IDENTITIES OF WEYL AND MACDONALD - LIE THEORY SEMINAR AT UC RIVERSIDE

1. Integrals in algebras

We work with an algebra H over a ground field k. Suppose Γ is the set of k-algebra maps : $H \to k$. Given $\lambda \in \Gamma$, we define the λ-*weight space* of any (H, μ_H)-module to be $\{m \in M : h \cdot m = \lambda(h)m \ \forall h \in H\}$.

Definition 1. An element $\Lambda_L^\lambda \in H$ is called a *left* λ*-integral* for H if $h\Lambda_L^\lambda = \lambda(h)\Lambda_L^\lambda \ \forall h \in H$. We denote the set of left λ-integrals in H by H_L^λ, say; similarly, we have the sets H_R^λ and H_{LR}^λ of *right* and *two-sided* λ*-integrals* in H respectively.

Proposition 1. *We work with a k-algebra H and a left H-module M.*

1. *Each H_L^λ, H_R^λ is a two-sided ideal in H, and $H_{LR}^\lambda := H_L^\lambda \cap H_R^\lambda$ is contained in the center of H.*
2. *If $\lambda \neq \nu$ in Γ, then $H_L^\lambda H_L^\nu = H_R^\lambda H_L^\nu = H_R^\lambda H_R^\nu = H_L^\lambda \cap H_L^\nu = H_R^\lambda \cap H_R^\nu = 0 = \nu(H_L^\lambda) = \nu(H_R^\lambda)$.*
3. *If $m \in M$, then $\Lambda_L^\lambda m \in M^\lambda$.*
4. *If $h \cdot m = cm$ (for a scalar c) but $\lambda(h) \neq c$, then $H_R^\lambda \cdot m = 0$.*

Proof. The proofs are short and simple.

1. We show this only for the left-integrals; the proofs are similar for the right-integrals. So given Λ_L^λ, we choose $a, b \in H$, and show that $a\Lambda_L^\lambda b$ is also a left-integral. For this, take any $h \in H$, and compute:
$$h(a\Lambda_L^\lambda b) = h\lambda(a)\Lambda_L^\lambda b = \lambda(h) \cdot \lambda(a)\Lambda_L^\lambda b = \lambda(h) \cdot a\Lambda_L^\lambda b$$
The rest is easy: if Λ is both a left and right λ-integral, then for any h, we have $h \cdot \Lambda = \lambda(h)\Lambda = \Lambda\lambda(h) = \Lambda \cdot h$.

2. We first show that the first two products vanish; the third is similar. So, given $\Lambda_L^\lambda, \Lambda_L^\nu$, choose h so that $\lambda(h) \neq \nu(h)$. Then we compute:
$$\lambda(h)\Lambda_L^\lambda\Lambda_L^\nu = h \cdot \Lambda_L^\lambda\Lambda_L^\nu = h\nu(\Lambda_L^\lambda)\Lambda_L^\nu = \nu(h)\nu(\Lambda_L^\lambda)\Lambda_L^\nu = \nu(h)\Lambda_L^\lambda\Lambda_L^\nu$$
and since $\nu(h) \neq \lambda(h)$, hence the product vanishes.
Similarly, for the second case, we once again take h as above, and compute:
$$\lambda(h)\Lambda_R^\lambda\Lambda_L^\nu = \Lambda_R^\lambda h \cdot \Lambda_L^\nu = \Lambda_R^\lambda \cdot h\Lambda_L^\nu = \Lambda_R^\lambda\nu(h)\Lambda_L^\nu = \nu(h)\Lambda_R^\lambda\Lambda_L^\nu$$
whence we are again done as above. The third product vanishes just as the first one did.

Next, we consider the two intersections, and the right-integral case proceeds similar to the left. Given $\Lambda \in H_L^\lambda \cap H_L^\nu$, we choose $h \in H$ so that $\lambda(h) \neq \nu(h)$, and we have

$$\lambda(h)\Lambda = h\Lambda^\lambda = h\Lambda^\nu = \nu(h)\Lambda$$

(where we write Λ^λ to emphasize that $\Lambda \in H_L^\lambda$ etc.). Thus $(\lambda(h) - \nu(h))\Lambda = 0$, and we are done.

Finally, $0 = \Lambda_R^\nu \Lambda_L^\lambda = \nu(\Lambda_L^\lambda)\Lambda_R^\nu$, so that the coefficient on the right-side also vanishes. Similarly, $\nu(\Lambda_R^\lambda) = 0$ since $\Lambda_R^\lambda \Lambda_L^\nu = 0$.

(3) Given $h \in H$, we have $h \cdot \Lambda_L^\lambda m = h\Lambda_L^\lambda \cdot m = \lambda(h)\Lambda_L^\lambda m$.

(4) This is because given any $\Lambda_R^\lambda \in H_R^\lambda$, we have

$$\lambda(h)\Lambda_R^\lambda m = \Lambda_R^\lambda h \cdot m = \Lambda_R^\lambda \cdot hm = c\Lambda_R^\lambda m$$

and since $c \neq \lambda(h)$, we get that $\Lambda_R^\lambda m = 0$.

□

Our next result talks about unimodularity.

Definition 2. H is said to be λ-*unimodular* if $H_L^\lambda = H_R^\lambda$.

Proposition 2. *We keep the same setup as in the previous proposition.*

(1) *Given $\lambda \in \Gamma$, if left λ-integrals exist, the following are equivalent:*
   (a) *There is some left λ-integral Λ_L^λ so that $\lambda(\Lambda_L^\lambda) \neq 0$.*
   (b) *There is some scalar $\alpha \in k^\times$ and some nonzero left λ-integral Λ_L^λ, so that $\alpha\Lambda_L^\lambda$ is an idempotent.*
   (c) *There is some scalar $\alpha \in k^\times$ and some nonzero left λ-integral Λ_L^λ, so that $\alpha\Lambda_L^\lambda$ is a projection : $M \to M^\lambda$ for any H-module M.*

(2) *Moreover, if $\lambda(\Lambda_L^\lambda) \neq 0$, and right λ-integrals also exist, then Λ_L^λ is a two-sided λ-integral, and $H_L^\lambda = H_R^\lambda = H_{LR}^\lambda = k \cdot \Lambda_L^\lambda$.*

Proof.

(1) We prove a series of cyclic implications:

(a) ⇒ (b): Set $\alpha = (\lambda(\Lambda_L^\lambda))^{-1} \in k^\times$. Then

$$(\alpha\Lambda_L^\lambda)^2 = \alpha^2\Lambda_L^\lambda\Lambda_L^\lambda = \alpha^2\lambda(\Lambda_L^\lambda)\Lambda_L^\lambda = \alpha\Lambda_L^\lambda$$

(b) ⇒ (c): Choose α to be the α from (b) (which is given to us). Now, $\alpha\Lambda_L^\lambda$ is an idempotent linear operator on any H-module M, whose image lies in M^λ by part (3) of Proposition 1. Thus, it remains to show that $\alpha\Lambda_L^\lambda$ fixes each $m^\lambda \in M^\lambda$. We first determine what α is. We have

$$\alpha\Lambda_L^\lambda = (\alpha\Lambda_L^\lambda)^2 = \alpha^2\Lambda_L^\lambda\Lambda_L^\lambda = \alpha^2\lambda(\Lambda_L^\lambda)\Lambda_L^\lambda$$

and $\Lambda_L^\lambda \neq 0$, so $\alpha \cdot (\alpha\lambda(\Lambda_L^\lambda) - 1) = 0$, whence $\alpha = (\lambda(\Lambda_L^\lambda))^{-1}$.

We now complete the argument: $\alpha\Lambda_L^\lambda m^\lambda = \alpha\lambda(\Lambda_L^\lambda)m^\lambda = m^\lambda$, where the first equality is by definition of M^λ.

(c) $\Rightarrow$ (a): Let $M = H$ and $m^\lambda = \Lambda_L^\lambda \in H^\lambda$. Then by definition,

$$\Lambda_L^\lambda = (\alpha\Lambda_L^\lambda) \cdot \Lambda_L^\lambda = \alpha\lambda(\Lambda_L^\lambda)\Lambda_L^\lambda$$

and since both sides are nonzero, we are done.

(2) If this happens, then we first claim that every right λ-integral is a scalar multiple of Λ_L^λ. (In particular, Λ_L^λ is a two-sided λ-integral.) This is because given any (nonzero) right λ-integral Λ_R^λ, we have

$$0 \neq \lambda(\Lambda_L^\lambda)\Lambda_R^\lambda = \Lambda_R^\lambda\Lambda_L^\lambda = \lambda(\Lambda_R^\lambda)\Lambda_L^\lambda$$

whence $\lambda(\Lambda_R^\lambda) \neq 0$, and we can deduce the claim.

But now a similar computation shows that every left λ-integral ${\Lambda_L^\lambda}'$ is also a scalar multiple of $\Lambda_L^\lambda = \Lambda_R^\lambda$ (call it this to avoid confusion, and also to explain the following calculation):

$$0 \neq \lambda(\Lambda_R^\lambda){\Lambda_L^\lambda}' = \Lambda_R^\lambda{\Lambda_L^\lambda}' = \lambda({\Lambda_L^\lambda}')\Lambda_R^\lambda$$

The rest of the claim now follows easily.

□

Henceforth, we assume that $H = kG$, for a *finite* group G. We now determine all weight spaces in any G-module M.

Proposition 3. *Define the operator $\theta_\chi \in kG$ for any character $\chi : G \to k^\times$, by: $\theta_\chi := \sum_{g\in G} \chi(g^{-1})g$.*

1. *θ_χ is both a left and a right χ-integral, and central in kG.*
2. *$\theta_\chi\theta_\lambda = \delta_{\chi,\lambda}|G|\theta_\chi$.*
3. *The χ-weight space $(kG)_\chi$ in the G-module kG is (one-dimensional and) spanned by θ_χ.*
4. *Suppose char $k \nmid |G|$. Now define $P_\chi = \frac{1}{|G|}\theta_\chi$. Then the P_χ's are idempotents in kG that pairwise multiply to zero. Moreover, they are the projection operators : $M \mapsto M_\chi$ for any G-module M.*

Proof. Most of this follows from the previous propositions.

(1) That θ_χ is central now follows from the earlier part of the statement, which is checked by direct verification and reindexing of the summation. For instance, θ_χ is a right-integral because

$$\theta_\chi \cdot h = \sum_g \chi(g^{-1})gh = \chi(h)\sum_g \chi(h^{-1})\chi(g^{-1})gh = \chi(h)\theta_\chi$$

(2) From above, $\theta_\chi\theta_\lambda = 0$ if $\chi \neq \lambda$. If they do agree, then since θ_χ is a left integral, we have

$$\theta_\chi\theta_\chi = \chi(\theta_\chi)\theta_\chi = \sum_g \chi(g^{-1})\chi(g) \cdot \theta_\chi = \sum_g 1 \cdot \theta_\chi = |G|\theta_\chi$$

(3) We show the result for the left-weight space; the proof is similar for the right-weight space - and these two coincide. So, if $\sum_g a_g g \in (kG)_\chi$ is a left-integral, we pre-multiply by $h \in G$ to get

$$\sum_g a_g(hg) = h\sum_g a_g g = \chi(h)\sum_g a_g g = \sum_g (a_g\chi(h))g$$

But the leftmost term can be rewritten by reindexing the variable of summation, to be $\sum_g a_{h^{-1}g} g$. Now comparing the coefficients on both sides, we get that $a_g\chi(h) = a_{h^{-1}g}$ for all g, h. Setting $g = h$ gives us that $a_g\chi(g) = a_1$ for all g, and so the original left-integral was of the form $a_1 \sum_g \chi(g^{-1})g = a_1\theta_\chi$.

(4) This follows from the preceding proposition and the previous parts.

□

2. Preliminaries on root systems

Setup for the results of Weyl and Macdonald: Suppose $\mathfrak{g}$ is a complex semisimple Lie algebra, with Weyl group W, root system Φ, and simple roots Δ corresponding to a choice of Cartan subalgebra $\mathfrak{h}$. We always consider the root system to span a *real* vector space V, so that $W \subset O(V)$. (Thus, complex numbers come into the picture only when we introduce $\mathfrak{g}$.)

We first interpret some of the conclusions in the above results, about the *group ring*. Define an $\mathbb{R}$-algebra $\mathcal{E}_V$ (V being the real span of the roots, as above) with vector space basis $\{e(v) : v \in V\}$, and multiplication given by: $e(v)e(v') = e(v + v')$.

We also have a W-action on $\mathcal{E}_V$, given by: $w(e(v)) = e(w \cdot v)$. Then $\mathcal{E}_V$ is a W-module, and we can talk of weight spaces $(\mathcal{E}_V)_\chi$. There are two special characters of W, the *trivial* character $\mathrm{id} : W \to \{1\}$, and the *determinant* $: W \to \{\pm 1\}$, given by $\det w = (-1)^{l(w)}$, where l is the *length*. We denote $\theta = \theta_{\det}$.

Lemma 1.

(1) *The set of alternating elements in the group ring is the image of* $\theta = \sum_{w\in W} \det w \cdot w$.
(2) *If* ν *is in a reflection hyperplane, then* $\theta(e(\nu)) = 0$.
(3) $(\mathcal{E}_V)_{\mathrm{id}}(\mathcal{E}_V)_{\det} \subset (\mathcal{E}_V)_{\det}$.

Proof. The first parts follow from the previous proposition; the second part follows from the last part of Proposition 1, by setting h to be the reflection with respect to that hyperplane, $m = \nu,\ c = 1,\ \lambda = \det$.

For the last part (which has elsewhere been termed as the *property of weights*), say $e_\chi = \sum_v a_v e(v)$ and $e_\lambda = \sum_x a_x e(x)$ are weight-vectors in $\mathcal{E}_V$ (for some scalar coefficients a_v, a_x, and characters χ, λ). Then we compute for any $w \in W$:

$$\begin{aligned} w \cdot e_\chi e_\lambda &= w \cdot \sum_{v,x} a_v a_x e(v+x) = \sum_{v,x} a_v a_x e(w(v+x)) \\ &= \sum_{v,x} a_v a_x e(wv) e(wx) = \left(\sum_v a_v w(e(v)) \right) \cdot \left(\sum_x a_x w(e(x)) \right) \\ &= (w \cdot e_\chi) \cdot (w \cdot e_\lambda) = \chi(w) e_\chi \cdot \lambda(w) e_\lambda \end{aligned}$$

and hence we are done. □

In this section, we present some results that shall be needed later. The first is [H2, Proposition 1.4].

Proposition 4. *s_α takes α to $-\alpha$, and permutes all other positive roots $\Phi^+ \setminus \{\alpha\}$. Hence if $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$, then $s_\alpha(\rho) = \rho - \alpha$ and $\langle \rho, \alpha \rangle = 1$.*

Proof. Suppose $\beta \neq \alpha$ is a positive root; write $\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma$, whence some $c_\gamma > 0$ for $\gamma \in W$. Now, $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$, so the coefficient of γ is still positive, whence $s_\alpha(\beta)$ must still be a positive root. It cannot be α, else $\beta = s_\alpha(s_\alpha(\beta)) = s_\alpha(\alpha) = -\alpha$, a contradiction. Since s_α is invertible, and maps $\Phi \to \Phi$, hence it is a bijection on $\Phi^+ \setminus \{\alpha\}$.

Next, given ρ as above, we see that $\rho - \alpha = s_\alpha(\rho) = \rho - \langle \rho, \alpha \rangle \alpha$, whence $\langle \rho, \alpha \rangle = 1$. □

The final result in this section, is crucial to both the "named" identities below:

Theorem 1. *Given $\Psi \subset \Pi = \Phi^+$, the vector $a_\Psi := \rho - \sum_{\alpha \in \Psi} \alpha$ is either of the form $w\rho$, with $l(w) = |\Psi|$, or else contained in a reflection hyperplane H_α. (Here, $\rho = \frac{1}{2}\sum_{\alpha \in \Pi} \alpha$.)*

Proof. Suppose the vector a_Ψ is not contained in any reflection hyperplane. Then it must lie in some chamber, whence (e.g. by [H2, Lemma 1.12]) there is a w such that wa_Ψ is in the fundamental domain.

We first claim that wa_Ψ is also of the form a_Ω for some $\Omega \subset \Pi$. This is because $a_\Psi = \frac{1}{2}(\sum_{\alpha \in \Pi \setminus \Psi} \alpha - \sum_{\alpha \in \Psi} \alpha)$, hence of the form $\sum_{\alpha > 0} \pm\alpha$. But since w permutes $\{\pm\alpha : \alpha > 0\}$, hence it permutes all expressions of the form $\sum_{\alpha > 0} \pm\alpha$ (one choice of sign for each positive root). In particular, wa_Ψ is also of the form a_Ω.

But we now claim that the only a_Ω to lie in the fundamental domain, is $a_\emptyset = \rho$, whence $a_\Psi = w^{-1}\rho$ as claimed. Moreover, (from the equation $wa_\Psi = \rho$) we would also get Ψ to be precisely the set of positive roots that w sends to $-\Pi$. Hence $|\Psi| = l(w) = l(w^{-1})$ as desired.

It thus remains to show the claim made in the previous paragraph. Define $\gamma = \rho - a_\Omega$ to be a nonnegative sum of positive roots. From Proposition 4 above, $\langle \rho, \alpha \rangle = 1$. Moreover, $\langle a_\Omega, \alpha \rangle \in \mathbb{N}$ for all $\alpha \in \Delta$ (by assumption, and since W is a Weyl group), hence $\langle \gamma, \alpha \rangle \leq 0$ for all $\alpha \in \Delta$.

But now if γ can be written as $\sum_{\alpha \in \Delta} n_\alpha \alpha$, then from above, $n_\alpha \geq 0$ for all α. Hence we get that $0 \leq (\gamma, \gamma) = \sum n_\alpha (\gamma, \alpha) \leq 0$, whence equality holds everywhere. In particular, $\gamma = 0$ and $\Omega = \emptyset$ as claimed. □

3. W-Invariance of characters of simple $\mathfrak{g}$-modules

The last preliminary result that we need, is the W-invariance of the character of simple $\mathfrak{g}$-modules:

Theorem 2. *Suppose $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra. If V is a finite-dimensional simple module of highest weight λ in $\mathcal{O}$, then the formal character of V is W-invariant.*

We first need a lemma about the "Ad-ad" correspondence; it can be found in [H1, §2.3]:

Lemma 2. *If x, y are endomorphisms of a vector space V, and x is nilpotent, then* $\operatorname{Ad}(\exp x)(y) = \exp(\operatorname{ad} x)(y)$.

Proof. The exponentials make sense since x is nilpotent. Now, $\operatorname{ad} x = \lambda_x + \rho_{-x}$, where λ_x is left-multiplication by x, and ρ_{-x} is right-multiplication by $-x$ - in the endomorphism ring of V. Since λ_x and ρ_{-x} obviously commute, hence we start from the right-hand side above, and compute:

$$\exp(\operatorname{ad} x)(y) = \exp(\lambda_x + \rho_{-x})(y) = \exp(\lambda_x) \exp(\rho_{-x})(y) = \exp(x) y \exp(-x)$$

where all of this makes sense in the endomorphism ring - and note that x is nilpotent, whence the exp's are all finite sums. □

In particular, if $\mathfrak{g}$ is as above, let us define τ_α for each simple root α, to be

$$\tau_\alpha = \exp(e_\alpha) \exp(-f_\alpha) \exp(e_\alpha)$$

where $\{e_\alpha, f_\alpha\}$ generate a copy of $\mathfrak{sl}_2$ corresponding to the simple root α. If e_α, f_α act nilpotently on a $\mathfrak{g}$-module M, then τ_α is a well-defined vector-space isomorphism on M. We now have the following result:

Lemma 3. *For such M, and for every $h \in \mathfrak{h}$, we have $\operatorname{Ad} \tau_\alpha(h) = h - \alpha(h) h_\alpha$ (as endomorphisms of M), where we define $h_\alpha = [e_\alpha, f_\alpha]$ for all $\alpha \in \Delta$. Therefore $\operatorname{Ad} \tau_\alpha^{-1}(h) = h - \alpha(h) h_\alpha = \operatorname{Ad} \tau_\alpha(h)$.*

Proof. We simply compute. Note that τ_α preserves the finite-dimensional $\mathfrak{h}$-semisimple module $\mathfrak{g}$, and e_α, f_α act by increasing or decreasing the weight, whence they are nilpotent. Thus, by the previous lemma, the computation is done as follows:

$$\exp(\operatorname{ad} e_\alpha)(h) = h - \alpha(h) e_\alpha$$

Next, we compute $\exp(\operatorname{ad} -f_\alpha)(\exp(\operatorname{ad} e_\alpha)(h))$; this equals

$$(h - \alpha(h)e_\alpha) - [f_\alpha, h - \alpha(h)e_\alpha] + \frac{1}{2}[f_\alpha, [f_\alpha, h - \alpha(h)e_\alpha]]$$

since the last term above is already in the span of f_α, and hence is killed by $(\operatorname{ad} f_\alpha)^3$. We now calculate the right-hand side, and get

$$\begin{aligned} &= h - \alpha(h)e_\alpha + (\alpha(h)f_\alpha - \alpha(h)h_\alpha) + \frac{1}{2}\alpha(h)2f_\alpha \\ &= h - \alpha(h)h_\alpha - \alpha(h)e_\alpha \end{aligned}$$

Finally, we apply one more adjoint-exponential, to get

$$\begin{aligned} \operatorname{Ad}\tau_\alpha(h) &= \exp(\operatorname{ad} e_\alpha)(h - \alpha(h)h_\alpha - \alpha(h)e_\alpha) \\ &= h - \alpha(h)h_\alpha - \alpha(h)e_\alpha - \alpha(h)e_\alpha + 2\alpha(h)e_\alpha = h - \alpha(h)h_\alpha \end{aligned}$$

This shows the first assertion. Moreover, plugging in h_α in place of h yields that $\tau_\alpha(h_\alpha) = h_\alpha - \alpha(h_\alpha)h_\alpha = (1-2)h_\alpha = -h_\alpha$. These two equations together give us the formula for τ_α^{-1}: simply apply τ_α to both sides and verify that the desired result holds. □

Proof of the Theorem. To show W-invariance, it suffices to show that the formal character is invariant under s_α. For this, we use τ_α defined above. This reduces the problem to an action of $\mathfrak{sl}_2$, spanned by $\{e_\alpha, f_\alpha, h_\alpha\}$ for each simple α.

We now note that $V = L(\lambda) \in \mathcal{O}$ is finite-dimensional and $\mathfrak{h}$-semisimple, whence e_α, f_α act nilpotently on it, and the above lemmas can be applied. Now, by the previous lemma, given $h \in \mathfrak{h}$ and $v_\mu \in V_\mu$ (for some weight $\mu \in \mathfrak{h}^*$), we have (omitting the Ad from in front of all the $\tau_\alpha^{\pm 1}$'s)

$$h \cdot \tau_\alpha v_\mu = \tau_\alpha \cdot \tau_\alpha^{-1}(h) \cdot v_\mu = \tau_\alpha \cdot (h - \alpha(h)h_\alpha) \cdot v_\mu = \mu(h) - \mu(h_\alpha)\alpha(h)(\tau_\alpha v_\mu)$$

But $\mu(h_\alpha) = 2(\mu, \alpha)/(\alpha, \alpha) = \langle \mu, \alpha \rangle$ (check with the special case $\mu = \alpha$!), whence we get that

$$h \cdot \tau_\alpha v_\mu = (\mu - \langle \mu, \alpha \rangle \alpha)(h)\tau_\alpha v_\mu = (s_\alpha \mu)(h)\tau_\alpha v_\mu$$

We thus conclude that $\tau_\alpha : V_\mu \to V_{s_\alpha(\mu)}$, and since it is an automorphism, s_α preserves the formal character of $V = L(\lambda)$. □

4. Results by Weyl

We now carry out some computations related to [H2, §3.20]. The approach we take, is as in [Cart, §10.1, 10.2] - and proves results by Weyl and Macdonald. In the framework of Weyl's character formula, first we have the following result.

Theorem 3 (Weyl's Identity)**.** $\theta(e(\rho)) = e(-\rho)\prod_{\alpha>0}(e(\alpha)-1)$.

Proof. We first show that the right hand side (we henceforth call it X) is indeed alternating. Given a simple reflection s_α, it clearly permutes all $e(\beta)-1$, for $\beta \neq \alpha$, by Proposition 4 above. From the proof of Theorem 1 above, $s_\alpha(\rho) = \rho - \alpha$. Thus, we get that for each simple root α, s_α acts on the right hand side X by

$$e(\alpha-\rho)(e(-\alpha)-1)\prod_{\beta\neq\alpha}(e(\beta)-1) = e(-\rho)(1-e(\alpha))\prod_{\beta\neq\alpha}(e(\beta)-1)$$

and hence $s_\alpha(X) = -X$ for each simple α. Since the s_α's generate W, hence X is indeed alternating. In particular, $X = \frac{1}{|W|}\theta(X)$, from Lemma 1.

Note that (the product in) X is a sum of terms of the form $e(-\rho + \sum_{\alpha\in\Psi}\alpha)(-1)^{|\Pi|-|\Psi|}$, for all $\Psi \subset \Pi$. Hence (upto scalars),

$$|W|X = \theta(X) = (-1)^{|\Pi|}\sum_{\Psi\subset\Phi}(-1)^{|\Psi|}\theta(e(-\rho+\sum_{\alpha\in\Psi}\alpha))$$

Now, the terms in the (outer) sum on the right, that are not of the form $\theta(w(e(-\rho)))$ are of the form $\theta(e(-\lambda))$ for λ in some hyperplane, by Theorem 1 above. But then $-\lambda$ is in the same hyperplane, and by Lemma 1 above, $\theta(e(-\lambda)) = 0$. Therefore we are left with expressions of the form $w\rho$.

By Proposition 4 above, ρ is in the fundamental domain (or dominant chamber), whence there is no repetition, and we are left with exactly $|W|$ elements. Thus we conclude that $|W|X = (-1)^{|\Pi|}\sum_{w\in W}(-1)^{l(w)}\theta(e(-w\rho))$. Now observe that the longest element $w_\circ$ takes Π to $-\Pi$, and is of length $|\Pi|$. Hence we can simplify the above expression (using different indices of summation, namely $w' = ww_\circ$):

$$\begin{aligned} |W|X &= \sum_{w\in W}\det w_\circ \det w \cdot \theta(e(w(w_\circ\rho))) = \sum_{w'\in W}\det w' \cdot \theta(e(w'\rho)) \\ &= \sum_{w'\in W}(\det w')^2\theta(e(\rho)) = |W|\theta(e(\rho)) \end{aligned}$$

whence we are done. ☐

We conclude by proving the Weyl Character Formula. For this, define the *μ-weight space* of an $\mathfrak{h}$-module M to be $M_\mu := \{m \in M : hm = \mu(h)m\ \forall h \in \mathfrak{h}\}$, and given a finite-dimensional $\mathfrak{h}$-semisimple module M, define its *(formal) character* to be $\mathrm{ch}_M = \sum_{\mu\in\mathfrak{h}^*}(\dim M_\mu)e(\mu) \in \mathcal{E}_V$. We then have

Theorem 4 (Weyl Character Formula)**.** *$\theta(e(\rho))\cdot ch_{L(\lambda)} = \theta(e(\rho+\lambda))$ for all dominant λ (i.e. finite-dimensional $L(\lambda)$).*

Proof. We need some results concerning the block decomposition of Category $\mathcal{O}$. Note that every Verma module $M(\mu)$ is of finite length, and its simple subquotients are of the form $L(\lambda)$ for $\chi_\lambda = \chi_\mu$, where χ denotes central characters. By Harish-Chandra's theorem, this means that $\lambda \in W \bullet \mu = W(\mu+\rho) - \rho$. Moreover, $[M(\mu) : L(\lambda)]$ is 1 if $\lambda = \mu$, and positive only if $\lambda - \mu = \sum_{\alpha \in \Delta} n_\alpha \alpha$ for some $n_\alpha \in \mathbb{N} \cup \{0\}$. Thus, the *decomposition matrix* $D_{\mu,\lambda} = (([M(\mu) : L(\lambda)]))$ is unipotent, whence its inverse is also unipotent. In the language of formal characters[1], this means that

$$\mathrm{ch}_{L(\lambda)} = \sum_{w \in W} a_w \, \mathrm{ch}_{M(w(\lambda+\rho)-\rho)} \tag{1}$$

where $a_1 = 1$, and $\lambda + \rho$ being in the fundamental chamber, the $w(\lambda+\rho)$'s are all distinct.

Moreover, by the structural properties of Verma modules, it is clear that $\mathrm{ch}_{M(\mu)} = e(\mu) \prod_{\alpha \in \Delta}(1 + e(-\alpha) + e(-\alpha)^2 + \dots)$, whence

$$\prod_{\alpha \in \Delta} (1 - e(-\alpha)) \cdot \mathrm{ch}_{M(\mu)} = e(\mu) \tag{2}$$

But by Weyl's identity, this "correction factor" is, indeed,

$$\prod_{\alpha \in \Delta} e(-\alpha)(e(\alpha) - 1) = e(-\sum_\alpha \alpha) \prod_\alpha (e(\alpha) - 1) = e(-2\rho) \prod_\alpha (e(\alpha) - 1)$$

$= e(-\rho)\theta(e(\rho))$. So we now multiply both sides of equation (1) above, by the correction factor $e(-\rho)\theta(e(\rho))$, and using equation (2) above, we get

$$e(-\rho)\theta(e(\rho)) \, \mathrm{ch}_{L(\lambda)} = \sum_{w \in W} a_w e(w(\lambda+\rho) - \rho) \tag{3}$$

We now multiply both sides by $e(\rho)$. The left side is now a product of two weight vectors: one of them is W-invariant, and the other is W-alternating. Hence by Lemma 1 above, the left side is W-alternating too, whence so is the right side. Applying any given $w' \in W$, we get the sum to equal

$$\sum_{w \in W} a_w w' e(w(\lambda+\rho)) = \sum_{w \in W} a_w e(w'w(\lambda+\rho))$$

By W-alternation, this equals $(-1)^{\det w'}$ times the original quantity, so a change of index of summation yields that $a_{w'w} = (-1)^{\det w'} a_w$ for all w, w'. Since $a_1 = 1$, we substitute $a_w = (-1)^{\det w}$ in equation (3) (multiplied by $e(\rho)$), to get

$$\theta(e(\rho)) \, \mathrm{ch}_{L(\lambda)} = \sum_{w \in W} (-1)^{\det w} e(w(\lambda+\rho)) = \theta(e(\lambda+\rho))$$

and this is the Weyl Character Formula. □

[1]In $\mathcal{O}$, we are often forced to work with infinite-dimensional $\mathfrak{h}$-semisimple modules (with finite-dimensional weight spaces). The characters of such modules lie in the completion $\overline{\mathcal{E}_{\mathfrak{h}^*}} := \prod_{\lambda \in \mathfrak{h}^*} \mathbb{Z}e(\lambda)$.

5. Macdonald's identity.

We conclude by proving the following remarkable theorem by Macdonald; the theorem is stated as equation (35) on [H2, Page 85].

Theorem 5 (Macdonald's identity)**.** $\displaystyle\sum_{w\in W}\prod_{\alpha>0}\frac{1-te(-w\alpha)}{1-e(-w\alpha)}=\sum_{w\in W}t^{l(w)}.$

Proof. We prove this result in several steps.

Step 1. Using that the right side in Weyl's identity is alternating, we now multiply both sides by $w\in W$ and compute:

$$\begin{aligned}\det w\cdot\theta(e(\rho)) &= \theta(w(e(\rho)))=w(\theta(e(\rho)))\\ &= w(e(-\rho))w\Big(\prod_{\alpha>0}e(\alpha)(1-e(-\alpha))\Big)\\ &= e(-w\rho)w\Big(\prod_{\alpha>0}e(\alpha)\prod_{\alpha>0}(1-e(-\alpha))\Big)\\ &= e(-w\rho)e(w\cdot 2\rho)\prod_{\alpha>0}(1-e(-w\alpha))\\ &= e(w\rho)\prod_{\alpha>0}(1-e(-w\alpha))\end{aligned}$$

Multiplying both sides by $e(-w\rho)$, we get the identity

$$\prod_{\alpha>0}(1-e(-w\alpha))=e(-w\rho)\det w\cdot\theta(e(\rho))$$

Step 2. Let us call the left side of Macdonald's identity above, Y. We now substitute the above identity into Y, and compute:

$$\begin{aligned}\theta(e(\rho))Y &= \theta(e(\rho))\sum_{w\in W}\prod_{\alpha>0}\frac{1-te(-w\alpha)}{1-e(-w\alpha)}\\ &= \sum_{w\in W}e(w\rho)\det(w)\prod_{\alpha>0}(1-te(-w\alpha))\\ &= \sum_{w\in W}e(w\rho)\det(w)\sum_{\Psi\subset\Pi=\Phi^+}(-t)^{|\Psi|}e(-w\sum_{\alpha\in\Psi}\alpha)\end{aligned}$$

where $|\Psi|$ denotes the cardinality of the set $\Psi\subset\Phi$. Now exchanging the order of summation, we get the desired expression to equal

$$\sum_{\Psi\subset\Pi}\sum_{w\in W}\det(w)\cdot w(e(\rho-\sum_{\alpha\in\Psi}\alpha))(-t)^{|\Psi|}=\sum_{\Psi\subset\Pi}(-t)^{|\Psi|}\theta(e(\rho-\sum_{\alpha\in\Psi}\alpha))$$

Step 3. Once again using Theorem 1 (and Lemma 1) above, we see that the only terms that are nonzero in this (outer) sum, are of the form

$(-t)^{l(w)}\theta(e(w\rho))$. Hence we make this substitution and compute the left hand side of Macdonald's identity to equal $\theta(e(\rho))Y$

$$= \sum_{w\in W} t^{l(w)} \det w \cdot \theta(e(w\rho)) = \sum_{w\in W} t^{l(w)} (\det w)^2 \theta(e(\rho)) = \theta(e(\rho)) \sum_{w\in W} t^{l(w)}$$

Since we are working in $B[t]$, with B an integral domain (cf. e.g. [H2, Page 85]), we can cancel $\theta(e(\rho))$ in the polynomial algebra $K(B)[t]$, where $K(B)$ is the field of fractions of B, to get Macdonald's identity. (We can make this cancellation, because $\theta(e(\rho)) \neq 0$ - either by Weyl's identity above, or because ρ is in the fundamental domain.) □

The results above and in [H2] give rise to a remarkable number of expressions for the same Poincaré polynomial of a Weyl group W. Namely,

$$\begin{aligned} W(t) &= \sum_{w\in W} t^{l(w)} = \prod_{i=1}^{n} \frac{1-t^{d_i}}{1-t} = \prod_{i=1}^{n} \frac{1-t^{m_i+1}}{1-t} = \prod_{\alpha>0} \frac{t^{\mathrm{ht}(\alpha)+1}-1}{t^{\mathrm{ht}(\alpha)}-1} \\ &= \sum_{w\in W} \prod_{\alpha>0} \frac{1-te(-w\alpha)}{1-e(-w\alpha)} \end{aligned}$$

where the d_i's and m_i's are the degrees and exponents of the Weyl group W, respectively.

References

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