Moreover, by Claims 12 and 16, the following two conditions must hold:

\[
\begin{align*}
p^A_1 &= Z(p^B_1 - p^A_1 + 1/2, m) \\
p^B_1 &= V(p^B_1 - p^A_1 + 1/2, m)
\end{align*}
\]

(EC 1.27)

By combining these two equations, it follows that \((p^A_1, p^B_1)\) is such that \( W(p^B_1 - p^A_1 + 1/2, m) = 0 \), where we define \( W : [0,1] \times \mathbb{R}^+ \to \mathbb{R} \) by

\[
W(x, m) = V(x, m) - Z(x, m) + \frac{1}{2} - x.
\]

(EC 1.28)

Now suppose that there exist two interior PSNE for \( G(m) \), say \((p^A_1, p^B_1)\) and \((p^A_1', p^B_1')\). It follows that \( X(p^A_1, p^B_1) = p^B_1 - p^A_1 + 1/2 \in (0, 1) \), and \( X(p^A_1', p^B_1') = p^B_1' - p^A_1' + 1/2 \in (0, 1) \). For the remainder of the proof, let us abbreviate these two quantities by \( \tilde{X} \) and \( \tilde{X}' \), respectively. By the previous argument, we have that \( W(\tilde{X}, m) = 0 \) and \( W(\tilde{X}', m) = 0 \). We will now prove that \( \tilde{X} = \tilde{X}' \) by leveraging the following result, which we prove in Section EC 1.1:

Claim 27. Given \( m > 0 \), there are at most two values of \( x < \tilde{x}(m) \) that solve \( W(x, m) = 0 \). Moreover, any such solution satisfies \( x < \tilde{x}(m) \).

It follows from the Claim that we have two cases to consider. If there exists only one solution to \( W(x, m) = 0 \) (with \( 0 < x < 1 \)), then it must be that \( \tilde{X} = \tilde{X}' \). Otherwise, if there are two solutions, we will show that one of the previous price pairs is not a Nash equilibrium. To do so, notice that by plugging in the expressions for \( Z(x, m) \) and \( V(x, m) \) as defined in (A.29) and (A.31) into (EC 1.28), we can write

\[
W(x, m) = \frac{\varphi(x, m)}{\psi_x(x, m)},
\]

(EC 1.29)

where

\[
\varphi(x, m) = 1 - 2\psi(x, m) + m\phi_x(x, m) - \left(x - \frac{1}{2}\right)\psi_x(x, m).
\]
Since both $\bar{X}$ and $\bar{X}'$ solve $W(x,m) = 0$, it follows that they also solve $\varphi(x,m) = 0$. Since $\varphi$ is continuous and differentiable in $x$ for $x < \hat{x}(m)$, and $\bar{X}', \bar{X}$ are the only two solutions of $\varphi(x,m) = 0$, it must be that $\varphi_x(\bar{X}, m) \varphi_x(\bar{X}', m) \leq 0$. Therefore, $\varphi_x(\bar{X}, m) \geq 0$ or $\varphi_x(\bar{X}', m) \geq 0$.

Assume w.l.o.g. that $\varphi_x(\bar{X}', m) \geq 0$, so that

$$-3\psi_x(\bar{X}', m) + m\phi_{xx}(\bar{X}', m) - \left(\bar{X}' - \frac{1}{2}\right) \psi_{xx}(\bar{X}', m) \geq 0$$

Since $p_1^A' = Z(\bar{X}', m)$ and $p_1^B' = \bar{X}' + p_1^A' - 1/2$. We can rewrite the previous expression as

$$[-2\psi_x(\bar{X}', m) + m\phi_{xx}(\bar{X}', m) - p_1^B' \psi_{xx}(\bar{X}', m)] + [p_1^A' \psi_{xx}(\bar{X}', m) - \psi_x(\bar{X}', m)] \geq 0.$$

The second term is always negative since $\psi$ is strictly increasing and strictly concave in $x$, so the first term must be positive. However, the second-order condition from Firm 2’s profit maximization is equivalent to having the first term to be non-positive, which implies that $p_1^B'$ does not maximize firm B’s profit given $p_1^A'$, contradicting that $(p_1^A', p_1^B')$ is a Nash equilibrium.

We have shown that $\bar{X} = \bar{X}'$. It follows from (EC 1.27) that $p_1^A = Z(\bar{X}, m) = Z(\bar{X}', m) = p_1^A'$. Similarly, we conclude that $p_1^B = p_1^B'$, and therefore there exists a unique interior PSNE for $G(m)$.

**Proof of Claim 19.** Note that setting a negative price is a strictly dominated strategy for firm A, as it may result in negative profits while setting a price of zero results in a profit of zero. Thus, we restrict our attention to the case where $p_1^A \geq 0$.

Given this constraint, note that setting a price below $-1/2$ is a dominated strategy for firm B, since for any choice of $p_1^A \geq 0$, setting a price equal to $-1/2$ makes firm B better off than any price $p_1^B < -1/2$. Indeed, we have that for $p_1^B < -1/2$

$$\pi^{B}(p_1^A, p_1^B, m) = p_1^B + m/2 < (m - 1)/2 = \pi^{B}(p_1^A, -1/2, m).$$

Thus, $\tilde{G}(m)$ has no PSNE in undominates strategies such that firm B sets a price below $-1/2$. The rest of the proof consists of two steps. First, we show that any PSNE of $G(m)$ is also a PSNE of $\tilde{G}(m)$. The second step consists of showing the converse proposition, i.e., that any PSNE of $\tilde{G}(m)$ (with $p_1^A \geq 0$) is a PSNE of $G(m)$. In what follows, we denote by $\tilde{BR}^A$ and $\tilde{BR}^B$ to be the firms’ best-response correspondences in $\tilde{G}(m)$.

**Step 1.** Fix $m > 0$ and suppose that $(p_1^A, p_1^B)$ is a PSNE of $G(m)$. We have two cases to cover. First, if $(p_1^A, p_1^B)$ is a corner equilibrium, then, by Claim 15, $(p_1^A, p_1^B) = (0, -1/2)$. Since given $p_1^B = -1/2$, firm A cannot make a profit larger than zero, we have that $\tilde{BR}^A(-1/2, m) = [0, \infty)$.

In addition, by following the same argument as in the proofs of Claims 13 and 14, we have that $-1/2 \in \tilde{BR}^B(0, m)$. Thus, $(p_1^A, p_1^B)$ is a PSNE in $\tilde{G}(m)$.

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59Note that $\varphi$ is differentiable everywhere except at $x = \hat{x}(m)$, since that is the case for $\phi_x$ and $\psi_x$, as shown in the proof of Claim 10. Then, by Claim 27, we are evaluating $\varphi_x$ at a point where it is defined.

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The second case corresponds to when \((p^A_1, p^B_1)\) is an interior equilibrium, i.e., \(p^A_1 > 0\) and \(p^B_1 > p^A_1 - 1/2\). By following the same argument as in the proof of Claim 12, we have that

\[
\max \left\{ \pi^A (y, p^B_1, m) : y \in [\max\{0, p^B_1 - 1/2\}, \min\{p^B_1 + 1/2, 1]\} \right\}
= \max \left\{ \pi^A (y, p^B_1, m) : y \in [0, p^B_1 + 1/2] \right\} = \max \left\{ \pi^A (y, p^B_1, m) : y \geq 0 \right\},
\]

where the last step follows since firm A makes a profit of zero when choosing any price above \(p^B_1 + 1/2\), and \(p^B_1 > -1/2\) by assumption. Moreover, \(BR^A(p^B_1, m)\) is single-valued and \(p^A_1 = BR^A(p^B_1, m)\) is the unique solution to firm A’s profit maximization problem above, by Claim 12. Hence, \(BR^A(p^B_1, m) = BR^A(p^B_1, m)\) and, since \((p^A_1, p^B_1)\) is a PSNE in \(G(m)\), \(p^A_1 \in BR^A(p^B_1, m)\).

We proceed similarly for firm B. By following the same argument as in step 1 of the proof of Claim 16, we have that

\[
\max \left\{ \pi^B (p^A_1, y, m) : y \in [p^A_1 - 1/2, \min\{p^A_1 + 1/2, 1]\} \right\}
= \max \left\{ \pi^B (p^A_1, y, m) : y \in [p^A_1 - 1/2, p^A_1 + 1/2] \right\} = \max \left\{ \pi^B (p^A_1, y, m) : y \geq -1/2 \right\},
\]

where the last equality follows since it is always suboptimal for firm B to set a price below \(p^A_1 - 1/2\), and setting a price of \(p^A_1 + 1/2\) or higher is also always suboptimal (by the same argument as in the proof of Claim 13). Then, since \((p^A_1, p^B_1)\) is a PSNE in \(G(m)\), we have that \(p^B_1 \in BR^B(p^A_1, m) = BRB(p^A_1, m)\). We have shown above that \(p^A_1 \in BR^A(p^B_1, m)\). Therefore, \((p^A_1, p^B_1)\) is a PSNE of \(\tilde{G}(m)\).

**Step 2.** Suppose now that \((p^A_1, p^B_1)\) is a PSNE in undominated strategies for \(\tilde{G}(m)\). Then, \(p^A_1 \geq 0\). Moreover, note that we must have that \(p^A_1 > p^B_1 - 1/2\) (since otherwise firm B receives no demand in the market for product 1 and is better off by matching firm A’s price), we must have \(p^B_1 \geq p^A_1 - 1/2\) (since otherwise firm B increases is better off by choosing \(p^A_1 - 1/2\)). Moreover, these two conditions can be written as \(0 \leq \tilde{X}(p^A_1, p^B_1) < 1\). We now consider two cases: \(\tilde{X}(p^A_1, p^B_1) = 0\), and \(\tilde{X}(p^A_1, p^B_1) > 0\) (i.e., the interior and corner cases).

First, suppose that \(\tilde{X}(p^A_1, p^B_1) = 0\). Then, \(p^B_1 = p^A_1 - 1/2\). By the same argument as in the proof of Claim 15, it follows that \((p^A_1, p^B_1) = (0, -1/2)\). Since \(BR^A(-1/2, m) = S^A\) (by Claim 12), it follows that \(0 \in BR^A(-1/2, m)\). In addition, by the same argument as in step 1, we have that \(BR^B(0, m) = BRB(0, m)\). Since \((p^A_1, p^B_1) = (0, -1/2)\) is a PSNE in \(\tilde{G}(m)\), it follows that \(-1/2 \in BRB(0, m)\). Thus, \((p^A_1, p^B_1) = (0, -1/2)\) is a PSNE in \(G(m)\).

Now consider the case with \(0 < \tilde{X}(p^A_1, p^B_1) < 1\), i.e., \(0 < p^B_1 - p^A_1 + 1/2 < 1\). Since \(p^A \geq 0\), this implies that \(p^B_1 > -1/2\). By following the same argument as in the proofs of Claims 12 and 16, we have that

\[
p^A_1 \in \arg \max \left\{ \pi^A (y, p^B_1, m) : y \in [0, p^B_1 + 1/2] \right\},
\]
\[
p^B_1 \in \arg \max \left\{ \pi^B (p^A_1, y, m) : y \in [p^A_1 - 1/2, p^A_1 + 1/2] \right\}.
\]

In particular, we have that \(p^A_1 \in (0, p^B_1 + 1/2)\), as choosing a price of zero leads to zero profits for firm A given that \(p^B_1 > -1/2\). Thus, \(p^A_1\) satisfies the first-order condition for profit maximization.
for firm A. By the proof of Claim 12, we can write this condition as $p_1^A = Z(p_1^B - p_1^A + 1/2, m)$, where $Z$ is defined as in (A.29).

For firm B, by following the same argument as in the proof of Claim 16, we have that $p_1^B \in (p_1^A - 1/2, p_1^A + 1/2)$, so $p_1^B$ satisfies firm B’s first-order condition for profit maximization, which, the proof of Claim 16, we can write as $p_1^B = V(p_1^B - p_1^A + 1/2, m)$, where $V$ is defined as in (A.31). Then, by following the same argument as in the proof of Claim 18, it follows that $W(p_1^B - p_1^A + 1/2, m) = 0$, where $W$ is defined as in (EC 1.28).

In addition, since $p_1^B \in \bar{B}R^B(p_1^A, m)$, we have that $\pi^B(p_1^A, p_1^B, m) \geq \pi^B(p_1^A, p_1^A - 1/2, m) = (m - 1)/2 + p_1^A$. Notice that we can write this condition as $T(p_1^B - p_1^A + 1/2, m) \geq 0$, where $^60$

$$T(x, m) = x - V(x, m)\psi(x, m) + m (\phi(x, m) - 1/2).$$

To summarize, we have that $0 < p_1^B - p_1^A + 1/2 < 1$, $T(p_1^B - p_1^A + 1/2, m) \geq 0$, and $W(p_1^B - p_1^A + 1/2, m) = 0$. By Claim 28 and Proposition 8 in Section EC 2, it follows that $(p_1^A, p_1^B)$ is a PSNE of $G(m)$, as desired.

EC 1.1 Proofs of the auxiliary results of Section EC 1 (Claims 24 – 27)

Proof of Claim 24. We want to show that for fixed $x \in (0, 1]$ and $p_1^A \in S^A$, $f(x, p_1^A, m)$ is strictly increasing in $m$, provided that $m > 4(1 - p_1^A)$, where

$$f(x, p_1^A, m) = \pi^B(p_1^A, p_1^A - 1/2, m) - \pi^B(p_1^A, x + p_1^A - 1/2, m)$$

$$= m - \frac{1}{2} + p_1^A - \left(p_1^A + x - \frac{1}{2}\right)(1 - \psi(x, m)) - m\phi(x, m).$$

From equations (A.2) and (A.7), we have that $\psi$ and $\phi$ are differentiable in $m$, except when $x = \tilde{x}(m)$, it suffices to show that the derivative of this expression w.r.t. $m$ is positive when $x \neq \tilde{x}(m)$ (since the expression is continuous in $m$, that would imply that it is increasing). Thus, we want to show that for all $p_1^A \in S^A$, $m > 4(1 - p_1^A)$, and $x \neq \tilde{x}(m)$, it holds that $f_m(x, p_1^A, m) > 0$, where

$$f_m(x, p_1^A, m) = 1/2 + \psi_m(x, m)(p_1^A + x - 1/2) - \phi(x, m) - m\phi_m(x, m). \quad (EC 1.30)$$

The rest of the proof consists of showing that $f_m(x, p_1^A, m) > 0$ in the aforementioned region by plugging in the functional forms of the terms in the previous expression. We consider two cases.

Case 1. $\tilde{x}(m) < x \leq 1$. First, notice that $\psi_m > 0$. To see this, simply note from (A.2) that $\tilde{\theta}_m > 0$, and recall that $\psi(x, m) = 2x\tilde{\theta}(x, m)$. Thus, we have from (EC 1.30) that $f_m(x, p_1^A, m)$ is increasing in $p_1^A$. This implies that

$$f_m(x, p_1^A, m) \geq f_m(x, 0, m) = \frac{1 + 4(3m^2 - 2)x + 30x^2 - 40x^3 + 17x^4}{24m^2(1 + x)},$$

See Section EC 2 for a detailed derivation of $T(x, m)$. 

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where the last equality follows by computing the partial derivatives of \( \psi \) and \( \phi \) w.r.t. \( m \) and plugging into (EC 1.30). We want to show that the numerator of the previous expression is positive for all \( m > 0 \), i.e., that

\[
1 + 4(3m^2 - 2)x + 30x^2 - 40x^3 + 17x^4 > 0.
\]

Notice that this expression is strictly increasing in \( m \), so it suffices to show that the inequality holds (even if weakly) for \( m = 0 \), i.e., that

\[
1 - 8x + 30x^2 - 40x^3 + 17x^4 = (x - 1)^2(17x^2 - 6x + 1) \geq 0,
\]

which holds, since both terms in the factorization are non-negative for all \( x \in \mathbb{R} \).

**Case 2.** \( 0 < x < \hat{x}(m) \). Notice that we can rewrite the constraint \( m > 4(1 - p_1^A) \) as \( p_1^A > \max\{1 - m/4, 0\} \). As in the previous case, since \( f_m(x, p_1^A, m) \) is increasing in \( p_1^A \), we have that\(^{61}\)

\[
f_m(x, p_1^A, m) > f_m(x, \max\{1 - m/4, 0\}, m) \geq f_m(x, 1 - m/4, m).
\]

We will now show that \( f_m(x, 1 - m/4, m) > 0 \) for \( m > 0 \). To do so, we obtain the following expression by computing the partial derivatives of \( \psi \) and \( \phi \) w.r.t. \( m \) and plugging into (EC 1.30):

\[
f_m(x, 1 - m/4, m) = \frac{x \left( (6m - 5m^2 + 24x - 8mx + 16x^2) x - \sqrt{2x(2x + m)} (8x^2 + 12x - 6mx - 3m^2) \right)}{6m^2 \sqrt{2x(2x + m)}}.
\]

Since the denominator of this expression is positive, it suffices to show that the numerator is positive. Indeed, we have that

\[
\begin{align*}
(6m - 5m^2 + 24x - 8mx + 16x^2) x &- \sqrt{2x(2x + m)} (8x^2 + 12x - 6mx - 3m^2) \\
&= (6m + 24x - 8mx + 16x^2) x - \sqrt{2x(2x + m)} (8x^2 + 12x - 6mx) + m^2 \left( 3\sqrt{2x(2x + m)} - 5x \right) \\
&> (6m + 24x + 16x^2) x - \sqrt{2x(2x + m)} (8x^2 + 12x) + m x \left( 6\sqrt{2x(2x + m)} - 8x \right) \\
&> (6m + 4mx + 24x + 16x^2) x - \sqrt{2x(2x + m)} (8x^2 + 12x) \\
&= 2x \left( 3 + 2x \right) \left( 4x + m - 2\sqrt{2x(2x + m)} \right) > 0.
\end{align*}
\]

Thus, \( f_m(x, p_1^A, m) > 0 \), as desired. \( \square \)

**Proof of Claim 25.** Recall from (A.6) that firm B’s profit function in \( \mathbf{G}(m) \) is

\[
\pi^B (p_1^A, p_1^B, m) = (1 - \psi(\bar{X}(p_1^A, p_1^B), m)) p_1^B + m \phi (\bar{X}(p_1^A, p_1^B), m).
\]

\(^{61}\)Formally, \( f_m \) is only defined when \( p_1^A \in S^A = [0, 1] \). For this proof, we simply extend the definition by plugging in \( p_1^A = 1 - m/4 \) to the original expression of \( f_m \) given in (EC 1.30).
By Claim 10, we have that for all \( m > 0 \), \( 1/4 \leq \phi(x, m) \leq 1/2 \) for all \( x \in [0, 1] \). Thus, we have that \( m\phi(\bar{X}(p^A_1, p^B_1), m) \rightarrow 0 \) as \( m \rightarrow 0^+ \). It remains to show that
\[
\lim_{m \rightarrow 0^+} \left( 1 - \psi(\bar{X}(p^A_1, p^B_1), m) \right) p^B_1 = p^B_1 \left( 1 - \bar{X}(p^A_1, p^B_1) \right).
\]
In particular, we just need to show that \( \lim_{m \rightarrow 0^+} \psi(x, m) = x \) for any \( x \in [0, 1] \). To do so, recall that \( \psi(x, m) = 2x \tilde{\theta}(x, m) \), where \( \tilde{\theta}(x, m) \) is given by
\[
\tilde{\theta}(x, m) = \begin{cases} 1, & \text{if } x \leq \tilde{x}(m) \\ \frac{1}{1 + x} \left[ 1 - \frac{1}{2m} (1 - x)^2 \right], & \text{if } x > \tilde{x}(m), \end{cases}
\]
where \( \tilde{x}(m) = \frac{1}{3} \left( \sqrt{(m-1)^2 + 3} - (m-1) \right) \). We now show that \( \lim_{m \rightarrow 0^+} \psi(x, m) = x \).

First, if \( x = 0 \), we have that \( \psi(0, m) = 0 \) for all \( m > 0 \). Thus, \( \lim_{m \rightarrow 0^+} \psi(0, m) = 0 \).

Next, we claim that for any \( x \in (0, 1) \), \( \lim_{m \rightarrow 0^+} \tilde{\theta}(x, m) = 1/2 \). To show this, first notice that \( \tilde{x}(m) \) is strictly decreasing in \( m \), and that \( \tilde{x}(0) = 1 \). Thus, for all \( m > 0 \), we have \( \tilde{\theta}(1, m) = 1/2 \) and therefore \( \lim_{m \rightarrow 0^+} \tilde{\theta}(1, m) = 1/2 \).

Now consider \( 0 < x < 1 \), and notice that \( x < \tilde{x}(m) \), for all small enough \( m > 0 \). Therefore
\[
\lim_{m \rightarrow 0^+} \tilde{\theta}(x, m) = \lim_{m \rightarrow 0^+} \sqrt{2x + m} \left[ \frac{\sqrt{2x + m} - \sqrt{2x}}{m} \right] = \sqrt{2x} \frac{1}{2\sqrt{2x}} = \frac{1}{2}.
\]
It follows that \( \lim_{m \rightarrow 0^+} \psi(x, m) = x \) for any \( x \in [0, 1] \), as desired.

**Proof of Claim 26.** The proof consists of three parts: first, we show that \( H(x, m) \) is decreasing in \( x \), for \( \tilde{x}(m) < x < 1 \). Then, we show that this is also the case for \( 0 < x < \tilde{x}(m) \) when \( m \geq 6 \). Finally, we show that when \( m < 6 \), \( H(x, m) \) is strictly concave in \( x \), in the region \( 0 < x < \tilde{x}(m) \).

Fix \( m > 0 \). To simplify the notation, we omit the dependency of \( H \) and other functions on \( m \), unless necessary. That is, we denote \( H(x) = H(x, m) \), \( \psi(x) = \psi(x, m) \), \( \psi'(x) = \psi(x, m) \), and so on.

**Part 1.** \( H(x) \) is decreasing in \( x \) for all \( \tilde{x}(m) < x < 1 \).

Notice that we can write \( H(x) = H_1(x) + H_2(x) \), where
\[
H_1(x) = \frac{1 - \psi(x)}{\psi'(x)}, \quad H_2(x) = \frac{m\phi'(x)}{\psi'(x)} - x.
\]
We will show that both \( H_1 \) and \( H_2 \) are decreasing in \( x \), for \( x \in (\tilde{x}(m), 1) \). We start by computing the derivative of \( H_1 \),
\[
H'_1(x) = -\frac{(\psi'(x)^2 + \psi''(x)(1 - \psi(x)))}{\psi'(x)^2}.
\]
We want to show that \( H_1(x) < 0 \) for \( \tilde{x}(m) < x < 1 \), which is equivalent to
\[
\psi'(x)^2 + \psi''(x)(1 - \psi(x)) > 0. \tag{EC 1.31}
\]

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By plugging in the expressions for the derivatives of \( \psi \) (from equations (EC 1.4) and (EC 1.5) in the proof of Claim 10) and simplifying the resulting expression, we can write the LHS of the previous inequality as

\[
\frac{4m^2x + (1-x)^2 (1 - x^2 + 6x^3 + 2x^4) + 2m (1 + 2x^2 - 4x^3 + x^4)}{m^2(1 + x)^4}.
\]

Since the denominator is positive, we just need to show that the numerator is positive for \( \tilde{x}(m) < x < 1 \). In fact, it is easy to show that it is positive for \( x \in (0, 1) \). Indeed, note by inspection that the first two terms in the denominator are positive. Simple calculus shows that the third term is positive as well.\(^{62}\) It follows that (EC 1.31) holds for \( \tilde{x}(m) < x < 1 \).

We now show that \( H_2(x) \) is also decreasing for \( \tilde{x}(m) < x < 1 \). The derivative of \( H_2 \) is

\[
H_2'(x) = \frac{m (\phi''(x)\psi'(x) - \phi'(x)\psi''(x))}{(\psi'(x))^2} - 1.
\]

We want to show that \( H_2'(x) < 0 \) for \( x \in (\tilde{x}(m), 1) \); which is equivalent to

\[
\frac{1}{m} \psi'(x)^2 - (\phi''(x)\psi'(x) - \phi'(x)\psi''(x)) > 0.
\]

By plugging in the derivatives of \( \psi \) and \( \phi \) (from equations (EC 1.4), (EC 1.5), (EC 1.13) and (EC 1.14) in the proof of Claim 10) and simplifying the resulting expression, we can write the LHS of the previous inequality as \( f(x)/(4m^3(1 + x)^4) \) where

\[
f(x) = 4m^2 \left( 2 + x + 3x^2 \right) + 4m(1 - x) \left( x^2 + 8x - 1 \right) + x(1 - x)^2 \left( 9x^3 + 27x^2 + 15x - 19 \right).
\]

So it suffices to show that \( f(x) > 0 \) for \( x \in (\tilde{x}(m), 1) \). To do so, we first obtain a lower bound for \( f(x) \) by removing some of the positive terms in the previous expression, so that:

\[
f(x) > 8m^2 - 4m(1 - x) - 19x(1 - x)^2 \geq 8m^2 - 4m(1 - x) - 19(1 - x)^2.
\]

We now show that \( 8m^2 - 4m(1 - x) - 19(1 - x)^2 > 0 \) for all \( x \in (\tilde{x}(m), 1) \). By changing variables to \( y = 1 - x \), we show that \( 8m^2 - 4my - 19y^2 \geq 0 \) for \( y \in [0, 1 - \tilde{x}(m)] \). This quadratic expression has one positive root, namely \( y(m) = 2m (\sqrt{39} - 1) / 19 \); so we just need to show that \( 1 - \tilde{x}(m) \leq y(m) \), where, as defined in (A.3), \( \tilde{x}(m) = \frac{1}{3} \left( \sqrt{(m - 1)^2 + 3} - (m - 1) \right) \).

Equivalently, we show that \( 1 \leq \tilde{x}(m) + y(m) \), for all \( m \geq 0 \). First note that both sides of this inequality are equal at \( m = 0 \). Moreover, since \( \tilde{x}(m) \) is convex and \( y(m) \) is linear in \( m \), their sum is convex and we just need to check that the derivative of that sum is positive at \( m = 0 \). Indeed, we have that \( y'(0) + \tilde{x}'(0) = 2 \left( \sqrt{39} - 1 \right) / 19 - 1/2 > 0 \).

Therefore, \( f(x) > 0 \) for \( x \in (\tilde{x}(m), 1) \), which implies that \( H_2'(x) < 0 \) for \( x \in (\tilde{x}(m), 1) \).

\(^{62}\) To see this, let \( f(x) = 1 + 2x^2 - 4x^3 + x^4 \), and observe that \( f(0) = 1 \) and \( f(1) = 0 \). By differentiation we have \( f'(x) = 4x (1 - 3x + x^2) \), which has two roots in \((0, 1)\), namely \( x_1 = 0 \) and \( x_2 = (3 - \sqrt{5})/2 \). Then, \( f(x) \) is increasing in \((0, x_2)\) and decreasing in \((x_2, 1)\). Therefore, \( \min_{x \in [0,1]} f(x) = f(1) = 0 \), and \( f(x) \) is positive for \( x \in (0, 1) \).
Part 2. Fix \( m \geq 6 \), then \( H(x, m) \) is strictly decreasing in \( x \) for all \( 0 < x < \hat{x}(m) \).

We show that the derivative of \( H \) w.r.t. \( x \) is negative under this conditions. Note that the derivative of \( H \) w.r.t. \( x \) is

\[
H_x(x,m) = \frac{m (\psi_x(x,m)\phi_{xx}(x,m) - \psi_{xx}(x,m)\phi_x(x,m)) - \psi_{xx}(x,m) (1 - \psi(x,m)) - 2(\psi_x(x,m))^2}{(\psi_x(x,m))^2}.
\]

The denominator is always positive since \( \psi(x,m) \) is strictly increasing in \( x \), by Claim 10. Then, we want to show that the numerator is negative for all \( m \geq 6 \) and \( 0 < x < \hat{x}(m) \), i.e., that

\[
2(\psi_x(x,m))^2 - m (\psi_x(x,m)\phi_{xx}(x,m) - \psi_{xx}(x,m)\phi_x(x,m)) + \psi_{xx}(x,m) (1 - \psi(x,m)) > 0.
\]

By plugging in the expressions for the derivatives of \( \psi \) and \( \phi \) for the region \( 0 < x < \hat{x}(m) \) (from equations (EC 1.4), (EC 1.5), (EC 1.13) and (EC 1.14) in the proof of Claim 10) and simplifying the resulting expression, we can write the LHS of this inequality as \( x f(x,m)/m^2 (2x(2x + m))^{3/2} \), where

\[
f(x,m) = \left( m^4 - 6m^3 - 48m^2 x - 28m^3 x - 64m^2 x^2 - 216m^2 x^2 - 576mx^3 - 512x^4 \right) + 4\sqrt{2x}(2x + m) \left( 2m^3 + 64x^3 + 8mx(1 + 7x) + m^2(4 + 17x) \right).
\]

Our claim reduces to proving that \( f(x,m) > 0 \) for \( x \in (0, \hat{x}(m)) \) and \( m \geq 6 \). To do so, we will first show that \( f(x,m) \) is increasing in \( m \) for all \( x > 0 \), and then show that \( f(x,6) > 0 \) for all \( 0 < x < \hat{x}(m) \). With some algebra, we can write the partial derivative of \( f \) w.r.t. \( m \) as \( A(x,m)/\sqrt{2x(2x + m)} \), where \( A(x,m) = A_1(x,m) - A_2(x,m) + A_3(x,m) \), and

\[
\begin{align*}
A_1(x,m) &= 4x \left( 14m^3 + 32x^2(1 + 9x) + 8mx(7 + 38x) + m^2(20 + 109x) \right), \\
A_2(x,m) &= 4x \sqrt{2x(2x + m)} \left( 16x(1 + 9x) + 12m(2 + 9x) + 21m^2 \right), \\
A_3(x,m) &= 2m^2 \sqrt{2x(2x + m)} (2m - 9).
\end{align*}
\]

Note that \( A_3(x,m) \) is positive when \( m > 9/2 \), and in particular when \( m \geq 6 \). Therefore, we just need to show that \( A_1(x,m) - A_2(x,m) > 0 \). Since both \( A_1(x,m) \) and \( A_2(x,m) \) are non-negative, it suffices to show that \( (A_1(x,m))^2 - (A_2(x,m))^2 > 0 \). By plugging in the expressions for \( A_1 \) and \( A_2 \) and simplifying the resulting expression, we have that

\[
(A_1(x,m))^2 - (A_2(x,m))^2 = 16mx^2 A_4(x,m),
\]

where \( A_4(x,m) \) is a polynomial in \( x \) and \( m \) with strictly positive coefficients:

\[
A_4(x,m) = 196m^5 + 1024x^4(1 + 9x) + 70m^4(8 + 31x) + m^3(4 + 19x)(100 + 503x) + 64mx^2(9 + x(131 + 346x)) + 16m^2x(68 + x(595 + 1298x)) > 0.
\]

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It follows that \( f(x, m) \) is strictly increasing in \( m \) when \( m \geq 6 \), for all \( x > 0 \). It remains to show that show that \( f(x, 6) > 0 \) for all \( 0 < x < \tilde{x}(m) \). By plugging in \( m = 6 \) into (EC 1.32) and rearranging terms we have

\[
f(x, 6) = 32 \left[ \sqrt{x(3+x)(16x^3 + 84x^2 + 165x + 144)} - x(16x^3 + 108x^2 + 255x + 243) \right].
\]

From where it follows that

\[
\frac{f(x, 6)}{x} = 32 \left[ \frac{3}{x} \left( 16x^3 + 84x^2 + 165x + 144 \right) - (16x^3 + 108x^2 + 255x + 243) \right] \\
> 32 \left[ 2(16x^3 + 84x^2 + 165x + 144) - (16x^3 + 108x^2 + 255x + 243) \right] \\
= 32 \left[ 16x^3 + 60x^2 + 75x + 45 \right] > 0.
\]

Thus, \( f(x, m) > 0 \) for all \( x > 0 \) and \( m \geq 6 \). In particular, this holds for \( x \in (0, \tilde{x}(m)) \). Therefore, \( H_x(x, m) < 0 \) for all \( x \in (0, \tilde{x}(m)) \) and all \( m \geq 6 \), as desired.

**Part 3.** Fix \( m < 6 \), then \( H(x, m) \) is strictly concave in \( x \) when \( 0 < x < \tilde{x}(m) \).

Recall that \( H(x, m) = \frac{1 - \psi(x,m) + m \phi_x(x,m)}{\psi_x(x,m)} - x \). By plugging in the expressions of \( \psi, \psi_x \) and \( \phi_x \) given in the proof of Claim 10 for the case when \( 0 < x < \tilde{x}(m) \), we have that

\[
H(x, m) = \frac{2x(m + 2x)(m + 4x) - \sqrt{2x(2x + m)}(m^2 - 2m + 8mx + 8x^2)}{4\sqrt{2x(2x + m)(4x + m)} - 4x(8x + 3m)} \quad \text{(EC 1.33)}
\]

We want to show that this expression is concave in \( x \), i.e., thet \( H_{xx}(x, m) < 0 \) for \( 0 < x < \tilde{x}(m) \).

By taking the second derivative w.r.t. \( x \) of \( H \) and simplifying the resulting expression, we can write

\[
H_{xx}(x, m) = mxB_1(x, m)/B_2(x, m),
\]

where

\[
B_1(x, m) = B_3(x, m) - \sqrt{2x(2x + m)}B_4(x, m),
\]

\[
B_2(x, m) = \sqrt{2}(x(2x + m))^{3/2} \left( \sqrt{2x(2x + m)}(8x + 2m) - 2x(8x + 3m) \right)^3,
\]

\[
B_3(x, m) = 9m^4x(6 - m) + 1136m^3x^2 + 52m^4x^2 + 6144m^2x^3 + 840m^3x^3 \\
+ 12288mx^4 + 2304m^2x^4 + 8192x^5 + 1792mx^5,
\]

\[
B_4(x, m) = m^4(6 - m) + 216m^3x - 8m^4x + 1920m^2x^2 + 216m^3x^2 + 5120mx^3 \\
+ 928m^2x^3 + 4096x^4 + 896mx^4.
\]

We will show that \( B_1(x, m) < 0 \) and \( B_2(x, m) > 0 \) for all \( 0 < x < 1 \) and \( 0 < m \leq 6 \). In particular, this implies the result for \( 0 < x < \tilde{x}(m) \). First, notice that \( B_2(x, m) > 0 \) if and only if

\[
\sqrt{2x(2x + m)}(8x + 2m) > 2x(8x + 3m). \quad \text{(EC 1.34)}
\]

By taking squares on both sides and rearranging terms, this inequality reduces to

\[
4mx(2m^2 + \ldots)
\]
11mx + 16x^2 > 0, which holds for all x, m > 0. Thus, B_2(x, m) > 0.

To show that B_1(x, m) < 0, first notice (by inspection) that both B_3(x, m) and B_4(x, m) are positive when x > 0 and m ∈ (0, 6). Then B_1(x, m) is negative if and only if B_5(x, m) > 0, where

\[ B_5(x, m) = (2x(2x + m)) B_4(x, m)^2 - B_3(x, m)^2. \]  

(EC 1.35)

By plugging in the expressions for B_3 and B_4 above and simplifying the resulting expression we can write B_5(x, m) = m^4 x B_6(x, m), where

\begin{align*}
B_6(x, m) &= 2m^7 - 32768x^5(2x - 3) - 3m^6(8 + 15x) - 1024mx^4 (81x + 14x^2 - 68) \\
&+ 12m^5 (6 - 11x + 22x^2) - 96m^3x^2 (74x^2 - 103x - 282) \\
&+ 4m^4x (603 + 828x + 80x^2) - 64m^2x^3 (300x + 309x^2 - 1664).
\end{align*}

Notice that B_6(0, m) = 2m^5(m - 6)^2 > 0, so it suffices to show that B_6(x, m) is increasing in x for all fixed 0 < m < 6. By taking the partial derivative of B_6 w.r.t. x and simplifying the resulting expression we can write

\[ \frac{\partial B_6}{\partial x}(x, m) = -3 \left[ B_7(x, m) + 32x^2B_6(x, m) \right], \]

where

\begin{align*}
B_7(x, m) &= (m - 6)m^4(134 + 15m) - 16m^3x (1128 + 138m + 11m^2) \\
B_8(x, m) &= 1024x^2(4x - 5) + m^3(296x - 309) - 10m^4 \\
&+ 32mx (28x^2 + 135x - 224) + 2m^2 (515x^2 + 400x - 1664).
\end{align*}

By inspection, we note that both B_7(x, m) and B_8(x, m) are strictly negative for 0 < m < 6 and 0 ≤ x ≤ 1, therefore B_6(x, m) is increasing in x and it follows that B_5(x, m) > B_5(0, m) = 0. Thus, B_1(x, m) < 0, which implies that H_{xx}(x, m) < 0 for m < 6 and 0 < x < \tilde{x}(m).

Finally, Claim 26 follows from the previous three parts. From part 1, we have that H(x, m) is strictly decreasing in x for \tilde{x}(m) < x < 1, and from parts 1 and 2 we know that H(x, m) is either strictly concave of strictly decreasing, for 0 < x < \tilde{x}(m). So H(x, m) is either strictly decreasing for all x, or strictly concave in (0, \tilde{x}(m)) and strictly decreasing afterwards. Since H is continuous in x, strict quasiconcavity follows.

\[ \square \]

Proof of Claim 27. As in the proof of Claim 26, we treat the cases where 0 ≤ x < \tilde{x}(m), and \tilde{x}(m) < x ≤ 1 separately. We prove the result in four steps.

**Step 1.** W(x, m) is decreasing in x for x > \tilde{x}(m).

Fix m > 0. From equations (EC 1.26) and (EC 1.28), we can write

\[ W(x, m) = H(x, m) - Z(x, m) + \frac{1}{2}. \]  

(EC 1.36)
By Part 1 in the proof of Claim 26, we have that $H(x,m)$ is decreasing in $x$ for $x \in (\bar{x}(m),1)$. In addition, $Z(x,m) = \psi(x,m)/\psi_x(x,m)$ is increasing in $x$ since, by Claim 10, $\psi(x,m)$ is strictly increasing and strictly concave in $x$. Thus, $W(x,m)$ is decreasing in $x$ for $x \in (\bar{x}(m),1)$.

**Step 2.** $W(\bar{x}(m),m) < 0$ for all $m > 0$.

Let $0 < x \leq \bar{x}(m)$. By plugging in the expression for $H(x,m)$, $\psi(x,m)$ and $\psi_x(x,m)$ (see equations (EC 1.1), (EC 1.4), (EC 1.33)) and simplifying the resulting expression, we can write

$$W(x,m) = -\frac{N(x,m) \left(\sqrt{2x(2x + m)} - 2x\right)}{D(x,m)},$$

(see Equation (EC 1.37))

where $N,D : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ are defined as

$$N(x,m) = m^2 - 4m + 20x^2 + 12mx - 10x + \sqrt{2x(2x + m)}(2x - 1),$$

$$D(x,m) = 2 \left(8x + 2m\right)\sqrt{2x(2x + m)} - 2x\left(8x + 3m\right)$$

(see Equation (EC 1.38)).

From (EC 1.34) in the proof of Claim 26, we have that $D(x,m) > 0$ for $x,m > 0$, and in particular for $x = \bar{x}(m)$. Since $\sqrt{2x(2x + m)} > 2x$, it suffices to show that $N(\bar{x}(m),m) > 0$.

Recall that $\bar{x}(m) = \frac{1}{3} \left(\sqrt{(m - 1)^2 + 3} - (m - 1)\right)$, and that (abbreviating $\bar{x} = \bar{x}(m)$), $2\bar{x}(2\bar{x} + m) = (1 + \bar{x})^2$, from (EC 1.2). Plugging this into $N(x,m)$ results in

$$N(\bar{x},m) = m^2 - 4m + 20\bar{x}^2 + 12m\bar{x} - 10\bar{x} - (1 + \bar{x})(1 - 2\bar{x})$$

$$= \frac{1}{9} \left(17m^2 - 61m + 74 + (17 - 8m)\sqrt{(m - 1)^2 + 3}\right).$$

We claim that this expression is positive. To see this, first notice that $17m^2 - 61m + 74$ is positive for all $m$, so $N(\bar{x},m)$ is clearly positive when $0 \leq m \leq 17/8$. If $m > 17/8$, then we can bound $N(\bar{x},m)$ as follows:

$$9N(\bar{x},m) = 17m^2 - 61m + 74 + (17 - 8m)\sqrt{(m - 1)^2 + 3}$$

$$\geq 17m^2 - 61m + 74 + (17 - 8m)\left(m - 1 + \sqrt{3}\right)$$

$$> 17m^2 - 61m + 74 + (17 - 8m)(m + 1)$$

$$= 9m^2 - 52m + 91 > 0.$$ 

Therefore, $N(\bar{x}(m),m) > 0$, which implies that $W(\bar{x}(m),m) > 0$ for all $m > 0$.

**Step 3.** If $x \in (0,1)$ solves $W(x,m) = 0$, then $x < \bar{x}(m)$. In addition, $x$ solves $W(x,m) = 0$ if and only if it solves $N(x,m) = 0$.

By steps 1 and 2, it follows that $W(x,m) < 0$ for all $\bar{x}(m) \leq x \leq 1$. Thus, if $W(x,m) = 0$, it must be that $x < \bar{x}(m)$. Moreover, it follows from equation (EC 1.37) that the roots of $W(x,m)$ with $0 < x < \bar{x}(m)$ for fixed $m$ are the same as for $N(x,m)$.

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Step 4. Given $m > 0$ fixed, $N(x, m)$ is strictly convex in $x$. Therefore, given $m > 0$ fixed, $N(x, m)$ has at most two roots.

Fix $m > 0$. Notice that the term without square root in $N(x, m)$, $m^2 - 4m + 20x^2 + 12mx - 10x$, is strictly convex in $x$. In addition, the remaining term $\sqrt{2x(2x + m)}(2x - 1)$ is easily verified to be convex in $x$ for $x > 0$, from where strict convexity of $N(x, m)$ follows. This implies that there are at most two values of $x \in [0, \tilde{x}(m))$ such that $N(x, m) = 0$.

To conclude the proof, note that by steps 1 and 2, $W(x, m) < 0$ for all $\tilde{x}(m) \leq x < 1$, so if $W(x, m) = 0$, we have that $x \in (0, \tilde{x}(m))$. By steps 3 and 4, there are at most two values of $x \in (0, \tilde{x}(m))$ such that $W(x, m) = 0$. \qed

EC 2 Characterization of interior equilibria in $G(m)$

In this section we provide a characterization of the set of values of $m$ for which an interior PSNE exists in $G(m)$. The main result in this section is Proposition 8, which provides a system of inequalities that allows to efficiently compute the quantity $\bar{X}^*(m) = p^B\star(m) - p^A\star(m) + 1/2$, if $G(m)$ admits an interior equilibrium $(p^A\star(m), p^B\star(m))$. This result allows us to efficiently compute the equilibrium outcomes of our model numerically. Indeed, once we know the value of $\bar{X}^*(m)$, it is easy to compute firms’ equilibrium prices for product 1 from the first-order conditions for profit maximization, and the price firm B sets for new customers in the second period as $m\tilde{\theta}(\bar{X}^*(m), m)$. In addition, Proposition 8 allows us to prove Remark 1 (see Section A.3), which provides numerical approximates for the constatns $m_L$ and $m_H$, and shows that $G(m)$ may admit no pure-strategy Nash equilibria for some values of $m$.

The rest of this section is organized as follows. We first establish some preliminary notation that we use in subsequent proofs, then state and prove two intermediate results (Claims 28 and 29) that then allow us to prove Proposition 8. We conclude with the proof of Remark 1.

Preliminary definitions As we have discussed in Section A.3, firm B’s profit function is in general not quasiconcave in its own price, so solving for the first order conditions for profit maximization is not sufficient to characterize its best-response correspondence. However, Claim 16 shows that firm B’s profit maximization problem admits at most one interior local maximum. So, to verify that a price $p^B_1$ is a best response to some firm A’s price $p^A_1$, it suffices to verify the second-order condition for profit maximization, and to check that such price results in a higher profit than choosing the “corner” price $p^A_1 - 1/2$ (which results in firm B capturing the entire demand for product 1). We define two auxiliary functions for this purpose.

Let $U : [0, 1] \times \mathbb{R}^{++} \to \mathbb{R}$ be defined by

$$U(x, m) = -2\psi_x(x, m) - V(x, m)\psi_{xx}(x, m) + m\phi_{xx}(x, m), \quad (EC\ 2.1)$$

where $\psi$ and $V$ are defined in (A.7) and (A.31), respectively.
Recall from Claim 16 that if $p_1^B$ solves firm B’s first-order condition for profit maximization, given $p_1^A \in S^A$, we have that $p_1^B = V(p_1^B - p_1^A + 1/2, m)$. If that is the case, $U(p_1^B - p_1^A + 1/2, m)$ is equal to the second-derivative of firm 2’s profit function with respect to its own price evaluated at $p_1^B$, which must be negative if $p_1^B$ indeed maximizes firm B’s profit.

Now, we define a function that allows us to check if a price that solves firm B’s first-order condition for profit maximization yields a higher profits than the price that captures all the market, i.e., if $\pi^B(p_1^A, p_1^B, m) \geq \pi^B(p_1^A, p_1^A - 1/2, m)$. We want to define a function that is positive if and only if this condition holds (for a price $p_1^B$ that satisfies firm B’s first-order condition for profit maximization). Let us then define $T : [0, 1] \times \mathbb{R^+} \rightarrow \mathbb{R}$ by

$$T(x, m) = x - V(x,m)\psi(x,m) + m(\phi(x,m) - 1/2), \tag{EC 2.2}$$

where $\psi$, $\phi$ and $V$ are defined in (A.7), (A.9) and (A.31), respectively. To see why $T$ serves the purpose we have described, notice that if $p_1^B = V(p_1^B - p_1^A + 1/2, m)$, we have that (abbreviating $\bar{X} = \bar{X}(p_1^A, p_1^B)$):

$$\pi^B(p_1^A, p_1^B, m) - \pi^B(p_1^A, p_1^A - 1/2, m) = p_1^B(1 - \psi(\bar{X}, m)) + m\phi(\bar{X}, m) - m/2 + 1/2 - p_1^A$$

$$= p_1^B - p_1^A + 1/2 - p_1^B\psi(\bar{X}, m) + m(\phi(\bar{X}, m) - 1/2)$$

$$= \bar{X} - V(\bar{X},m)\psi(\bar{X}, m) + m(\phi(\bar{X}, m) - 1/2)$$

$$= T(\bar{X}, m).$$

The next claim provides a system of equations and inequalities that characterize interior PSNE in $G(m)$, based on the functions we have defined.

**Claim 28.** Let $p_1^A \geq 0$ and $p_1^B \geq -1/2$. Then, $(p_1^A, p_1^B)$ is an interior pure-strategy Nash Equilibrium for $G(m)$ if and only if $X^* = p_1^B - p_1^A + 1/2 \in (0, 1)$, and the following conditions hold

$$Z(X^*, m) = p_1^A, \tag{EC 2.3}$$
$$V(X^*, m) = p_1^B, \tag{EC 2.4}$$
$$T(X^*, m) \geq 0, \tag{EC 2.5}$$
$$U(X^*, m) < 0. \tag{EC 2.6}$$

**Proof of Claim 28.** If $(p_1^A, p_1^B)$ is an interior PSNE in $G(m)$, then $p_1^A \in S^A$ and $p_1^B \in S^B$ by definition of $G(m)$. In addition, both firms’ first order conditions for profit maximization must be satisfied (Equations (EC 2.3) and (EC 2.4)). Since $p_1^B$ is a best response to $p_1^A$ for firm B, conditions (EC 2.5) and (EC 2.6) must also hold by the preceding discussion. Finally, to see that $X^* \in (0, 1)$ holds, notice that, since the equilibrium is interior, $p_1^B > p_1^A - 1/2$. In addition, we have that $p_1^A > p_1^B - 1/2$ since otherwise firm B’s price is not a best response to $p_1^A$ by Claim 16.

Now suppose that $X^*, p_1^A, p_1^B$ solve the system above and that $X^* = p_1^B - p_1^A + 1/2 \in (0, 1)$. We will show that $p_1^A \in S^A = [0, 1]$ and $p_1^B \in S^B = [-1/2, 1]$, and that $p_1^A = BR^A(p_1^B, m)$ and
$p^B_1 \in BR^B(p^A_1, m)$. To prove this, note that by conditions (EC 2.3) and (EC 2.4), we have that

$$p^A_1 = \frac{\psi(X^*, m)}{\psi_x(X^*, m)}, \quad p^B_1 = \frac{1 - \psi(X^*, m) + m\phi_x(X^*, m)}{\psi_x(X^*, m)}.$$

By adding these two conditions, we have that

$$p^A_1 + p^B_1 = \frac{1 + m\phi_x(X^*, m)}{\psi_x(X^*, m)} < \frac{1}{\psi_x(X^*, m)} < 2,$$

where the last two inequalities follow since, by Claim 10, $\phi(x, m)$ is strictly decreasing in $x$, $\psi(x, m)$ is strictly concave in $x$ and $\psi_x(1, m) = 1/2$. Therefore, at least one of the inequalities $p^A_1 < 1$ or $p^B_1 < 1$ holds. We now cover these two cases.

First suppose that $p^B_1 < 1$. Since $\bar{X}^* = p^B_1 - p^A_1 + 1/2 > 0$, and $p^A_1 > 0$ (as $Z(\bar{X}^*, m) > 0$), we have that $p^B_1 > -1/2$. Since $p^A_1 = Z(p^B_1 - p^A_1 + 1/2, m)$ (by (EC 2.3)), it follows from the proof of Claim 12 that $p^A_1 = BR^A(p^B_1, m)$, and that $p^A_1 \in S^A$.

Now suppose that $p^A_1 < 1$. Then, by following the same argument as in the proof of Claim 16, and by conditions (EC 2.4) and (EC 2.6), it follows that $p^B_1$ is a local maximizer of the problem

$$\max \{ \pi^B(p^A_1, y, m) : y \in [p^A_1 - 1/2, p^A_1 + 1/2] \} = \max \{ \pi^B(p^A_1, y, m) : y \in S^B \},$$

where the equality follows from the proof of Claim 16 and by (EC 1.18).

But we know, by the proof of Claim 16, that this problem has at most one local maximum, which must be $p^B_1$. In fact, we claim that $p^B_1$ is a global maximizer of the problem. To verify this, note that since conditions (EC 2.4) and (EC 2.6) hold, $p^B_1$ is a global maximizer of firm B’s profit maximization problem (given $p^A_1$) if and only if it results in higher profits than choosing $\hat{p}^B_1 = p^A_1 - 1/2$, which holds by (EC 2.5). Therefore, $p^B_1 \in BR^B(p^A_1, m)$, and by Claim 16, $p^B_1 \in S^B$.

By the previous two cases, it follows that $p^A_1 \in S^A$ and $p^B_1 \in S^B$. In addition, $p^A_1 = BR^A(p^B_1, m)$ and $p^B_1 = BR^B(p^A_1, m)$. Thus $(p^A_1, p^B_1)$ is a PSNE of $G(m)$. \[ \square \]

Next, we show that the system above can be simplified.

**Claim 29.** Conditions (EC 2.4) and (EC 2.5) imply condition (EC 2.6).

**Proof of Claim 29.** Equation (EC 1.25) in the proof of Claim 16 implies that firm B’s first-order condition for profit maximization (i.e., condition (EC 2.4)) can be written as $H(p^B_1 - p^A_1 + 1/2, m) = p^A_1 - 1/2$. By Claim 26, we know that $H(x, m)$ is a strictly quasiconcave function of $x$ for fixed $m > 0$. Therefore, given $p^A_1 \in S^A$, there can be at most two values of $p^B_1 \in S^B$ that satisfy

$$H(p^B_1 - p^A_1 + 1/2, m) = p^A_1 - 1/2.$$

If there is only one such value, it must be a local maximizer of firm B’s profit maximization problem, since choosing $\hat{p}^B_1 = p^A_1 - 1/2$ is suboptimal by (EC 2.5). Thus, we must have that $\frac{\partial^2}{\partial (p^B_1)^2} \pi^B(p^A_1, p^B_1, m) < 0$, which implies that condition (EC 2.6) holds.
Now assume that there are two values of \( x \in [0, 1] \) that satisfy \( H(x, m) = p_1^A - 1/2 \), say \( x_1 < x_2 \). Notice that \( \pi^B(p_1^A, p_1^B, m) \) is decreasing at \( p_1^B \) if and only if \( H(p_1^B - p_1^A + 1/2, m) < p_1^A - 1/2 \). Since \( H(x, m) \) is strictly quasiconcave in \( x \), it follows that \( H(x, m) > p_1^A - 1/2 \) for \( x_1 < x < x_2 \), and \( H(x, m) < p_1^A - 1/2 \) otherwise. This implies that \( \pi^B(p_1^A, p_1^A - 1/2 + x, m) \) is increasing for \( x \in (x_1, x_2) \) and decreasing otherwise. In particular, we have that

\[
\pi^B(p_1^A, p_1^A - 1/2, m) > \pi^B(p_1^A, p_1^A - 1/2 + x_1, m),
\]

which implies that \( T(x_1, m) < 0 \). By (EC 2.5), it follows that \( x_2 = p_1^B - p_1^A + 1/2 \). In addition, \( x = x_2 \) is a local maximizer of \( \pi^B(p_1^A, p_1^A - 1/2 + x, m) \), which implies that the second-order condition for firm B’s profit maximization holds and therefore \( U(p_1^B - p_1^A + 1/2, m) < 0 \). \( \square \)

With the previous two claims, we can now characterize the values of \( m \) for which \( G(m) \) has an interior equilibrium.

**Proposition 8.** Fix \( m > 0 \) and consider the following system for \( x \in [0, 1] \)

\[
\begin{align*}
W(x, m) &= 0, \\
T(x, m) &\geq 0,
\end{align*}
\]

(EC 2.7)

where \( W \) and \( T \) are defined in (EC 1.28) and (EC 2.2), respectively. Then \( G(m) \) has an interior PSNE if and only if some \( x \in (0, 1) \) satisfies system (EC 2.7), and in that case, \( \bar{x}^*(m) = x \).

**Proof of Proposition 8.** First assume \( G(m) \) has an interior Nash Equilibrium \( (p_1^A, p_1^B) \) and take \( x = \bar{x}(p_1^A, p_1^B) = p_1^B - p_1^A + 1/2 > 0 \). By Claim 28, \( x \) satisfies equations (EC 2.3) and (EC 2.4), which implies that \( W(x, m) = 0 \), by (EC 1.28). In addition, \( x \) satisfies (EC 2.5), and therefore system (EC 2.7) is satisfied.

Now assume that some \( x \in (0, 1) \) satisfies system (EC 2.7). Define \( p_1^A = Z(x, m) \) and \( p_1^B = V(x, m) \). Since \( W(x, m) = 0 \) and \( W(x, m) = V(x, m) - Z(x, m) + 1/2 - x \), we have that \( x = p_1^B - p_1^A + 1/2 \). Since \( T(x, m) \geq 0 \), it follows from Claims 28 and 29 that \( (p_1^A, p_1^B) \) is a Nash Equilibrium for \( G(m) \). Since \( 0 < x < 1 \), the equilibrium is interior, and by Claims 15 and 18, it is the unique PSNE for \( G(m) \). Thus, \( \bar{x}^*(m) = x \). \( \square \)

Finally, we leverage the characterization that we have established to provide bounds for \( m_L \) and \( m_H \), given in Lemma 3, and in particular we show that \( G(m) \) admits no PSNE when \( m = 4 \).

**Proof of Remark 1.** To show that \( m_L < 4 \), we first notice by computation that \( \lim_{x \to 0^+} W(x, 4) = 0 \). By Claim 27, there is at most one value of \( x \in (0, 1) \) such that \( W(x, 4) = 0 \). We solve this equation numerically and obtain that this solution is \( x_{sol} \approx 0.0054 \). We compute that \( T(x_{sol}, 4) \approx -6.01 \times 10^{-6} < 0 \). Therefore, System (EC 2.7) has no solution for \( m = 4 \), which implies, by Proposition 8, that \( G(4) \) has no interior PSNE. By the definition of \( m_L \) given in (A.32), it follows that \( m_L < 4 \).
To show that that $G(4)$ has no corner PSNE, by Claim 15, it suffices to show that $-1/2 \notin BR^B(0,4)$, i.e., that there exists some price $p^B_1 > -1/2$ such that $\pi^B(0, -1/2, 4) < \pi^B(0, p^B_1, 4)$. Numerical computation shows that this holds for $p^B_1 = -0.48$, which implies that $-1/2 \notin BR^B(0,4)$. Therefore, $G(4)$ has no pure-strategy Nash equilibria and $m_H > 4$.

We now provide an approximation for $m_L$. By following a similar argument as in the proofs of Claims 22 and 27, we have that since $m_L < 4$, given a fixed $0 < m < 4$, there exists a unique $x = x(m) \in (0,1)$ that satisfies $W(x,m) = 0$. In particular, we have that $x(m) = \bar{X}(m)$ for $m < m_L$. Then, by Proposition 8 and the definition of $m_L$ given in (A.32), we have that

$$m_L = \sup_{0 \leq m \leq 4} \{ m : T(x(m'), m') \geq 0, \text{ for all } 0 < m' \leq m \}.$$ 

By following a similar argument as in the proof of Claim 23, one can show that $T(x(m), m) > 0$ for all $m \in (0, 3.98)$. By computation, we obtain that $T(x(3.995), 3.995) < 0$, so we have that $3.98 < m_L < 3.995$. We then use $m_L \approx 3.98$ as an approximation.

To approximate the value of $m_H$, one can show by computation that $-1/2 \notin BR^B(0, 4.01)$, and, similarly, that $BR^B(0, 4.03) = -1/2$. By Claim 13, we have that $4.01 < m_H \leq 4.03$, so we approximate $m_H \approx 4.02$.

EC 3 Proof of Proposition 1

To prove Proposition 1, we proceed by backwards induction to establish necessary conditions that any equilibrium $(\gamma, \sigma^A, \sigma^B, \mu)$ must satisfy in the restricted setting, and then show that these conditions are sufficient. In what follows, we present a series of claims that state these conditions and conclude with the proof of Proposition 1.

First, Claim 30 characterizes the beliefs in the second period and firm B’s pricing strategy for product 2 in any equilibrium. Then, Claim 31 provides the form of the consumer’s purchasing strategy for product 1. This is complemented with Claim 1, which is easily seen to hold in the restricted setting as well, and determines the consumer’s purchasing strategy for product 2. Then, Claim 32 provides the form of firms’ expected profit functions in terms of product 1 prices, and Claim 33 leverages these functions to show that both firms’ set a price of $1/2$ for product 1 in equilibrium. Finally, we argue that an equilibrium for the restricted setting exists for any value of $m > 0$ in Claim 34, and conclude with the proof of Proposition 1.

The following claim shows that the beliefs for firm B in the second period are equal to the prior distribution of consumer types, since the consumer’s action in the first period result in no information for firm B in the restricted setting. In addition, since the prior distribution of consumer types is the uniform distribution on the unit square, firm B sets a price of $m/2$ for product 2, regardless of the actions played in the first period.

Claim 30. In any equilibrium in the restricted setting, we have that $\mu_2(p^A_1, p^B_1) = \mu_0$ for any product 1 prices $p^A_1$ and $p^B_1$. In addition, firm B sets the price of product 2 as $\sigma^B_2(p^A_1, p^B_1) = m/2$. 

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Proof of Claim 30. Fix any product 1 prices \( p^A \) and \( p^B \). Then, the information available to firm B after any history of the form \( h = (s, \theta, p^A, p^B, a_1) \in H^f_2 \) is \( J(h) = (p^A, p^B) \). Since these prices are set independently of the consumer’s type, the only consistent belief system is such that \( \mu_2(p^A, p^B) \) is equal to the prior distribution of consumer types, \( \mu_0 \).

Now consider firm B’s pricing decision for product 2. First note that Claim 1 also holds in the restricted setting. Moreover, since \( \mu_2(p^A, p^B) = \mu_0 \), the firm’s profit maximization problem when setting the price of product 2 is

\[
\max_{p^B_2 \in \mathbb{R}} p^B_2 \mu_2 (\left[ 0, 1 \right] \times \left[ p^B_2 / m, 1 \right] | p^A, p^B) = \max_{p^B_2 \in \mathbb{R}} p^B_2 \min \left\{ (1 - p^B_2 / m)^+, 1 \right\}, \tag{EC 3.1}
\]

where the equality follows since \( \mu_2(p^A, p^B) = \mu_0 \). The unique solution to problem (EC 3.1) is \( p^B_2 = m/2 \). Thus, in any equilibrium, \( \sigma^B_2(p^A, p^B) = m/2 \). \hfill \Box

Next, we show that since the consumer’s actions have no implications for pricing in the second period, the consumer decides which firm to buy product 1 from by maximizing her utility associated to product 1.

Claim 31. In any equilibrium for the restricted setting, given a consumer history \( h \in H^c_1 \), \( \gamma_1 \) must take the following form:

\[
\gamma_1(h) = \begin{cases} 
1, & \text{if } s(h) > p^B_1(h) - p^A_1(h) + 1/2 \\
\beta(h) \in [0, 1], & \text{if } s(h) = p^B_1(h) - p^A_1(h) + 1/2 \\
0, & \text{if } s(h) < p^B_1(h) - p^A_1(h) + 1/2.
\end{cases} \tag{EC 3.2}
\]

Proof of Claim 31. Fix a history \( h = (s, \theta, p^A, p^B) \in H^c_1 \). First note that Claim 1 also holds in the restricted setting. Moreover, the price firm B sets for product 2, is independent of the consumer’s action in the first period (by definition of the restricted setting). Therefore, regardless of her action in the first period, the consumer’s utility associated to product 2 in any equilibrium is \((m\theta - \sigma^B_2(p^A, p^B))^+ \). Thus, given the history, the consumer’s total utility as a function of her action in the first period is

\[
u_1(a_1; h) + (m\theta - \sigma^B_2(p^A, p^B))^+.
\]

Therefore, in equilibrium, the consumer chooses \( a_1 \) to maximize

\[
u_1(a_1; h) = \bar{u} + a_1 \left( s - p^B_1 + p^A_1 - 1/2 \right).
\]

The consumer is only indifferent between either action if \( s = p^B_1 - p^A_1 + 1/2 \), therefore any sequentially rational strategy for the consumer takes the form of \( \gamma_1 \) given in (EC 3.2). \hfill \Box

The next claim provides expressions for the firms’ expected profit functions in terms of product 1 prices, assuming that the consumer and firm B play equilibrium strategies following firms’ choices for product 1 prices.
Claim 32. Fix an assessment \((\gamma, \sigma_A^1, \sigma_B^1, \mu)\) that satisfies the conditions of Claims 1, 30 and 31, and let \(p_1^A = \sigma_A^1(\emptyset)\) and \(p_1^B = \sigma_B^1(\emptyset)\). Then,

\[
\Pi_1^A (\gamma, \sigma_A^1, \sigma_B^1, \mu \mid \emptyset) = p_1^A X (p_1^A, p_1^B), \quad \text{and} \quad \Pi_1^B (\gamma, \sigma_A^1, \sigma_B^1, \mu \mid \emptyset) = p_1^B (1 - \bar{X} (p_1^A, p_1^B)) + m/4,
\]

where, as before, \(\bar{X} (p_1^A, p_1^B) = \max \{0, \min \{p_1^B - p_1^A + 1/2, 1\}\}\).

Proof of Claim 32. Fix product 1 prices \(p_1^A, p_1^B\). By Claim 31, we have that the expected demand to buy product 1 from firm A is\(^{63}\)

\[
\mu_0 (\{\gamma_1(s, \theta, p_1^A, p_1^B) = 1\}) = \mu_0 (\{s < p_1^B - p_1^A + 1/2\}) = \bar{X} (p_1^A, p_1^B),
\]

where the last equality follows since \(\mu_0\) is the uniform distribution on the unit square. Then, it follows that firm A’s expected profit in terms of product 1 prices is \(p_1^A \bar{X} (p_1^A, p_1^B)\).

Moreover, by the previous argument, the expected demand to buy product 1 from firm B is \(1 - \bar{X} (p_1^A, p_1^B)\), and therefore its expected profit associated to product 1 is \(p_1^B (1 - \bar{X} (p_1^A, p_1^B))\). In addition, by Claims 1 and 30, firm B sets the price of product 2 as \(m/2\) which the consumer buys with probability \(1/2\). Thus, firm B’s expected profit associated to product 2 is \(m/4\) regardless of the prices set for product 1, and its total expected profit is \(p_1^B (1 - \bar{X} (p_1^A, p_1^B)) + m/4\), as desired. \(\square\)

Now, based on the profit functions given in Claim 32, we show that both firms set a price of \(1/2\) for product 1 in any equilibrium.

Claim 33. In any equilibrium in the restricted setting, both firms set their product 1 price as \(1/2\), i.e., \(\sigma_A^1(\emptyset) = \sigma_B^1(\emptyset) = 1/2\).

Proof of Claim 33. By the previous three claims, firms expected profit functions in terms of product 1 prices are given by

\[
\pi_R^A (p_1^A, p_1^B) = p_1^A \bar{X} (p_1^A, p_1^B), \quad \pi_R^B (p_1^A, p_1^B) = p_1^B (1 - \bar{X} (p_1^A, p_1^B)) + m/4. \quad \text{(EC 3.3)}
\]

Thus, in any equilibrium the prices \(p_j^1 = \sigma_j^1 (\emptyset)\), for \(j = A, B\) must form a Nash equilibrium in the two-player normal form game with profit functions \(\pi_R^A, \pi_R^B\), and where we take firms’ action spaces as\(^ {64}\) \(S^A = [0, 1]\) and \(S^B = [-1/2, 1]\). Note that this game is equivalent to the classic Hotelling model with linear transportation costs with firms being located at the extremes of the unit interval,\(^ {65}\) and therefore the unique equilibrium is for both firms to set prices equal to the unit transportation cost, i.e., \(p_1^A = p_1^B = 1/2\). \(\square\)

The following claim constructs an equilibrium for the restricted setting that follows the same structure as established by the necessary conditions in Claims 30 - 33.

---

\(^{63}\)Observe that the event \(s = p_1^B - p_1^A + 1/2\) occurs with probability zero under \(\mu_0\).

\(^{64}\)As in the forward-looking setting, the argument still works even if we take firms’ action spaces to be \(\mathbb{R}\).

\(^{65}\)d’Aspremont et al. (1979); Osborne and Pitchik (1987).
Claim 34. For any $m > 0$, there exists an equilibrium in the restricted setting,

Proof of Claim 34. Following a similar argument as in Claim 9, we will construct an assessment that follows the same structure given in Claims 30 - 33 and show that it is indeed an equilibrium in the restricted setting.

Let $\sigma^A_1(\emptyset) = \sigma^B_1(\emptyset) = 1/2$, and for any product 1 prices $p^A_1$ and $p^B_1$, let $\sigma^B_2(p^A_1, p^B_1) = m/2$ and $\mu_2(p^A_1, p^B_1) = \mu_0$. In addition, define the beliefs in the first period as $\mu_1(\emptyset) = \mu_0$. Moreover, define $\gamma_2$ as in (A.10) with $q(h) = 1$ for any $h \in H^c_2$ such that $m\theta(h) = p^B_2(h)$, and define $\gamma_1$ as in (EC 3.2), with $\beta(h) = 1$ for any $h \in H^c_1$ such that $s(h) = p^B_1(h) - p^A_1(h) + 1/2$.

We claim that $(\gamma, \sigma^A_1, \sigma^B, \mu)$ is an equilibrium in the restricted setting. To see this, observe that by the same argument as in the proof of Claim 30, $\mu$ is consistent with $(\gamma, \sigma^A_1, \sigma^B)$. In addition, by the proof of Claim 30, $\sigma^B_2$ is sequentially rational at time $t = 2$ for firm B given the other players’ strategies and the belief system $\mu$. By Claims 32 and the proof of Claim 33, $\sigma^A_1$ and $\sigma^B_1$ are sequentially rational for firms at time $t = 1$. Finally, it follows from the same arguments as in the proofs of Claims 1 and 31 that $\gamma$ is sequentially rational for the consumer at times $t = 1, 2$.

Finally, given that we know that the restricted setting admits an equilibrium, and we have established necessary conditions for equilibrium, we can show that these are also sufficient and prove Proposition 1.

Proof of Proposition 1. By Claim 34, there exists an equilibrium $(\gamma, \sigma^A_1, \sigma^B, \mu)$ for the restricted setting. By Claim 33, we have that in any equilibrium, both firms set the price of product 1 as $1/2$. By Claim 31, in the equilibrium path, the consumer strictly prefers to buy product 1 from firm A if $s < 1/2$, and strictly prefers to buy from firm B if $s > 1/2$. Finally, by Claim 30, firm B sets a price of $p^B_2 = m/2$ for product 2, which, by Claim 1, the consumer strictly prefers to buy if $\theta > 1/2$.

EC 4 Proof of Proposition 2

To prove Proposition 2, we follow a similar argument as in Section EC 3. We consider the myopic setting and derive necessary conditions that any equilibrium $(\gamma, \sigma^A_1, \sigma^B, \mu)$ must satisfy, and then show that these conditions are sufficient. In what follows, we state a series of claims that derive these conditions and conclude with the proof of Proposition 2.

Briefly, our claims are as follows. Claim 35 establishes the form of the consumer’s strategy in any equilibrium. Then, Claim 36 pins down firm B’s beliefs in the second period, following a history where the consumer does not buy product 1 from firm B, assuming that this occurs with positive probability. Then, given these beliefs, Claim 37 provides firm B’s pricing strategy for product 2 following such histories. This is complemented with Claim 2, which also holds in the myopic setting. Then, Claim 38 provides the form of firms’ expected profit functions in terms of product 1 prices, assuming that an equilibrium is played following such prices, and Claim 39 derives the Nash equilibrium in a game with these functions to obtain the equilibrium product 1 prices. Then,
we show that an equilibrium for the myopic setting exists for any value of $m > 0$ in Claim 40, and finally conclude with the proof of Proposition 2.

First, we show that the consumer’s strategy is as in the restricted setting, in any equilibrium. This follows from noting that in the myopic setting, the consumer maximizes her current period, rather than her total payoff.

**Claim 35.** In any equilibrium for the myopic setting, $\gamma_1$ and $\gamma_2$ take the forms given in (EC 3.2) and (A.10), respectively.

*Proof of Claim 35.* By the same logic as in Claim 1, $\gamma_2$ take the form given in (A.10) in equilibrium. In addition, recall that we define equilibrium in the myopic setting with the consumer deciding which firm to buy product 1 from in order to maximize the utility associated to product 1 (instead of the total utility). Thus, by the proof of Claim 31, $\gamma_1$ has the form given in (EC 3.2). \qed

The next claim characterizes the beliefs for firm B in the second period, following a history where the consumer does not buy product 1 from firm B, assuming that this occurs with positive probability (which, as we will see, is equivalent to the condition $p_1^B - p_1^A + 1/2 > 0$).

**Claim 36.** Fix prices $p_1^A$, $p_1^B$ and suppose that $\gamma_1$ satisfies (EC 3.2). Let $I = (p_1^A, p_1^B, 0) \in \mathcal{I}^2_0$, and suppose that $p_1^B - p_1^A + 1/2 > 0$. Then, $\mu$ is consistent with $(\gamma, \sigma)$ at $I$ if and only if $\mu_2(\cdot \mid I)$ is the uniform probability measure on $[0, \bar{X}(p_1^A, p_1^B)] \times [0, 1]$.

*Proof of Claim 36.* Since $\gamma_1$ satisfies (EC 3.2), by following the same argument as in the proof of Claim 32, the expected demand to buy product 1 from firm A if prices $p_1^A$ and $p_1^B$ are set is $\bar{X}(p_1^A, p_1^B) = \max \{0, \min \{p_1^B - p_1^A + 1/2, 1\}\}$. In particular, this expected demand is positive if and only if $p_1^B - p_1^A + 1/2 > 0$.

In that case, the consumer buys product 1 from firm A if her type satisfies $s < \bar{X}(p_1^A, p_1^B)$, which occurs with positive probability.\footnote{We ignore the event in which the consumer is indifferent between buying from either firm, $\{s = \bar{X}(p_1^A, p_1^B)\}$, which occurs with probability zero under $\mu_0$.} Therefore, we can pin down the beliefs in the second period given information vector $I = (p_1^A, p_1^B, 0)$ by Bayes’ rule as follows. For any Borel set $B$ in the unit square we have

$$
\mu_2(B \mid I) = \mathbb{P}_{\mu_0} \left[ (s, \theta) \in B \mid I, \gamma, \sigma \right] = \frac{\mu_0 \left( B \cap \{s < \bar{X}(p_1^A, p_1^B)\} \right)}{\mu_0 \left( \{s < \bar{X}(p_1^A, p_1^B)\} \right)} = \frac{\int_B 1_{\{s < \bar{X}(p_1^A, p_1^B)\}}(s, \theta) \, d(s, \theta)}{\bar{X}(p_1^A, p_1^B)}.
$$

Thus, $\mu_2(\cdot \mid I)$ is the uniform probability measure on $[0, \bar{X}(p_1^A, p_1^B)] \times [0, 1]$, as desired. \qed

Next, taking the beliefs characterized in the previous claim, we derive firm B’s pricing strategy for histories where it does not perfectly observe the consumer’s type, when this occurs with positive probability.

**Claim 37.** Fix prices $p_1^A$, $p_1^B$ and suppose that $\gamma$ and $\sigma^B$ satisfy the conditions in Claims 2, 35, and 36. Let $I = (p_1^A, p_1^B, 0) \in \mathcal{I}^2_0$, and suppose that $p_1^B - p_1^A + 1/2 > 0$. Then, in any equilibrium for the myopic setting, we must have that $\sigma^B(I) = m/2$ with probability 1.

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Proof of Claim 37. Consider firm B’s pricing decision for product 2 given information vector $I = (p_A^1, p_B^1, 0)$. By Claim 36, in any equilibrium, the marginal distribution for $\theta$ induced by $\mu_2(B \mid I)$ is the uniform distribution in $[0, 1]$. It follows that we can write firm B’s profit maximization problem when choosing the price of product 2 as in (EC 3.1), which is maximized by choosing $p_B^2 = m/2$. Thus, in any equilibrium, we must have $\sigma_B^2(I) = m/2$ by sequential rationality for firm B.

The next claim provides expressions for the firms’ expected profit functions in terms of product 1 prices, assuming that agents play an equilibrium after these prices are set.

Claim 38. Fix an assessment $(\gamma, \sigma_1^A, \sigma_B^B, \mu)$ that satisfies the conditions of Claims 2, 35, 36, and 37. Let $p_A^1 = \sigma_1^A(\emptyset)$ and $p_B^1 = \sigma_B^B(\emptyset)$. Then, firms’ expected profit functions in terms of product 1 prices are, respectively,

$$
\Pi_A^1 (\gamma, \sigma_1^A, \sigma_B^B, \mu \mid \emptyset) = p_A^1 \bar{X}(p_A^1, p_B^1)
$$

$$
\Pi_B^1 (\gamma, \sigma_1^A, \sigma_B^B, \mu \mid \emptyset) = (p_B^1 + m/4)(1 - \bar{X}(p_A^1, p_B^1)) + m/4,
$$

where, as before, $\bar{X}(p_A^1, p_B^1) = \max \{0, \min \{p_B^1 - p_A^1 + 1/2, 1\}\}$.

Proof of Claim 38. By the same argument as in the proof of Claim 32, firm A’s expected profit in terms of product 1 prices is $p_A^1 \bar{X}(p_A^1, p_B^1)$. To obtain firm B’s profit function, note that since, by Claim 35, $\gamma_1$ satisfies (EC 3.2), we have two cases:

1. If this consumer’s type is such that $s < \bar{X}(p_A^1, p_B^1)$, she buys product 1 from firm A. By Claim 37, firm B then offers a price of $m/2$ for product 2, which the consumer buys with probability 1/2 (only if $\theta > 1/2$, by Claim 35). Thus, if $s < \bar{X}(p_A^1, p_B^1)$, firm B’s expected profit is $m/4$.

2. On the other hand, if $s > \bar{X}(p_A^1, p_B^1)$, the consumer buys product 1 from firm B, and, by Claim 2, buys product 2 at a price of $p_B^2 = m\theta$. Therefore, if $s > \bar{X}(p_A^1, p_B^1)$, firm B’s total expected profit is

$$
p_B^1 + \mathbb{E}_{\mu_0} [m\theta \mid s > \bar{X}(p_A^1, p_B^1)] = p_B^1 + m/2,
$$

where the equality follows since $\mu_0$ is the uniform distribution in the unit square.

By taking these two cases into account, it follows that firm B’s expected profit in terms of product 1 prices is

$$
\Pi_B^1 (\gamma, \sigma_1^A, \sigma_B^B, \mu \mid \emptyset) = \mu_0 \{s > \bar{X}(p_A^1, p_B^1)\}(p_B^1 + m/2) + \mu_0 \{s < \bar{X}(p_A^1, p_B^1)\} \cdot m/4
$$

$$
= (1 - \bar{X}(p_A^1, p_B^1))(p_B^1 + m/2) + \bar{X}(p_A^1, p_B^1) \cdot m/4
$$

$$
= (1 - \bar{X}(p_A^1, p_B^1))(p_B^1 + m/4) + m/4,
$$

as desired. 

\footnote{We do not consider the event \{$s = \bar{X}(p_A^1, p_B^1)$\} as it occurs with probability zero under $\mu_0$, and therefore has no impact in the calculations.}
Given these profit functions, the next claim derives the pure-strategy Nash equilibrium of the simulatenous-move game where firms set the price of product 1. As in the forward-looking setting, we restrict our attention to equilibria in undominated strategies (which implies that firm A chooses non-negative prices).

Claim 39. In any equilibrium in the myopic setting such that firms play according to undominated strategies, firms’ set their product 1 prices as

\[ \sigma_A^1(\emptyset) = p_{A,M}^1(m), \quad \sigma_B^1(\emptyset) = p_{B,M}^1(m), \]

where

\[ p_{A,M}^1(m) = \max \left\{ \frac{1}{2} - \frac{m}{12}, 0 \right\}, \quad p_{B,M}^1(m) = \max \left\{ \frac{1}{2} - \frac{m}{6}, -\frac{1}{2} \right\}. \]

Proof of Claim 39. Consider the two-player normal-form game with action spaces \( S_A, S_B \) (which we define below) and profit functions given by

\[ \pi_A^M(p_A^1, p_B^1) = p_A^1 \bar{X}(p_A^1, p_B^1), \quad \pi_B^M(p_A^1, p_B^1) = (p_B^1 + m/4) \left( 1 - \bar{X}(p_A^1, p_B^1) \right) + m/4, \]

respectively. Note that these are the expected profit functions given in Claim 38, so in any equilibrium for the myopic setting, firms’ prices for product 1 must form a Nash equilibrium in the game that we just described.

Note that in this game, setting any negative price is a strictly dominated strategy for firm A (as it is better off by setting a price of zero and receiving zero profits). Thus, to focus in undominated strategies, we define the game’s action spaces as \( S_A = [0, \infty) \) and \( S_B = \mathbb{R} \). Considering this choice for action spaces, it is easy to show that firms’ best-response correspondences are, respectively,

\[ BR_A^M(p_B^1) = \begin{cases} p_B^1/2 + 1/4 & \text{if } p_B^1 > -1/2, \\ [0, \infty) & \text{if } p_B^1 \leq -1/2. \end{cases} \quad \text{and} \quad BR_B^M(p_A^1) = \max \left\{ -1/2, (p_A^1 - m/4 + 1/2) / 2 \right\}. \]

One can verify that the only pure-strategy Nash equilibrium induced by these correspondences is \( (p_{A,M}^1(m), p_{B,M}^1(m)) \).

Finally, we show that the myopic setting admits an equilibrium. To prove this, we construct an equilibrium that follows the structure established in the previous claims.

Claim 40. For any \( m > 0 \), there exists an equilibrium in the myopic setting,

Proof of Claim 40. We follow a similar argument as in the proof of Claim 9, i.e., we will construct an assessment that follows the same structure given in Claims 35 - 39 and show that it is an equilibrium in the myopic setting.

Firms’ strategies. Let \( \sigma_A^1(\emptyset) = p_{A,M}^1(m) \) and \( \sigma_B^1(\emptyset) = p_{B,M}^1(m) \), where these prices are given in Claim 39. Moreover, define firm B’s pricing strategy for product 2, given \( I \in I_2 \) as

\[ \sigma_B^2(I) = \begin{cases} m \theta(I), & \text{if } I \in I_2^1, \\ m/2, & \text{if } I \in I_2^0. \end{cases} \]

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Consumer’s strategy. As in the proof of Claim 34, define $\gamma_2$ as in (A.10) with $q(h) = 1$ for any $h \in H^c$ such that $m\theta(h) = p^B_2(h)$, and define $\gamma_1$ as in (EC 3.2), with $\beta(h) = 1$ for any $h \in H^c$ such that $s(h) = p^B_1(h) - p^A_1(h) + 1/2$.

Belief system. Define the beliefs in the first period as $\mu_1(\emptyset) = \mu_0$. For the second period, given $I \in T_2$, define $\mu_2(\cdot \mid I)$ as follows:

1. If $I \in T_2^1$, let $\mu(\cdot \mid I)$ be the probability distribution that assigns probability 1 to the true consumer type $(s(I), \theta(I))$. That is, for any Borel set $B \subseteq T$, let $\mu(B \mid I) = 1_B(s(I), \theta(I))$.

2. If $I \in T_2^0$, let $\mu(\cdot \mid I)$ be the uniform probability distribution on $[0, \bar{X} (p^A_1, p^B_1)] \times [0, 1]$.

We claim that $(\gamma, \sigma^A_1, \sigma^B_2, \mu)$ is an equilibrium in the myopic setting. To see this, first observe that $\mu$ is consistent with $(\gamma, \sigma^A_1, \sigma^B_2)$ – indeed, if $I \in T_2^1$, $\mu$ is consistent by definition, and if $I \in T_2^0$, $\mu$ is consistent by the same argument.68 as in the proof of Claim 36.

In addition, by the proofs of Claims 2 and 37, $\sigma^B_2$ is sequentially rational at time $t = 2$ for firm B given the other players’ strategies and the belief system $\mu$. By Claims 38 and 39, $\sigma^A_1$ and $\sigma^B_2$ are sequentially rational for firms at time $t = 1$. Finally, it follows from the same arguments as in the proofs of Claims 1 and 31 that $\gamma$ is sequentially rational for the consumer at times $t = 1, 2$.

To conclude we complete this Section by proving Proposition 2, leveraging the series of claims we have established.

Proof of Proposition 2. By Claim 40, there exists an equilibrium $(\gamma, \sigma^A_1, \sigma^B_2, \mu)$ for the myopic setting. By Claim 39, we have that in any equilibrium, firms’ prices for product 1 are $p^{A,M}_1(m)$ and $p^{B,M}_1(m)$, respectively. Let us denote $\bar{X}^M(m) = p^{B,M}_1(m) - p^{A,M}_1(m) + 1/2$.

Then, by Claim 35, in the equilibrium path, the consumer buys product 1 from firm A if $s < \bar{X}^M(m)$, in which case, it follows from Claim 37 that firm B sets a price of $p^{B,N}_2 = m/2$ for product 2, which, by Claim 35, the consumer strictly prefers to buy if $\theta > 1/2$.

On the other hand, if the consumer’s type satisfies $s > \bar{X}^M(m)$, by Claim 35, in the equilibrium path, the consumer buys product 1 from firm B. Due to data tracking, firm B observes the consumer’s type and (by Claim 2) sets a price of $p^B_2 = m\theta$ for product 2, which the consumer buys with probability 1.

Finally, notice that the expected demand to buy product 1 from firm A is $\bar{X}^M(m)$, which can easily shown to be equal to $p^{A,M}_1(m)$ by straightforward computation.

EC 5 Proof of Proposition 4

In order to prove Proposition 4, we first establish the following claim.

Claim 41. Fix $0 < m < m_L$, and let $W$ and $\bar{X}$ be defined as in (EC 1.28) and (A.3), respectively. Suppose that $x \in (0, \bar{x}(m))$ is such that $W(x, m) < 0$, then $x > \bar{X}^*(m)$.

---

68 If $\bar{X} (p^A_1(I), p^B_1(I)) \equiv 0$, the beliefs cannot be obtained by Bayes’ rule since the consumer buys product 1 from firm B with probability 1 (see the proof of Claim 36). Thus, we can define beliefs arbitrarily in this case.
Proof of Claim 41. From Equation (EC 1.37) in the proof of Claim 27, we can write for $0 < x < \bar{x}(m)$,

$$W(x, m) = -\frac{N(x, m) \left(\sqrt{2x(2x + m)} - 2x\right)}{D(x, m)},$$

where

$$N(x, m) = m^2 - 4m + 20x^2 + 12mx - 10x - \sqrt{2x(2x + m)}(1 - 2x),$$

$$D(x, m) = 2 \left((8x + 2m)\sqrt{2x(2x + m)} - 2x(8x + 3m)\right).$$  \hspace{1cm} (EC 5.1)

We have shown that $D(x, m) > 0$ for $x, m > 0$ in (EC 1.34) in the proof of Claim 26. Moreover, we know by Claim 27 that $W(\bar{X}^*(m), m) = 0$ and $\bar{X}^*(m) < \bar{x}(m)$. In particular, it follows that $N(\bar{X}^*(m), m) = 0$. Then, it suffices to show that if $N(x, m) > 0$, then $x > \bar{X}^*(m)$. To do so, first notice that $N(0, m) = m^2 - 4m$, and since $m_L < 4$ (by Remark 1), we have that $N(0, m) < 0$ for all $m \in (0, m_L)$.

By Step 4 in the proof of Claim 27, $N(x, m)$ is strictly convex in $x$ for $m > 0$ fixed. Therefore, $x = \bar{X}^*(m)$ is the unique root of $N(x, m)$, which implies that $N(x, m) > 0$ for $x > \bar{X}^*(m)$.

Given this result, we now prove Proposition 4.

Proof of Proposition 4. We want to show that for $0 < m < m_L$, we have

$$0 < p^A_M(m) - p^B_M(m) < p^A_4(m) - p^B_4(m),$$

where $p^A_M(m)$ and $p^B_M(m)$ are firms’ equilibrium prices for product 1 in the myopic setting, and $p^A_4(m)$ and $p^B_4(m)$ are the corresponding equilibrium prices in the forward-looking setting. From Proposition 2, it is easy to see that $p^A_M(m) > p^B_M(m)$. To show that the second inequality holds, it is equivalent to prove that $\bar{X}^M(m) > \bar{X}^*(m)$, where, by Proposition 2,

$$\bar{X}^M(m) = p^B_M(m) - p^A_M(m) + \frac{1}{2} = \max\left\{\frac{1}{2} - \frac{m}{12}, 0\right\},$$

and $\bar{X}^*(m) = p^B_4(m) - p^A_4(m) + 1/2$.

First, recall that, by Claim 27, $\bar{X}^*(m)$ satisfies $W(\bar{X}^*(m), m) = 0$, where $W$ is defined in (EC 1.28), and that $\bar{X}^*(m) < \bar{x}(m)$. So, if $\bar{X}^M(m) \geq \bar{x}(m)$, the result follows.

Consider then the case where $\bar{X}^M(m) < \bar{x}(m)$. By Claim 41, it suffices to show that $W(\bar{X}^M(m), m) < 0$, which we do now.\(^{69}\) By the proof of Claim 41, we just need to show that $N(\bar{X}^M(m), m) > 0$. By plugging in $\bar{X}^M(m) = 1/2 - m/12$ and simplifying the resulting expression we have that

$$N(\bar{X}^M(m), m) = \frac{m}{36} \left(42 + 5m - \sqrt{(6-m)(6+5m)}\right).$$

Straightforward algebra shows that this expression is positive for $m \in (0,6]$. In particular, since

\(^{69}\)Notice that since $m_L < 4$ (by Remark 1), we have that $\bar{X}^M(m) > 0$ for $0 < m < m_L$.
$m_L < 4$ (by Remark 1), it follows that $N(\bar{X}^M(m), m) > 0$ for $m \in (0, m_L)$. Therefore, $\bar{X}^M(m) > \bar{X}^*(m)$ for any $m \in (0, m_L)$, as desired.

\section*{EC 6 Proof of Proposition 5}

In this section, we compare the equilibrium expected profit for firm B across our three settings, for the range of parameters in which the forward-looking setting admits an interior equilibrium (i.e., $0 < m < m_L$). First, we show that firm B’s equilibrium expected profit is lower in the restricted than in the forward-looking setting (Claim 42), and then that this expected profit is lower than in the myopic setting (Claim 43). Both propositions result in Proposition 5. In what follows, we refer to expected profit simply as “profit”, and denote firm B’s equilibrium expected profit on each setting by $\Pi^B_R(m)$, $\Pi^B_N(m)$, and $\Pi^B_F(m)$ respectively.

\textbf{Claim 42.} For all $m \in (0, m_L)$, we have that $\Pi^B_F(m) > \Pi^B_R(m)$.

\textit{Proof of Claim 42.} Firm B’s equilibrium profit in the restricted setting is $\Pi^B_R(m) = (m + 1)/4$, since it sells product 1 at a price of 1/2 with probability 1/2, and independently, sells product 2 for $m/2$, also with probability 1/2. In the forward-looking setting, for $0 < m < m_L$, we have that $\Pi^B_F(m) = \pi^B(\pi^A_1(m), \pi^B_1(m), m)$ by Claim 8, where $(\pi^A_1(m), \pi^B_1(m))$ are the unique equilibrium prices for $G(m)$. We now compare $\Pi^B_R(m)$ and $\Pi^B_F(m)$, considering two cases.

\textbf{Case 1.} \hspace{1em} $3 \leq m < m_L$. Since $\pi^B_1(m)$ is firm B’s best response to $\pi^A_1(m)$ in $G(m)$ and $\pi^A_1(m) > 0$ (since, by Theorem 1, $G(m)$ admits an interior equilibrium), we have that

$$\pi^B_F(m) = \pi^B(\pi^A_1(m), \pi^B_1(m), m) \geq \pi^B(\pi^A_1(m), \pi^A_1(m) - 1/2, m) = \pi^A_1(m) + \frac{m - 1}{2} > \frac{m - 1}{2} \geq \frac{m + 1}{4} = \pi^B_R(m),$$

where the last inequality follows since $m \geq 3$.

\textbf{Case 2.} \hspace{1em} $0 < m < 3$. Since $\pi^B_1(m)$ is firm B’s best response to $\pi^A_1(m)$ in $G(m)$, firm B is (weakly) better off when choosing $\pi^B_1(m)$ instead of $\pi^A_1(m) - m/6$, if firm A sets $\pi^A_1(m)$. Therefore,

$$\pi^B_F(m) = \pi^B(\pi^A_1(m), \pi^B_1(m), m) \geq \pi^B(\pi^A_1(m), \pi^A_1(m) - m/6, m) = (\pi^A_1(m) - m/6) (1 - \psi(1/2 - m/6, m)) + m\phi(1/2 - m/6, m).$$

In order to give a lower bound for the RHS of the last inequality, we will establish a bound for $\pi^A_1(m)$. We claim that for $0 < m < 3$, we have that $\pi^A_1(m) > Z(1/2 - m/6, m)$, where $Z$ is defined as in (A.29).

To prove this, recall that by Claim 12, $\pi^A_1(m)$ satisfies firm A’s first order condition for profit maximization, $\pi^A_1(m) = Z(\bar{X}, m)$. Since $Z$ is strictly increasing in its first argument,

\begin{footnote}
To see this, note that $Z(x, m) = \psi(x, m)/\psi_3(x, m)$. Since $\psi(x, m)$ is non-negative, strictly increasing and strictly concave in $x$ (by Claim 10), it follows that $Z(x, m)$ is strictly increasing in $x$.
\end{footnote}
suffices to show that $\bar{X}(m) > 1/2 - m/6$. The proof of Claim 41 implies that for $m \in (0, 3)$, $N(x, m) < 0$ if and only if $x < \bar{X}(m)$, where $N$ is defined as in (EC 5.1). Plugging in $x = 1/2 - m/6$ to $N(x, m)$ (see equation (EC 5.1)) results in

$$N(1/2 - m/6, m) = \frac{m}{9} \left( 3 - 4m - \sqrt{(3 - m)(2m + 3)} \right) < 0.$$  

Therefore, $p_A^*(m) > Z(1/2 - m/6, m)$ for $0 < m < 3$. Combining this with (EC 6.1) results in $\pi_B^{\hat{P}}(m) > b(m)$, where we define

$$b(m) = (Z(1/2 - m/6, m) - m/6)(1 - \psi(1/2 - m/6, m)) + m\phi(1/2 - m/6, m).$$

Therefore, it suffices to show that $b(m) > \pi_B^{\hat{P}}(m) = (m + 1)/4$ to complete the proof. To do so, define $\bar{b}(m) = b(m) - (m + 1)/4$. After plugging in the functional forms for $Z, \psi, \phi$ (see Equations (A.7), (EC 1.4), (EC 1.12)) into $b(m)$ and simplifying the resulting expression we have:

$$\bar{b}(m) = \frac{(m - 3)^2 \left( 324m + 153m^2 - 25m^3 - 216 + 2(m + 6)(6 - 11m)\sqrt{(3 - m)(3 + 2m)} \right)}{324m \left( (m - 3)(12 + 5m) + (12 + 2m)\sqrt{(3 - m)(3 + 2m)} \right)}$$

(EC 6.2)

We want to show that this expression is positive for $m \in (0, 3)$. First, we show that the denominator is positive. Note that we can write the term inside parentheses as

$$\sqrt{3 - m} \left[ (12 + 5m)\sqrt{(3 + 2m)} - (12 + 5m)\sqrt{3 - m} \right],$$

which is positive for $m \in (0, 3)$ since

$$(12 + 2m)^2(3 + 2m) - (12 + 5m)^2(3 - m) = 216m + 153m^2 + 33m^3 > 0.$$  

Now consider the numerator of (EC 6.2), and note that it suffices to show that for $0 < m < 3$,

$$h(m) \equiv \frac{324m + 153m^2 - 25m^3 - 216 + 2(m + 6)(6 - 11m)\sqrt{(3 - m)(3 + 2m)}}{P_1(m) P_2(m)} > 0.$$  

We have that $h(0) = 0$ and $h(3) = 1458$, so we just need to show that $h(m)$ has no roots in $(0, 3)$. If $h$ had such a root, it would satisfy the following equation:

$$\left( 324m + 153m^2 - 25m^3 - 216 \right)^2 = 4(m + 6)^2(6 - 11m)^2(3 - m)(3 + 2m),$$

which, after expanding terms, simplifies to

$$27m^3(59m^3 + 54m^2 + 351m - 216) = 0.$$

\textsuperscript{71}Note that $1/2 - m/6 < \bar{X}(m) < \tilde{x}(m)$, so we use the functional forms for $Z, \psi,$ and $\phi$ for the case when $x < \tilde{x}(m)$.  

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Since \(m > 0\), the equation is equivalent to \(\hat{h}(m) \equiv 59m^3 + 54m^2 + 351m - 216 = 0\). Computation shows that \(\hat{h}(1/2) < 0\) and \(\hat{h}(6/11) > 0\), so \(\hat{h}\) admits a root in \((1/2, 6/11)\). Denote this root by \(m_*\). Furthermore, since \(\hat{h}\) is strictly increasing for positive \(m\), \(m_*\) is the only positive root of \(\hat{h}\). However, we claim that \(h(m_*) > 0\). To prove this, we show that \(P_1(m_*), P_2(m_*) > 0\). First, we have that

\[
P_1(m) - \hat{h}(m) = 3m(4m - 3)(3 - 7m),
\]

so \(P_1(m) - \hat{h}(m) > 0\) for \(m \in (3/7, 3/4)\). In particular, \(m_* \in (3/7, 3/4)\), thus \(P_1(m_*) > 0\).

For \(P_2(m) = 2(m + 6)(6 - 11m)\), we have that \(P_2(m) > 0\) for all \(m \in (0, 6/11)\), and therefore \(P_2(m_*) > 0\). It follows that \(h(m_*) > 0\), so \(h\) has no roots in \((0, 3)\).

Therefore, the numerator of (EC 6.2) is positive for \(0 < m < 3\), and it follows that \(\bar{h}(m) > 0\), as desired.

\[\square\]

The next proposition compares firm B’s profits in the myopic and forward-looking settings, when the forward-looking setting admits an interior equilibrium \((0 < m < m_L)\).

**Claim 43.** For all \(m \in (0, m_L)\), we have that \(\Pi^B_M(m) > \Pi^B_{FL}(m)\).

**Proof of Claim 43.** We first derive the equilibrium profit for firm B in the myopic setting. Recall from (EC 4.1) that firm B’s expected profit function in terms of product 1 prices is

\[
\pi^B_M(p_1^A, p_1^B) = (1 - \bar{\bar{X}}(p_1^A, p_1^B)) \left(p_1^B/m + m/4\right) + m/4.
\]

By Proposition 2, the unique equilibrium product 1 prices in the myopic setting are \(p_{1,M}^A(m) = \max\{1/2 - m/12, 0\}\) and \(p_{1,M}^B(m) = \max\{1/2 - m/6, -1/2\}\). By plugging in these prices, we have that firm B’s equilibrium expected profit is

\[
\Pi^B_M(m) = \begin{cases} 
\frac{m}{4} + \left(\frac{1}{2} + \frac{m}{12}\right)^2, & \text{if } m < 6, \\
\frac{m - 1}{2}, & \text{if } m \geq 6.
\end{cases} \tag{EC 6.3}
\]

Now fix \(m \in (0, m_L)\). We will show that \(\Pi^B_M(m) > \Pi^B_{FL}(m)\).

By Remark 1, we know that \(m_L < 4\) and therefore \(\Pi^B_M(m) = \frac{1}{4} + \frac{m}{3} + \frac{m^2}{114}\), so it suffices to show that \(\frac{1}{4} + \frac{m}{3} > \pi^B_{FL}(m)\). Let us define

\[
F(m) = \frac{1}{4} + \frac{m}{3} - \Pi^B_{FL}(m) = \frac{1}{4} + \frac{m}{3} - p_{1,M}^B(m) \left(1 - \psi(\bar{\bar{X}}^*(m), m)\right) - m\phi(\bar{\bar{X}}^*(m), m).
\]

We want to show that \(F(m) > 0\) for all \(m \in (0, m_L)\). As in the proof of Claim 23, we will show that \(F(m)\) (i) is continuous for \(m \in (0, m_L)\), (ii) has no roots in \((0, m_L)\), and (iii) \(F(m) > 0\) for some \(m \in (0, m_L)\).

To prove (i), recall that by Claim 16, \(p_{1,M}^B(m) = V(\bar{\bar{X}}^*(m), m)\), where \(V\) is defined as in (A.31). Therefore, we can write \(F(m) = f(\bar{\bar{X}}^*(m), m)\) where

\[
f(x, m) = \frac{1}{4} + \frac{m}{3} - V(x, m) \left(1 - \psi(x, m)\right) - m\phi(x, m). \tag{EC 6.4}
\]

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Then, \( f(x,m) \) is continuous since both \( V(x,m) \) and \( \phi(x,m) \) are continuous functions\(^{72}\) of \((x,m) \in [0,1] \times \mathbb{R}^{++}\). By the same argument given in the proof of Claim 23, we know that \( \bar{X}^*(m) \) is continuous for \( m \in (0,m_L) \). Thus, \( F(m) = f(\bar{X}^*(m), m) \) is continuous in \((0,m_L)\).

To show (ii), we follow the same approach as in the proof of Claim 23. That is, we show that \( F(m) = f(\bar{X}^*(m), m) \) has no roots in \((0,m_L)\) by showing that the following system admits no solution with \( 0 < x < \min\{1/2, \tilde{x}(m)\} \) and \( 0 < m < m_L \).

\[
\begin{align*}
f(x,m) &= 0, \quad (\text{EC 6.5}) \\
N(x,m) &= 0,
\end{align*}
\]

where \( N(x,m) = m^2 - 4m + 20x^2 + 12mx - 10x - \sqrt{2x(2x + m)}(1 - 2x) \).

By plugging in the expressions for \( V(x,m) = (1 - \psi(x,m) + m\phi_x(x,m))/\psi_x(x,m) \) and \( \psi(x,m) \) for \( x < \tilde{x}(m) \) into (EC 6.4) (from equations (A.7), (EC 1.4), (EC 1.13)) and simplifying the resulting expression, we can write \( f(x,m) = K_1(x,m)/K_2(x,m) \), where

\[
\begin{align*}
K_1(x,m) &= xK_3(x,m) + \sqrt{2x(2x + m)}K_4(x,m), \\
K_2(x,m) &= 12m \left( \sqrt{2x(2x + m)}(4x + m) - x(8x + 3m) \right), \\
K_3(x,m) &= 128x^4 - 2m^3x + 8mx(16x^2 - 3) + m^2(32x^2 - 8x - 9), \\
K_4(x,m) &= m^3 - 64x^4 - m^2(12x^2 - 16x + 3) + 12mx(1 - 4x^2).
\end{align*}
\]

It is straightforward to show that \( K_2(x,m) > 0 \) for all \( x, m > 0 \). Therefore, system (EC 6.5) can be written equivalently as

\[
\begin{align*}
K_1(x,m) &= 0, \quad (\text{EC 6.6}) \\
N(x,m) &= 0.
\end{align*}
\]

We now show that this system has no solution with \( 0 < x < 1/2 \) and \( 0 < m < m_L \). As in the proof of Claim 23, we change variables to \( w = \sqrt{2x} \), \( z = \sqrt{2x + m} \), so that \( \sqrt{2x(2x + m)} = wz \).

By plugging the change of variables to the previous equations we can write

\[
\begin{align*}
K_1(w^2/2, z^2 - w^2) &= w(w - z)^2Q_0(w, z)/2, \\
N(w^2/2, z^2 - w^2) &= Q_2(w, z),
\end{align*}
\]

where,

\[
\begin{align*}
Q_0(w, z) &= 3w^3 - 4w^5 + w^7 - 12w^2z + 6w^4z - 21wz^2 + 24w^3z^2 - 4w^5z^2 - 6z^3 \\
&\quad + 16w^2z^3 - 8w^4z^3 + 4wz^4 - w^3z^4 + 2z^5, \\
Q_2(w, z) &= -w^2 - wz + w^3z - 4z^2 + 4w^2z^2 + z^4.
\end{align*}
\]

\(^{72}\)See the proof of Claim 10.
Showing that system (EC 6.6) has no solutions with $0 < x < 1/2$ and $0 < m < m_L$ is then equivalent to showing that the transformed system has no solutions with $0 < w < 1$ and $0 < z^2 - w^2 < m_L$. Moreover, since we look for solutions where $w > 0$ and $w > z$, it suffices to analyze the solutions of the following system:

$$Q_0(w, z) = 0,$$
$$Q_2(w, z) = 0.$$  

This system consists of two polynomial equations with integer coefficients in two variables, and it can be solved by computing the Gröbner basis of $Q_0, Q_2$, and solving the resulting triangular system (see, e.g., Cox et al. (2015), Sturmfels (2002), Sturmfels (2005)). We find that the system has seven real solutions, but none of them satisfy $0 < w < 1$ and $z > 0$. It follows that system (EC 6.6) has no solutions with $0 < x < 1/2$ and $0 < m < m_L$ and, thus, $F$ has no roots in $(0, m_L)$.

Finally, to show (iii), we evaluate $F$ at $m = 2$. We have that $\Pi_{FL}(2) \approx 0.834716$. Therefore, $F(2) \approx 1/4 + 2/3 - 0.834716 \approx 0.08195 > 0$. It follows that $F(m) > 0$ for $m \in (0, m_L)$.  

We conclude this Section with the proof of Proposition 5, which follows directly from the previous two claims.

**Proof of Proposition 5.** Let $m \in (0, m_L)$. By Claims 42 and 43, we have that $\Pi_B^M(m) > \Pi_B^{FL}(m) > \Pi_B^R(m)$.  

### EC 7  Proof of Proposition 6

In this Section we compare firm A’s equilibrium expected profit in our three settings, focusing on the values of $m$ for which the forward-looking setting admits an interior equilibrium $(0 < m < m_L)$. These comparisons derive in the proof of Proposition 6.

We proceed in two steps. First, Claim 44 establishes that firm A’s equilibrium profit is highest in the restricted setting than in any of the other two settings. Then, in Claim 45, we show that firm A’s profit is higher in the forward-looking than in the myopic setting for small values of $m$, but the opposite occurs for large values. These comparisons result in Proposition 6. At the end of the Section, we state and prove Claim 46, an auxiliary result that we use in the proof of Claim 44.

In what follows, we denote by $\Pi_{R}^{A}(m)$, $\Pi_{M}^{A}(m)$, and $\Pi_{FL}^{A}(m)$ the corresponding equilibrium expected profit for firm A in the three settings.

**Claim 44.** For any $m \in (0, m_L)$, firm A’s equilibrium expected profit is maximal in the restricted setting.

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73 We obtain the solutions by using the Solve routine in Mathematica. The real solutions to system are $(0, 0)$, $\pm(1, 1)$, $\pm(0.884, -1.14201)$, $\pm(1.27263, 0.626246)$.  

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Proof of Claim 44. First, we claim that the corresponding expressions for firm A’s equilibrium profit on each setting are as follows:

\[ \Pi^A_R(m) = \frac{1}{4}, \quad \Pi^A_M(m) = \left( \max \left\{ 0, \frac{1}{2} - \frac{m}{12} \right\} \right)^2, \quad \Pi^A_{FL}(m) = p_1^A(m)p^*(m). \quad (EC\ 7.1) \]

In the restricted setting, by Proposition 1, we know that in equilibrium, firm A sets a price of 1/2 for product 1 and the consumer buys with probability 1/2. Therefore, \( \Pi^A_R(m) = 1/4 \).

For the myopic setting, by Proposition 2, the corresponding equilibrium outcome is such that firm A sets a price of \( p_1^{A,M}(m) = \max \{1/2 - m/12, 0\} \) and the consumer buys from it with probability \( \bar{X}^M(m) = p_1^{A,M}(m) \). Thus, \( \Pi^A_M(m) = \left(p_1^{A,M}(m)\right)^2 = \left(\max \{1/2 - m/12, 0\}\right)^2 \).

Finally, for the forward-looking setting we have that, by Claim 8,

\[ \Pi^B_{FL}(m) = \pi^B \left(p_1^A(m), p_1^{B*}(m), m\right) = p_1^A(m)p^*(m), \]

where \( (p_1^A(m), p_1^{B*}(m)) \) are the unique equilibrium prices for \( G(m) \), and we denote \( p^*(m) = \psi \left( \bar{X}^*(m), m \right) \).

The comparison between the restricted and myopic settings is immediate as for any \( m > 0 \) we have \( \max \left\{ 0, \frac{1}{2} - \frac{m}{12} \right\} < 1/2 \), and by taking squares on both sides of this inequality we obtain that \( \Pi^A_M(m) < \Pi^A_R(m) \).

We now compare the restricted with the forward-looking setting. When \( m \to 0^+ \), we have that \( p_1^A(m), p^*(m) \to 1/2 \) as formally established in Claim 46. Therefore, \( \lim_{m \to 0^+} \Pi^A_{FL}(m) = \Pi^A_R(0) = 1/4 \). Moreover, note that \( \Pi^A_{FL}(m) \) is strictly decreasing for \( m \in (0, m_L) \) since both \( p_1^A(m) \) and \( p^*(m) \) are positive and strictly decreasing in \( m \) (by Claim 46). Thus, for \( 0 < m < m_L \), \( \Pi^A_{FL}(m) < 1/4 = \Pi^A_R(m) \).

Now that we have established that firm A’s is better off in the restricted setting than when data tracking is available, we compare firm A’s equilibrium profit in the forward-looking and myopic settings.

Claim 45. There exists a cutoff \( \bar{m} \in (0, m_L) \) such that \( \Pi^A_{FL}(m) > \Pi^A_M(m) \) when \( m \in (0, \bar{m}) \) and vice versa when \( m \in (\bar{m}, m_L) \).

Proof of Claim 45. We proceed in a similar fashion to the proofs of Claims 23 and 43, but rather than showing that the function defined the difference in firm A profit between the forward-looking and myopic scenarios has no roots in \((0, m_L)\), we will show that only one such root exists.

Let \( F(m) = \Pi^A_{FL}(m) - \Pi^A_M(m) \). Since \( 0 < m < m_L \), we know that \( (p_1^A(m), p_1^{B*}(m)) \) is an interior PSNE in \( G(m) \). By Claim 12, we have that \( p_1^A(m) = Z(\bar{X}^*(m), m) \), where \( Z \) is defined as in (A.29). Thus, since \( \Pi^A_{FL}(m) = p_1^A(m)p^*(m) \) and \( \Pi^A_M(m) = \left(1/2 - m/12\right)^2 \) (as \( m < m_L < 6 \), by Remark 1), we can write \( F(m) = \Omega(\bar{X}^*(m), m) \), where

\[ \Omega(x, m) = Z(x, m)p(x, m) - \left(\frac{1}{2} - \frac{m}{12}\right)^2. \]
Following a similar argument as in the proofs of Claims 23 and 43, we know that $F(m) = \Omega(\bar{X}^*(m), m)$ is continuous in $(0, m_L)$.

We will now show that $F$ has a unique root $\bar{m} \in (0, m_L)$, and then that $F$ changes sign at $\bar{m}$. Following the same argument as in proof of Claim 23, recall that $x = \bar{X}^*(m)$ is the unique solution in $(0, 1/2)$ to $N(x, m) = 0$ given fixed $m > 0$, where $N$ is defined in (EC 1.38). Therefore, $m \in (0, m_L)$ solves $F(m) = 0$ if and only if the following system has a solution with $0 < x < \min\{1/2, \bar{x}(m)\}$ and $0 < m < m_L$.

\[
\begin{align*}
\Omega(x, m) &= 0, \\
N(x, m) &= 0. 
\end{align*}
\] (EC 7.2)

By plugging in the functional forms for $Z$ and $\psi$ (from equations (A.7), (EC 1.4)) and simplifying the resulting expression, we have that for all $x < \bar{x}(m)$,

\[
\Omega(x, m) = \frac{4x^2(2x + m)^2}{2(2x + m)^2 + (4x + m)(2x + m)} - \frac{1}{144} (m - 6)^2. 
\]

We claim that System (EC 7.2) has a unique solution such that $x \in (0, 1/2)$ and $m \in (0, m_L)$. To do so, we change of variables by letting $w = \sqrt{2x}$, $z = \sqrt{2x + m}$, so that $\sqrt{2x(2x + m)} = wz$. Then, we look for solutions such that $0 < w < 1$, $z > 0$ and $0 < z^2 - w^2 < m_L$ to the following transformed system:

\[
\begin{align*}
\Omega(w^2/2, z^2 - w^2) &= 0, \\
N(w^2/2, z^2 - w^2) &= 0. 
\end{align*}
\] (EC 7.3)

By plugging in the change of variables, we have that $N(w^2/2, z^2 - w^2) = Q_2(w, z)$ and $\Omega(w^2/2, z^2 - w^2) = Q_3(w, z)$ where

\[
\begin{align*}
Q_2(w, z) &= -w^2 - wz + w^3 z - 4z^2 + 4w^2 z^2 + z^4, \\
Q_3(w, z) &= \frac{w^4 z^3}{w^3 + wz^2 + 2z^3} - \frac{1}{144} (z^2 - w^2 - 6)^2. 
\end{align*}
\]

Note that even though $Q_3$ is not a polynomial, the denominator of the first term is always positive for $0 < w < 1$ and $z > 0$. Therefore the roots of $Q_3(w, z)$ are the same of $\tilde{Q}_3(w, z)$ where

\[
\tilde{Q}_3(w, z) = 144 \left( w^3 + wz^2 + 2z^3 \right) Q_3(w, z) = 144w^4 z^3 - (w^3 + wz^2 + 2z^3) (z^2 - w^2 - 6)^2. 
\]

We then look for solutions to the following system of equations:

\[
\begin{align*}
Q_2(w, z) &= 0, \\
\tilde{Q}_3(w, z) &= 0. 
\end{align*}
\] (EC 7.3)

System (EC 7.3) is a system of two polynomial equations with integer coefficients in two vari-
ables. We can compute all the solutions to the system by computing the Gröbner basis of the system and solving the resulting triangular system (as in the proofs of Claims 23 and 43). We find that there exists a unique solution to (EC 7.3) such that\(^{74}\) \(0 < w < 1\) and \(z > 0\), which is approximately \((w_{sol}, z_{sol}) \approx (0.818686, 1.3689)\). By reversing the change of variables, we let \(\bar{m} = z_{sol}^2 - w_{sol}^2 \approx 1.20365 < m_L\). It follows that \(F\) has a unique root \(\bar{m} \in (0, m_L)\) which is close to 1.2.

Finally, it remains to show that \(F\) changes its sign for some \(m \in (0, m_L)\). To see this, we simply compute \(F(1/2) \approx 0.0092 > 0\) and \(F(3) \approx -0.0337 < 0\). This implies that \(F(m) > 0\) for \(0 < m < \bar{m}\) and \(F(m) < 0\) for \(\bar{m} < m < m_L\), as desired.

\(\blacksquare\)

Proposition 6 now follows directly from the previous two claims.

**Proof of Proposition 6.** Let \(m \in (0, m_L)\). By Claim 44, firm A’s expected equilibrium profit is higher in the restricted setting than in the two scenarios with data tracking. Among the two other settings, by taking \(\bar{m}\) as in Claim 44, we have that firm A’s profits are higher when consumers are forward-looking if \(m < \bar{m}\), while profits are higher with myopic consumers if \(m > \bar{m}\).

To end this Section, we state and prove the following result, which was used in the proof of Claim 44.

**Claim 46.** Consider \(0 < m < m_L\), let \(p_1^A(m)\) and \(p_1^B(m)\) the unique product 1 equilibrium prices in the forward-looking setting, and let \(\bar{X}(m) = p_1^B(m) - p_1^A(m) + 1/2\). In addition, let \(\psi(m) = \psi(\bar{X}(m), m)\) be the expected product 1 demand for firm A in equilibrium. Then, the following properties hold:

(i) \(\bar{X}(m), p_1^A(m), \) and \(\psi(m)\) are strictly decreasing in \(m\).

(ii) \(\lim_{m \to 0^+} \bar{X}(m) = \lim_{m \to 0^+} p_1^A(m) = \lim_{m \to 0^+} \psi(m) = 1/2\).

**Proof of Claim 46.** We first prove (i), which involves a series of algebraic arguments. We will prove that \(\bar{X}(m)\) is strictly decreasing in \(m\). The same can be shown for \(p_1^A(m)\), and \(\psi(m)\) using similar arguments, which we omit for brevity.

By the proof of Claim 18, it follows that \(W(\bar{X}(m), m) = 0\), where \(W\) is defined in (EC 1.28). In addition, by Claim 18, we know that \(\bar{X}(m) < \bar{x}(m)\), which implies that \(W(x, m)\) is differentiable at \((\bar{X}(m), m)\), so we can compute the derivative of \(\bar{X}(m)\) as follows by the implicit function theorem:

\[
\bar{X}'(m) = -\frac{W_m(\bar{X}(m), m)}{W_x(\bar{X}(m), m)}.
\]  

(EC 7.4)

We claim that \(\bar{X}'(m) < 0\). To prove this, we will show in two steps that \(W_x(\bar{X}(m), m) < 0\) and \(W_m(\bar{X}(m), m) < 0\).

**Step 1.** \(W_x(\bar{X}(m), m) < 0\).

\(^{74}\)There are 7 real solutions to system (EC 7.3): \((0, 0), \pm(1, 1), \pm(0.651494, -1.49704), \pm(0.818686, 1.3689)\).
Write \( W(x, m) = \varphi(x, m)/\psi_x(x, m) \), where \( \varphi \) is defined by

\[
\varphi(x, m) = 1 - 2\psi(x, m) + m\phi_x(x, m) - \left(x - \frac{1}{2}\right)\psi_x(x, m).
\]

By Claim 10, both \( \varphi(x, m) \) and \( \psi_x(x, m) \) are differentiable in \( x \) when \( x < \tilde{x}(m) \) and, by taking the partial derivative w.r.t. \( x \) we have

\[
W_x(x, m) = \frac{\varphi_x(x, m)\psi_x(x, m) - \varphi(x, m)\psi_{xx}(x, m)}{(\psi_x(x, m))^2}.
\]

In particular, \( W_x(\bar{X}^*(m), m) \) is well-defined since \( \bar{X}^*(m) < \tilde{x}(m) \), by Claim 27. In addition, since \( W(\bar{X}^*(m), m) = 0 \), it follows that \( \varphi(\bar{X}^*(m), m) = 0 \). Plugging back in the previous equation yields

\[
W_x(\bar{X}^*(m), m) = \frac{\varphi_x(\bar{X}^*(m), m)}{\psi_x(\bar{X}^*(m), m)}.
\]

The denominator is positive since \( \psi \) is strictly increasing in \( x \) (by Claim 10), and, by following the same argument as in the proof of Claim 18, we conclude that the numerator is negative. Thus, \( W_x(\bar{X}^*(m), m) < 0 \).

**Step 2.** \( W_m(\bar{X}^*(m), m) < 0 \).

We will show that \( W_m(x, m) < 0 \) for all \( m > 0 \) and \( 0 < x < \tilde{x}(m) \). Taking the derivative of \( W \) w.r.t. \( m \) results in

\[
W_m(x, m) = \frac{\varphi_m(x, m)\psi_x(x, m) - \varphi(x, m)\psi_{xm}(x, m)}{(\psi_x(x, m))^2},
\]

(\text{EC } 7.5)

By computing the derivatives involved in this expression (based on equations (A.7), (EC 1.4), (EC 1.12), (EC 1.13)) and simplifying the resulting expression, we can write

\[
W_m(x, m) = x \frac{C_1(x, m)}{C_2(x, m)},
\]

where

\[
C_1(x, m) = C_3(x, m) - C_4(x, m) + \frac{\sqrt{2x(2x + m)}}{(C_6(x, m) - C_5(x, m))},
\]

\[
C_2(x, m) = \sqrt{2x(2x + m)} \left( \frac{\sqrt{2x(2x + m)}(8x + 2m) - 2x(8x + 3m)^2}{(C_6(x, m) - C_5(x, m))} \right)^2,
\]

\[
C_3(x, m) = x\left(11m^3 + 100m^2x + 352mx^2 + 384x^3\right),
\]

\[
C_4(x, m) = 2x\left(3m^2 + 24mx + 32x^2\right),
\]

\[
C_5(x, m) = 2m^3 + 26m^2x + 128mx^2 + 192x^3,
\]

\[
C_6(x, m) = 16x(2x + m).
\]

From (EC 1.34) in the proof of Claim 26, we know that \( C_2(x, m) > 0 \), so it remains to show that
\( C_1(x, m) < 0 \). We show this by proving the following two inequalities:

\[
C_3(x, m) - \sqrt{2x(2x + m)}C_5(x, m) < 0
\]
\[-C_4(x, m) + \sqrt{2x(2x + m)}C_6(x, m) < 0
\]

By inspection, we note that \( C_i(x, m) \geq 0 \) for \( x > 0 \) and \( i = 3, 4, 5, 6 \). The first inequality is then equivalent to \( C_7(x, m) < 0 \) where

\[
C_7(x, m) = C_3(x, m)^2 - 2x(2x + m)C_5(x, m)^2.
\]

Indeed, by plugging in from (EC 7.6), we have that

\[
C_7(x, m) = -m^2x \left( 8m^5 + 103m^4x + 592m^3x^2 + 1856m^2x^3 + 3584mx^4 + 3072x^5 \right) < 0.
\]

Similarly, for the second inequality we show that \( C_8(x, m) < 0 \) where

\[
C_8(x, m) = 2x(2x + m)C_6(x, m)^2 - C_4(x, m)^2.
\]

By plugging in \( C_4 \) and \( C_6 \) from (EC 7.6), we have that \( C_8(x, m) = -4m^2x^2(9m + 16x) < 0 \). Adding the two inequalities that we just proved results in \( C_1(x, m) < 0 \), as desired. This implies that \( W_m(x, m) < 0 \) for all \( 0 < x < \tilde{x}(m) \). In particular, since \( \bar{X}^*(m) < \tilde{x}(m) \), we have that \( W_m(\bar{X}^*(m), m) < 0 \).

Finally, by (EC 7.4) and steps 1 and 2, it follows that \( \bar{X}^*(m) < 0 \), i.e., \( \bar{X}^*(m) \) is strictly decreasing in \( m \).

Now, we prove (ii). We start by showing that \( \lim_{m \to 0^+} \bar{X}^*(m) = 1/2 \).

From Step 3 in the proof of Claim 27, we have that \( x = \bar{X}^*(m) \) solves \( N(x, m) = 0 \) given fixed \( m \), where

\[
N(x, m) = m^2 - 4m + 20x^2 + 12mx - 10x - \sqrt{2x(2x + m)}(1 - 2x).
\]

Since \( N(x, 0) = 12x(2x - 1) \), we have by continuity that \( N(2/5, m) < 0 \) for all small \( m > 0 \). Moreover \( N(1/2, m) = m^2 + 2m > 0 \), for all \( m > 0 \). In addition, we have that \( N(0, m) = m^2 - 4m < 0 \) for all small \( m > 0 \).

Since \( N(x, m) \) is convex in \( x \), it follows that given small enough \( m > 0 \), there is a unique \( \tilde{x} \in (0, 1/2) \) such that \( N(\tilde{x}, m) = 0 \), and in addition, \( \tilde{x} \in (2/5, 1/2) \). Moreover, by Step 3 in the proof of Claim 27, we know that \( \tilde{x} = \bar{X}^*(m) \). Thus, for all small \( m > 0 \), it holds that \( 2/5 < \bar{X}^*(m) \leq 1/2 \), and \( N(\bar{X}^*(m), m) = 0 \). Let \( \bar{X}^*(0) = \lim_{m \to 0^+} \bar{X}^*(m) \). By continuity of \( N(x, m) \), we have that \( N(\bar{X}^*(0), 0) = 0 \), and \( 2/5 \leq \bar{X}^*(0) \). Therefore, \( \bar{X}^*(0) = 1/2 \).

Next, we show that \( \lim_{m \to 0^+} \psi^*(m) = 1/2 \). Note that

\[
\lim_{m \to 0^+} \psi^* \left( \bar{X}^*(m), m \right) = \lim_{m \to 0^+} 2\bar{X}^*(m)\bar{D}^* \left( \bar{X}^*(m), m \right) = \bar{X}^*(0) = 1/2,
\]
where the second to last equality follows since \( \lim_{m \to 0^+} \bar{\theta}(x,m) = 1/2 \) for all \( x \in (0,1) \), by the proof of Claim 25, and the fact that \( \bar{X}^*(0) = 1/2 \).

Finally, we show that \( \lim_{m \to 0^+} \tilde{p}_1^* (m) = 1/2 \). By Claim 12, we have that \( p_1^* (m) = Z(X^*(m),m) \) for \( 0 < m < m_L \), where \( Z(x,m) = \psi(x,m)/\psi_x(x,m) \). Then, since \( \psi^*(m) \to 1/2 \) as \( m \to 0^+ \), it suffices to show that \( \lim_{m \to 0^+} \psi_x(X^*(m),m) = 1 \). To prove this, we will first show that for all \( x \in (0,1) \), \( \lim_{m \to 0^+} \psi_x(x,m) = 1 \). Indeed, for fixed \( x \in (0,1) \) and small enough \( m > 0 \), it follows from (EC 1.4) and (EC 1.1) that

\[
\psi_x(x,m) = 1 - \left(1 - \bar{\theta}(x,m)\right) \left[2 - \frac{4x + m}{\sqrt{4x^2 + 2mx}}\right].
\]

By taking the limit as \( m \to 0^+ \), we have that \( \psi_x(x,m) \to 1 \). Moreover, by Claim 10, we know that \( \psi_x(x,m) \) is decreasing in \( x \) for fixed \( m > 0 \). Thus, for all small enough \( m > 0 \) we have that

\[
\psi_x(2/5,m) \geq \psi_x(X^*(m),m) \geq \psi_x(3/5,m).
\]

By taking the limit as \( m \to 0^+ \), we conclude that \( \lim_{m \to 0^+} \psi_x(X^*(m),m) = 1 \). It follows that \( \lim_{m \to 0^+} \tilde{p}_1^*(m) = 1/2 \), as desired.

**EC 8 The model with a monopoly in both markets**

In this section, we modify our original model to consider the case where firm B is a monopoly in both markets. We study this setting using a similar approach to that used in the model with competition: we characterize the equilibrium of the model in the forward-looking, restricted and myopic settings, and compare their corresponding equilibrium consumer surplus. This comparison results in Proposition 3, which states that, in contrast to the model with competition, consumer surplus is always lower in the myopic than in the restricted setting. Therefore, *the presence of competition in product market 1 is essential to argue that myopic consumers can be better off with data tracking than without it.*

This section is organized as follows: we modify the original model to the monopoly context, and characterize the equilibrium outcomes in our three settings in Section EC 8.1 (see Propositions 9, 10 and 11). Then, based on these characterizations, we compare the equilibrium levels of consumer surplus across of the two settings with data tracking with the restricted setting and contrast the implications of these comparisons with the ones in the original model in Section EC 8.2. These comparisons result in the proof of Proposition 3.

**EC 8.1 Characterization of equilibria in the monopoly model**

**EC 8.1.1 The monopoly model**

We consider the original model, described on section 2, but removing firm A from the first product market. That is, with firm B as a monopoly for both products. Throughout this section, we refer
to firm B simply as “the firm” or “the monopolist”. In this context, we reinterpret the consumer’s action in the first period $a_1 \in \{0, 1\}$ to denote whether it buys product 1 from the monopolist ($a_1 = 1$) or not ($a_1 = 0$). Thus, the consumer’s utility associated with product 1 given her action, her type, and the price for product 1 is equal to

$$u_{1\text{mon}}(a_1; s, \theta, p_1) = a_1 \left(\bar{u} - (1 - s)/2 - p_1\right). \quad \text{(EC 8.1)}$$

Notice that we could rewrite this utility function by defining the consumer’s valuation for product 1 as $v_1 = \bar{u} - (1 - s)/2$, and simply write the payoff as $a_1(v_1 - p_1)$. With this parametrization, the prior distribution of the consumer’s valuation for product 1 is the uniform distribution in $[\bar{u} - 1/2, \bar{u}]$. In what follows, we assume that the consumer’s valuation for product 1 is always positive, i.e., that

**Assumption 1.** $\bar{u} \geq 1/2$.

The consumer’s utility associated to product 2 remains as in the original model, i.e., we have

$$u_{2\text{mon}}(a_2; s, \theta, p_2) = a_2 (m\theta - p_2), \quad \text{(EC 8.2)}$$

where $a_2 = 1$ ($a_2 = 0$) denotes the decision to (not) buy product 2 from the monopolist. As in the original model, the consumer’s total utility is defined as the sum of the utilities for each product.

The monopolist profit function is as firm B’s profit function in the original model. That is, the profits associated to product $i = 1, 2$ is

$$\pi_{i\text{mon}}(p_i, a_i) = a_i p_i,$$

and the monopolist’s total profit is the sum of the profit associated to each product. The timeline, histories, strategies, expected payoffs and the equilibrium in this model are defined similarly to the original model, but removing firm A. In what follows, we characterize the equilibria in the forward-looking, restricted, and myopic settings of the monopoly model. Afterwards, we compare the resulting equilibrium consumer surplus across these scenarios, which allows us to obtain the comparisons established in Proposition 3.

**EC 8.1.2 Forward-looking setting**

We now characterize the equilibrium of the setting with data tracking and forward-looking consumers in the monopoly model. Proposition 9 establishes existence of an equilibrium in this setting and characterizes the resulting equilibrium path, which follows a similar structure to the corresponding one in the duopoly model, as given in Theorem 1.

**Proposition 9.** For any $m > 0$ and $\bar{u} \geq 1/2$, there exists an equilibrium in the forward-looking setting of the monopoly model. In addition, for a fixed $\bar{u} \geq 1/2$, there exists a constant $K(\bar{u})$ such that the equilibrium path can be characterized as follows:
(a) If $0 < m < K(\bar{u})$, there exists a unique price $p_{1}^{FL,mon} = p_{1}^{FL,mon}(m, \bar{u}) > \bar{u} - 1/2$ such that the monopolist sets $p_{1}^{FL,mon}$ as the price of product 1. Moreover, there exists a constant $\bar{\theta}^{mon} = \bar{\theta}^{mon}(m, \bar{u}) \in (1/2, 1)$, and a function $g_{mon}^{*} : [0, 1] \to \mathbb{R}$ defined by

$$g_{mon}^{*}(t) = 1 - 2 (\bar{u} - p_{1}^{FL,mon}) + 2m (t - \bar{\theta}^{mon})^+,$$

such that

(i) If the consumer’s type $(s, \theta)$ is such that $s > g_{mon}^{*}(\theta)$, then, with probability one, the consumer buys product 1; the firm perfectly observes the type of the consumer and, in the second period, sets a price equal to $m\theta$ for product 2, which the consumer also buys.

(ii) If the consumer’s type $(s, \theta)$ is such that $s < g_{mon}^{*}(\theta)$, then, with probability one, the consumer chooses not to buy product 1. In the second period, the firm sets a price equal to $p_{2} = m\bar{\theta}^{mon}$ for product 2, which the consumer buys if $\theta > \bar{\theta}^{mon}$.

(b) If $m > K(\bar{u})$, in any equilibrium, the firm’s price for product 1 is $p_{1}^{FL,mon} = \bar{u} - 1/2$. The consumer buys product 1 with probability one. In addition, the firm perfectly observes the consumer’s type and, in the second period, sets a price equal to her valuation for product 2, which the consumer buys.

The proof of this result follows a very similar structure to the proof of Theorem 1. We first characterize the equilibrium in the subgame that follows the monopolist’s choice for product 1’s price in Claim 47. Then, Claim 48 provides the form of the monopolist’s total expected profit function in terms of the price of product 1. Finally, Claim 49 establishes that the firm’s profit maximization problem in the first period has a unique solution (except for at most one value of $m$), which fully determines the equilibrium path.

We start by characterizing the equilibrium in the subgame that follows the firm’s choice of price for product 1. To do so, let us define some auxiliary notation. Let us define

$$\bar{X}^{mon}(p_{1}) = \min\{(1 - 2(\bar{u} - p_{1}))^+, 1\}. \quad \text{(EC 8.3)}$$

This quantity plays the role of $\bar{X}$ in the original model (see (A.1)). To see this, note that if $\bar{X}^{mon}(p_{1}) \in (0, 1)$, it holds that $p_{1} = \bar{u} - (1 - \bar{X}^{mon}(p_{1}))/2$. Thus, $\bar{X}^{mon}(p_{1})$ denote the location of the consumer that is indifferent between buying product 1 or not without taking into account the utility derived from product 2. In addition, for a fixed $p_{1}$ and $t \in [0, 1]$, we define

$$g_{mon}(t \mid p_{1}) = 1 - 2(\bar{u} - p_{1}) + 2m (t - \bar{\theta} (\bar{X}^{mon}(p_{1}), 2m))^+ \quad \text{(EC 8.4)}$$

where $\bar{\theta}$ is defined in (A.2). The function $g_{mon}$ plays the same role as that of the function $g$ (see (A.4)) in the original model. With these auxiliary functions, we can now characterize the equilibrium in the subgame that follows the firm’s choice of $p_{1}$ as follows.
Claim 47. In any equilibrium where the monopolist chooses \( p_1 \) as the price of product 1, we must have that:

(a) If \( p_1 \leq \bar{u} - 1/2 \), the consumer buys product 1 from the firm with probability one; the firm perfectly observes the type of the consumer and, in the second period, sets a price equal to the consumer’s valuation for product 2, that the consumer buys.

(b) If \( p_1 > \bar{u} - 1/2 \), we have that:

(i) If the consumer’s type \((s, \theta)\) is such that \( s > g_{\text{mon}}(\theta \mid p_1) \), then, the consumer buys product 1 from the firm with probability one; the firm perfectly observes the consumer’s type and, in the second period, sets a price equal to \( m\theta \) for product 2, which the consumer buys.

(ii) If the consumer’s type \((s, \theta)\) is such that \( s < g_{\text{mon}}(\theta \mid p_1) \), then, the consumer chooses not to buy product 1 with probability one. In the second period, the firm sets the price for product 2 equal to \( p_2 = m\bar{\theta} (\bar{X}_{\text{mon}}(p_1), 2m) \), which the consumer buys if \( \theta > \bar{\theta} (\bar{X}_{\text{mon}}(p_1), 2m) \),

where \( \bar{\theta}, \bar{X}_{\text{mon}} \) and \( g_{\text{mon}} \) are defined in (A.2), (EC 8.3) and (EC 8.4), respectively.

Proof of Claim 47. Follows the same argument as the proof of Lemma 1.

Claim 48. Suppose that the monopolists sets \( p_1 \) as the price of product 1, and the subsequent play follows the path given in Claim 47. Then, the monopolist’s total expected profit is

\[
\pi_{\text{mon}}^F(p_1, m, \bar{u}) = (1 - \psi(\bar{X}_{\text{mon}}(p_1), 2m)) p_1 + m\phi(\bar{X}_{\text{mon}}(p_1), 2m),
\]

where \( \psi, \phi \) and \( \bar{X}_{\text{mon}} \) are given in (A.7), (A.9), and (EC 8.3), respectively.

Proof of Claim 48. Follows the same argument as in the proof of Claim 8.

Now, we show that the monopolist’s profit maximization problem when choosing the price of product 1 always has a solution, which is unique except for at most one value of \( m \) (given fixed \( \bar{u} \)).

Claim 49. For fixed \( m > 0 \) and \( \bar{u} \geq 1/2 \), consider the firm’s profit maximization problem when setting the price of product 1:

\[
\max_{p_1 \in \mathbb{R}} \pi_{\text{mon}}^F(p_1, m, \bar{u}).
\]

Then, this problem has a solution. In addition, there exists a constant \( K(\bar{u}) \) such that if \( m \neq K(\bar{u}) \), the solution is unique.

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75As in Lemma 3, we do not explicitly characterize the equilibrium outcomes when the consumer is indifferent between buying either product. This event occurs with probability zero and has no impact in subsequent results.
Proof of Claim 49. First note that any price \( p_1 < \bar{u} - 1/2 \) is dominated by \( \bar{u} - 1/2 \) as \( X^{mon}(p_1) = 0 \) for any such price. In addition any price larger than \( \bar{u} \) yields the same profit as \( p_1 = \bar{u} \), from where it follows that
\[
\max_{p_1 \in \mathbb{R}} \pi_{mon}^{FL}(p_1, m, \bar{u}) = \max_{\bar{u} - 1/2 \leq p_1 \leq \bar{u}} \pi_{mon}^{FL}(p_1, m, \bar{u}).
\] (EC 8.7)

Since \([\bar{u} - 1/2, \bar{u}]\) is compact and \( \pi_{mon}^{FL}(p_1, m, \bar{u}) \) is continuous in \( p_1 \) (by the same argument as in Claim 11), the problem above always has a solution. It remains to show that this solution is unique for all values of \( m > 0 \), except perhaps for a single value.

Next note that \( p_1 = \bar{u} \) is suboptimal as it results in a profit of zero from product 1 and minimizes the profit associated to product 2. It follows that the solution of (EC 8.6) is either \( p_1 = \bar{u} - 1/2 \), or some \( p_1 \in (\bar{u} - 1/2, 1) \) such that
\[
\frac{\partial}{\partial p_1} \pi_{mon}^{FL}(p_1, m, \bar{u}) = 0.
\] (EC 8.8)

Let us define \( K(\bar{u}) = \inf \{ m \geq 0 : p_1 = \bar{u} - 1/2 \text{ solves problem (EC 8.6)} \} \). By the same reasoning as in the proof of Claim 13, it follows that if \( m > K(\bar{u}) \), \( p_1 = \bar{u} - 1/2 \) is the unique solution of (EC 8.6).

Now consider \( m < K(\bar{u}) \). By definition of \( K(\bar{u}) \), any solution to the optimization problem (EC 8.6) lies in \((\bar{u} - 1/2, 1)\) and satisfies (EC 8.8). As in the proof of Claim 16, by changing variables to \( x = 1 - 2(\bar{u} - p_1) \), we can write this condition as \( V(\bar{x}, 2m) - \bar{x} + (1 - 2u) = 0 \) where, \( V(x, m) \) is defined in (A.31). It follows from Claim 26 that for any fixed \( m > 0 \), \( V(x, 2m) - x \) is a strictly quasiconcave function of \( x \). Therefore, equation (EC 8.8) has at most two solutions. Furthermore, if it has two solutions, one of them most correspond to a local minimum since \( \pi_{mon}^{FL}(p_1, m, \bar{u}) \) is continuously differentiable in \( p_1 \) (by Claim 10). Therefore, at only one solution to (EC 8.8) corresponds to a local maximum, which must be a global maximum since the corner points are not solutions to problem (EC 8.6), by the assumption that \( m < K(\bar{u}) \). Therefore, problem (EC 8.6) admits a single solution when \( m < K(\bar{u}) \).

Finally, we provide the proof of the equilibrium characterization given in Proposition 9.

Proof of Proposition 9. Fix \( m > 0 \), \( \bar{u} \geq 1/2 \), and let \( K(\bar{u}) \) be as in Claim 49. Suppose that \( m > K(\bar{u}) \). Then, by the proof of Claim 49, the unique solution to (EC 8.6) is \( p_1 = \bar{u} - 1/2 \). By Claim 48 and by sequential rationality for the firm, this is the price set for product 1 in any equilibrium. The characterization of the equilibrium path following such price choice follows from Claim 47.

Suppose instead that \( m < K(\bar{u}) \). Let \( p_1^{FL,mon}(m, \bar{u}) \) be the solution to (EC 8.6), which we know is unique by Claim 49. By Claim 48 and by sequential rationality for the firm, this is the price set for product 1 in any equilibrium. Let \( g_{mon}^*(t) = g \left( t \mid p_1^{FL,mon}(m, \bar{u}) \right) \) and \( \theta_{mon}(m, \bar{u}) = \tilde{\theta} \left( X^{mon} \left( p_1^{FL,mon}(m, \bar{u}) ; 2m \right) \right) \). The form of the equilibrium path that follows the choice of the price of product 1 follows from Claim 47.
EC 8.1.3 Restricted setting

In this section we characterize the equilibrium in the restricted setting of the monopoly model. Recall that in this setting, the consumer’s purchase decision for product 1 does not reveal anything about her valuation for product 2. Therefore, as in the original model, both markets are entirely decoupled and we can obtain the equilibrium by looking at two separate games – one for each market. Proposition 10 summarizes the equilibrium outcome in this setting.

**Proposition 10.** For any \( m > 0 \) and \( \bar{u} \geq 1/2 \), there exists an equilibrium in the restricted setting of the monopoly model. In addition, the equilibrium path is essentially unique and takes the following form:

(i) The monopolist sets \( p_{1,\text{mon}} = \max\{\bar{u}/2, \bar{u} - 1/2\} \) as the price of product 1.

(ii) The consumer buys product 1 if her type satisfies \( s > \max\{1 - \bar{u}, 0\} \).

(iii) The monopolist sets a price of \( p_{2,\text{mon}} = m/2 \) for product 2, which the consumer buys if \( \theta > 1/2 \).

Moreover, in equilibrium, the expected consumer surplus is

\[
CS_{R,\text{mon}}(m, \bar{u}) = \begin{cases} 
1/4 + m/8, & \text{if } \bar{u} \geq 1, \\
\bar{u}^2/4 + m/8, & \text{if } \bar{u} < 1.
\end{cases}
\]

(EC 8.9)

**Proof of Proposition 10.** We proceed by backwards induction, following the same approach as in the restricted setting for the original model. As in the proof of Proposition 1, in any equilibrium, the monopolist’s beliefs over the consumer’s type when setting the price of product 2 are equal to the prior distribution \( \mu_0 \). Thus, in equilibrium, the monopolist sets the price of product 2 assuming that \( \theta \) is uniformly distributed in \([0, 1]\), which results in a price of \( p_{2,\text{mon}} = m/2 \) for product 2. As in Proposition 1, the consumer prefers to buy product 2 if and only if \( \theta \geq 1/2 \), with indifference if \( \theta = 1/2 \). Finally, note that this outcome results in an expected consumer surplus of \( m/8 \) associated to product 2.

For product 1, denote \( v_1(s) = u - (1 - s)/2 \), so that given a price \( p_1 \), the consumer strictly prefers to buy product 1 if and only if \( p_1 < v_1(s) \) (since the utility derived from product 2 is independent of the purchase decision in the first period). Then, the monopolist’s profit maximization problem is

\[
\max_{p_1} p_1 \mathbb{P}_{\mu_0} [v_1(s) \geq p_1] = \max_{p_1 \in [\bar{u} - 1/2, \bar{u}]} p_1 \max \{2(\bar{u} - p_1), 1\}.
\]

(EC 8.10)

It follows that the profit maximizing price for product 1 is \( p_{1,\text{mon}} = \max\{\bar{u}/2, \bar{u} - 1/2\} \), and the consumer is indifferent between buying product 1 or not if and only if her type satisfies \( s = \max\{1 - \bar{u}, 0\} \).

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\(^{76}\)As in the original model, the consumer’s strategy is not uniquely pinned down when she is indifferent between buying a product or not, but these events occur with probability zero.

\(^{77}\)Since \( s \) is uniformly distributed in \([0, 1]\), the event \( \{v_1(s) = p_1\} \) occurs with probability zero.
Finally, to obtain the expressions for the equilibrium consumer surplus given in (EC 8.9), we add the surplus associated to product 2 (which is \(m/8\)) and the corresponding one for product 1, which is:

\[
CS_{\text{mon}}^1 = \mathbb{E}_{\mu_0} \left[ \left( \bar{u} - (1-s)/2 - p_{1,\text{mon}}^R \right)^+ \right] = \begin{cases} 
1/4, & \text{if } \bar{u} \geq 1, \\
\bar{u}^2/4, & \text{if } \bar{u} < 1.
\end{cases}
\]

\[\]  

EC 8.1.4 Myopic setting

We now consider the setting with myopic consumers in the monopoly model. Recall that in this setting, the firm has data tracking ability but consumers act as in the restricted setting, i.e., they decide whether to buy product 1 by maximizing the utility associated to that product only. Proposition 11 summarizes the equilibrium outcome in this setting.

**Proposition 11.** For any \(m > 0\) and \(\bar{u} \geq 1/2\), there exists an equilibrium in the myopic setting of the monopoly model. In addition, the equilibrium path is essentially unique and takes the following form:

(i) The monopolist sets \(p_{1,\text{mon}}^M(m, \bar{u}) = \max\{\bar{u}/2 - m/8, \bar{u} - 1/2\}\) as the price of product 1.

(ii) The consumer buys product 1 if her type \((s, \theta)\) satisfies \(s > \max\{1 - \bar{u} - m/4, 0\}\). In that case, the firm perfectly observes the type of the consumer and, in the second period, sets a price equal to \(m\theta\) for product 2, which the consumer also buys.

(iii) The consumer does not buy product 1 if her type \((s, \theta)\) satisfies \(s < \max\{1 - \bar{u} - m/4, 0\}\). In that case, the firm sets a price of \(p_{2,\text{mon}}^M = m/2\) for product 2, which the consumer buys if \(\theta > 1/2\).

Moreover, in equilibrium, the expected consumer surplus is

\[
CS_{\text{mon}}^M(m, \bar{u}) = \begin{cases} 
1/4, & \text{if } \bar{u} \geq 1 - m/4, \\
\frac{1}{4} (\bar{u} + m/4)^2 + \frac{m}{32} (4(1 - \bar{u}) - m), & \text{if } \bar{u} < 1 - m/4.
\end{cases}
\]

Proof of Proposition 11. We proceed by backwards induction, following a similar approach to the proof of Proposition 2. As in the proof of Proposition 10, denote \(v_1(s) = u - (1-s)/2\), so that given a price \(p_1\), the (myopic) consumer strictly prefers to buy product 1 if and only if her type satisfies \(p_1 < v_1(s)\). In addition, note that the game is identical to the original model after the consumer has decided whether to buy product 1 or not. Therefore, by following the same argument as in the proof of Proposition 2, we have that the equilibrium path after the price of product 1 has been set is as follows:

1. If the consumer’s type \((s, \theta)\) satisfies \(p_1 < v_1(s)\), the consumer buys product 1 and the monopolist observes the value of \(\theta\). The monopolist then sets the price of product 2 to be
equal to the consumer’s valuation, i.e., $p_2 = m \theta$, and the consumer buys the product with probability 1. Therefore, if $p_1 < v_1(s)$, the monopolist’s total profit is $p_1 + m \theta$.

2. If $p_1 > v_1(s)$, the consumer does not buy product 1. Since the purchase decision for product 1 is independent of $\theta$, the monopolist’s beliefs on the consumer’s type are such that the marginal distribution of $\theta$ is the uniform distribution in $[0, 1]$. Given these beliefs, the monopolist sets $p_2 = m/2$ and receives a total expected profit of $m/4$.

Taking these two cases into account, we can write the monopolist’s expected profit function in terms of $p_1$ as follows:

$$\pi_{mon}^M(p_1, m, \bar{u}) = \Pr[p_1 < v_1(s)] (p_1 + \mathbb{E}[m \theta | p_1 < v_1(s)]) + \Pr[p_1 > v_1(s)] \cdot m/4$$

$$= \Pr[p_1 < v_1(s)] (p_1 + m/2) + \Pr[p_1 > v_1(s)] \cdot m/4$$

$$= \Pr[p_1 < v_1(s)] (p_1 + m/4) + m/4$$

$$= (1 - \bar{X}_{mon}^{\text{mon}}(p_1)) (p_1 + m/4) + m/4,$$

where $\bar{X}_{mon}^{\text{mon}} = \min\{(1 - 2(\bar{u} - p_1))^+, 1\}$. The monopolist’s profit maximization problem is given by $\max_{p_1 \in \mathbb{R}} \pi_{mon}^M(p_1, m, \bar{u})$. Simple calculus shows that there exists a unique optimal solution given by

$$p_{1,mon}^M(m, \bar{u}) = \max\{\bar{u}/2 - m/8, \bar{u} - 1/2\}.$$  \hspace{1cm} (EC 8.13)

To complete the characterization of the equilibrium path, note that the consumer is indifferent between buying product 1 or not if and only if her type satisfies $v_1(s) = p_{1,mon}^M(m, \bar{u})$. By plugging in (EC 8.13), we can rewrite this condition as $s = \bar{X}_{mon}^{\text{mon}}$, where

$$\bar{X}_{mon}^{\text{mon}} = \bar{X}_{mon}^{\text{mon}}(p_{1,mon}^M(m, \bar{u})) = \max\{1 - u - m/4, 0\}.$$  \hspace{1cm} (EC 8.14)

Finally, to compute consumer surplus, notice that if the the monopolist chooses $p_1 = \bar{u} - (1 - \bar{x})/2$ (so that the location of the indifferent consumer is $s = \bar{x}$), the expected consumer surplus associated to product 1 is $\frac{1}{4}(1 - \bar{x})^2$ as in the restricted setting. Moreover, since the consumer’s purchase decisions for products 1 and 2 are independent in the myopic setting, the consumer surplus associated to product 2 is $\bar{x} m/8$, since a fraction $\bar{x}$ of consumers get an expected surplus of $m/8$ (as in the restricted setting), and the rest get zero (as they pay their valuation for product 2). Therefore, the total consumer surplus is $(1 - \bar{x})^2/4 + \bar{x} m/8$. Plugging in $\bar{x} = \bar{X}_{mon}^{\text{mon}}$ results in (EC 8.11).

**EC 8.2 Consumer surplus comparisons across the three settings in the monopoly model**

In this section, we compare the expected consumer surplus in the three settings of the monopoly model, to obtain the proof of Proposition 3. To do so, we proceed as follows. Claim 50 establishes that in contrast with the original model, consumer surplus is larger in the restricted than in the
myopic setting in the monopoly context. Next, Claim 51 compares the firm’s expected profit as a function of the price of product 1 across the myopic and forward-looking settings. This result is an auxiliary step to later show, in Claim 52, that if the parameters of the model satisfy $m > 4(1 - \bar{u})$, consumer surplus is larger in the restricted than in the forward-looking setting. These claims then allow us to prove Proposition 3.

Finally, we complement this discussion by comparing consumer surplus in the restricted and forward-looking setting numerically when the parameters satisfy $m < 4(1 - \bar{u})$. We find that neither setting dominates in this region, i.e., there are parameter choices such that forward-looking consumers are on average better off if the monopolist having access to data tracking (see Figure 8).

**Claim 50.** In the monopoly model, consumer surplus is larger in the restricted than in the myopic setting for all $m > 0$ and $\bar{u} \geq 1/2$.

*Proof of Claim 50.* We consider three cases separately, depending on the parameters of the model.

**Case 1.** $\bar{u} \geq 1$. From equations (EC 8.9) and (EC 8.11) we have that $CS^{M,\text{mon}}(m, \bar{u}) = 1/4 < 1/4 + m/8 = CS^{R,\text{mon}}(m, \bar{u})$.

**Case 2.** $1 - m/4 \leq \bar{u} < 1$. In this case we have $m \geq 4(1 - \bar{u})$, therefore

$$CS^{R,\text{mon}}(m, \bar{u}) = \frac{\bar{u}^2}{4} + \frac{m}{8} \geq \frac{\bar{u}^2}{4} + \frac{1 - \bar{u}}{2} > \frac{1}{4} = CS^{M,\text{mon}}(m, \bar{u}),$$

where the second inequality follows since the expression $\bar{u}^2/4 + (1 - \bar{u})/2$ is strictly decreasing for $\bar{u} \in [1/2, 1]$.

**Case 3.** $\bar{u} < 1 - m/4$. From equations (EC 8.9) and (EC 8.11) we have

$$CS^{M,\text{mon}}(m, \bar{u}) = \frac{1}{4} \left( \bar{u} + \frac{m}{4} \right)^2 + \frac{m}{32} (4(1 - \bar{u}) - m) = \frac{\bar{u}^2}{4} + \frac{m}{8} - \frac{m^2}{64} = CS^{R,\text{mon}}(m, \bar{u}) - \frac{m^2}{64},$$

which implies that $CS^{M,\text{mon}}(m, \bar{u}) < CS^{R,\text{mon}}(m, \bar{u})$, as desired. \hfill \Box

Our next claim shows that, as one would expect, the monopolist’s expected profit in terms of the price of product 1 is larger when facing myopic than forward-looking consumers.\textsuperscript{78}

**Claim 51.** Fix $m > 0$ and $\bar{u} \geq 1/2$. Then, for any $p_1 \in [\bar{u} - 1/2, \bar{u}]$, we have that $\pi^{M,\text{mon}}(p_1, m, \bar{u}) \geq \pi^{FL,\text{mon}}(p_1, m, \bar{u})$, where $\pi^{FL,\text{mon}}$ and $\pi^{M,\text{mon}}$ are given in (EC 8.5) and (EC 8.12), respectively.

*Proof of Claim 51.* From equations (EC 8.5) and (EC 8.12), we have that

$$\pi^{M,\text{mon}}(p_1, m, \bar{u}) - \pi^{FL,\text{mon}}(p_1, m, \bar{u}) = (\psi(\bar{X}^{\text{mon}}(p_1), 2m) - \bar{X}^{\text{mon}}(p_1)) p_1$$

$$+ m \left( \frac{1}{2} - \frac{1}{4} \bar{X}^{\text{mon}}(p_1) - \phi(\bar{X}^{\text{mon}}(p_1), 2m) \right).$$

Since $\psi(x, 2m) \geq x$ for all $x \in [0, 1]$ (see Equation (A.19)) and $p_1 \geq \bar{u} - 1/2 \geq 0$, it follows that $(\psi(\bar{X}^{\text{mon}}(p_1), 2m) - \bar{X}^{\text{mon}}(p_1)) p_1 \geq 0$. To see that the second term is also positive let

\textsuperscript{78}Note that we require that $\bar{u} - 1/2 \leq p_1 \leq \bar{u}$. However, we know from (EC 8.7) and (EC 8.13) that any equilibrium price for product 1 lies in this region.
\[ f(x) = 1/2 - x/4. \] Notice that \( f(0) = 1/2 \) and \( f(1) = 1/4 \). We know from Claim 10 that \( \phi(x, 2m) \) is strictly convex in \( x \); and since \( f \) is linear it follows that \( f(x) > \phi(x, 2m) \) for all \( x \in (0, 1) \). In particular, we have that \( 1/2 - X_{mon}^m(p_1)/4 \geq \phi (X_{mon}^m(p_1), 2m) \). Thus, \( \pi_{mon}^M(p_1, m, \bar{u}) \geq \pi_{mon}^F(p_1, m, \bar{u}) \), as desired.

We now show that if the parameters of the model satisfy \( m \geq 4(1 - \bar{u}) \), then consumer surplus is larger in restricted than in the forward-looking setting. As we show in the proofs, in this region of the parameter space, the equilibrium price for product 1 in the forward-looking setting is \( p_1 = \bar{u} - 1/2 \), which induces the consumer to buy product 1 from the firm with probability 1. The resulting equilibrium path is identical to the corresponding one in the myopic setting, which, by Claim 50 leads to an inferior consumer surplus compared to the restricted setting.

Claim 52. Suppose that \( m \geq 4(1 - \bar{u}) \). Then, in equilibrium, consumer surplus is larger in the restricted than in the forward-looking setting.

Proof of Claim 52. Let \( K(\bar{u}) \) be as in Proposition 9. We claim that \( K(\bar{u}) < 4(1 - \bar{u}) \). To show this, note that for any \( p_1 \in (\bar{u} - 1/2, \bar{u}] \), we have from Proposition 11 and Claim 51 that

\[ \pi_{mon}^M(\bar{u} - 1/2, m, \bar{u}) > \pi_{mon}^M(p_1, m, \bar{u}) \geq \pi_{mon}^N(p_1, m, \bar{u}). \]

However, if the firm sets a price of \( p_1 = \bar{u} - 1/2 \) for product 1, the resulting expected profit is the same for the myopic and forward-looking settings, i.e., \( \pi_{mon}^F(\bar{u} - 1/2, m, \bar{u}) = \pi_{mon}^M(\bar{u} - 1/2, m, \bar{u}) = \bar{u} - 1/2 + m/2 \). Plugging into the previous inequality yields that for any \( p_1 \in (\bar{u} - 1/2, \bar{u}] \),

\[ \pi_{mon}^F(p_1, m, \bar{u}) \geq \pi_{mon}^F(\bar{u} - 1/2, m, \bar{u}). \]

Then, by (EC 8.7), we have that \( p_1 = \bar{u} - 1/2 \) is the unique equilibrium price for product 1 in the forward-looking setting (i.e., that \( m > K(\bar{u}) \)). By Proposition 9, the resulting equilibrium path is identical to the corresponding one for the myopic setting, which implies that the expected consumer surplus is 1/4 (by equation (EC 8.11)). By the proof of Claim 50, the restricted setting results in a higher consumer surplus.

We now provide the proof of Proposition 3.

Proof of Proposition 3. The comparison between the myopic and restricted setting follows from Claim 50, and the one involving the forward-looking setting follows from Claim 52.

Finally, we consider the case with with \( m < 4(1 - \bar{u}) \), which is a subset of the rectangle given by \( Z = \{(m, \bar{u}) : 0 \leq m \leq 2, 1/2 \leq \bar{u} \leq 1 \} \) (taking into account Assumption 1). Working with the consumer surplus expression analytically turns out to be complicated,\(^79\) so we compute

\(^79\)Similar computations to the corresponding ones to obtain (B.3) show that the expression for consumer surplus
the consumer surplus values numerically for a grid of values in this rectangle, and compare it with the expression for the restricted setting given in (EC 8.9). This comparison is illustrated on Figure 8, which splits the region \( Z \) in two subsets defined by whether the forward-looking or the restricted setting dominates in terms of consumer surplus. We find that consumers can be better off (on average) in the forward-looking than in the restricted setting when both \( m \) and \( \bar{u} \) are not large relative to the rectangle \([0, 2] \times [1/2, 1]\). Nonetheless, the subset of the parameter space where this occurs is smaller than the corresponding one for the original model, which is defined by \( m \in (0, m_L) \cup (m_H, 7) \), regardless of the value of \( \bar{u} \). Indeed, this is a larger region than the subset of \( 0 < m < 2 \) that for which this occurs in the monopoly model (see the region shaded in grey in Figure 8). In particular, there are parameter values for which forward-looking consumers benefit from data tracking in a competitive environment, but not when facing a monopoly.

Figure 8: Numerical computations show that, in the monopoly model, consumer surplus is higher in the forward-looking than in the restricted setting only when \((m, \bar{u})\) belongs to the region shaded in grey.

in the monopoly model is (abbreviating \( \bar{x}^* = \bar{X}_{mon}^{F_L,mon}(m, \bar{u}) \) and \( \bar{\theta}^* = \bar{\theta}_{mon}^{F_L,mon}(m, \bar{u}) \)):

\[
CS^{F_L,mon}(m, \bar{u}) = \begin{cases} 
\frac{1}{4} \bar{\theta} (1 - \bar{x}^*)^2 + \frac{1}{8} (1 - \bar{\theta}^*)^2 + \frac{1}{4} m (1 - \bar{x}^*)^3, & \text{if } \bar{x}^* \geq \bar{x}(2m), \\
\frac{1}{4} (1 - \bar{x}^*)^2 + \frac{1}{2} m \bar{x}^* (1 - \bar{\theta})^2 + \frac{1}{2} m^2 (1 - \bar{\theta})^3, & \text{if } \bar{x}^* < \bar{x}(2m). 
\end{cases}
\]

\(^{80}\)We compute the equilibrium and the resulting consumer surplus for parameters \((m, \bar{u})\) with \( m \in \{0, 0.001, 0.002, ..., 2\} \) and \( \bar{u} \in \{0.5, 0.501, 0.502, ..., 1\} \).