# THE BGG CATEGORY $\mathcal{O}$ OVER A SKEW GROUP RING

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**References**:
A. Khare, *Axiomatic Framework for the BGG Category* $\mathcal{O}$, `math.RT/0502227`
A. Khare, *Category* $\mathcal{O}$ *over skew group rings*, `math.RT/0504371`

## 0.1. **Hopf regular triangular algebras.**

We consider the category $\mathcal{O}$ over a general class of algebras.

**Definition.** A *Hopf regular triangular algebra* (or *HRTA*) is an algebra $A$ over a field $k$ of characteristic zero, defined by the following axioms:

1. $A \cong B_- \otimes_k H \otimes_k B_+$, where $H, B_\pm, H \otimes B_\pm$ are associative unital $k$-subalgebras of $A$. Moreover, $H$ is a commutative Hopf algebra.
2. The set (a group, actually !) $G := \mathrm{Hom}_{k-alg}(H, k)$ contains a free abelian group with finite basis $\Delta$, so that $B_\pm = \bigoplus_{\lambda \in \pm \mathbb{N}_0 \Delta} (B_\pm)_\lambda$. Each summand here is a finite-dimensional weight space for the adjoint action of $H$, and $(B_\pm)_0 = k$.
3. $A$ has an anti-involution $i$ such that $i|_H = \mathrm{id}|_H$, and $i : B_+ \to B_-$.

## 0.2. **Examples.**

1. *Lie algebras with regular triangular decomposition*:
    - semisimple Lie algebras $\mathfrak{g}$ (say over $\mathbb{C}$)
    - (symmetrizable) Kac-Moody Lie algebras
    - contragredient Lie algebras (introduced in [Kac-Kazhdan 1979])
    - some Borcherds Lie algebras
    - (centerless) Virasoro and (extended) Heisenberg algebras
    - affine Lie algebras (e.g. $\widehat{sl_2}(\mathbb{C})$)
    - certain quotients of preprojective algebras of loop-free quivers

    In each case, $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$, and we take $B_\pm = \mathfrak{U}\mathfrak{g}_\pm,\ H = \mathfrak{U}\mathfrak{h}$.

**Remark.** For such Lie algebras, ad : $H \to \mathrm{End}(\mathfrak{U}\mathfrak{g})$ is an algebra map, and we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, $\mathfrak{g}_\alpha V_\beta \subset V_{\alpha+\beta}$ for a $\mathfrak{g}$-module $V$. Similarly, for Hopf RTA's in general, we have

**Proposition 1.** ad : $H \to$ *End(A) is an algebra map. For all* $\alpha, \beta \in (G, *)$ *and an* $A$*-module* $V$*, we have*

$$A_\alpha \cdot A_\beta \subset A_{\alpha * \beta}, \qquad A_\alpha V_\beta \subset V_{\alpha * \beta}$$

*Moreover,* $i(A_\mu) = A_{-\mu}$ *for each* $\mu$*.*

(2) Non-Lie-algebra examples:
  - quantum groups (i.e. $U_q(\mathfrak{g})$ for $\mathfrak{g}$ a symmetrizable Kac-Moody algebra)
  - symplectic oscillator algebras and their quantized analogues
  - deformations of Kleinian singularities (the group therein is replaced by an infinite subgroup of the 2-torus)

(3) *Functoriality*: If $A_i$ are HRTA's, then so is their "symmetric tensor product" $A = \otimes_{i=1}^n A_i$; moreover, $G = \prod_i G_i$.

0.3. **Skew group rings and the Category $\mathcal{O}$.** Suppose a finite group $\Gamma$ acts on $A$ by automorphisms, each of which also preserves $B_\pm, N_\pm, H, k \subset A$. We define the *skew group ring* $A \rtimes \Gamma$ to be $A \otimes_k k[\Gamma]$, via the relations $(a \otimes \gamma)(a' \otimes \gamma') = a\gamma(a') \otimes \gamma\gamma'$.

**Assumptions.** $\Gamma$ now acts on the group $(G, *)$, via group automorphisms. Assume that

- $\gamma(\alpha) \geq 0$ if $\alpha \in \Delta$.
- If $\Gamma \neq 1$, then $k$ is algebraically closed.

Examples include HRTA's (set $\Gamma = 1$) and *wreath products* $A^{\otimes n} \rtimes S_n$ for an HRTA $A$.

**Definitions.**

1. The *Harish-Chandra Category* $\mathcal{H}$ is the full subcategory of all $H$-semisimple $A \rtimes \Gamma$-modules, with finite-dimensional weight spaces.
2. The *BGG Category* $\mathcal{O}$ is the full subcategory of $\mathcal{H}$, with all finitely generated $(A \rtimes \Gamma)$-modules, with a locally finite $B_+$-action.
3. Define a *duality functor* $F : \mathcal{H} \to \mathcal{H}$, using the anti-involution $i$ above: $F(M)$ contains all weight vectors in $M^*$, and for $a \otimes \gamma \in A \rtimes \Gamma$,
$$((a \otimes \gamma)m^*)(m) = m^*(\gamma^{-1} i(a)m)$$

Then $F$ is an exact, contravariant, involutive functor that preserves the length of a module, as well as its *formal character*.

0.4. **Simple modules.** Let $\mathcal{C}$ be the category of finite-dimensional $H$-semisimple $H \rtimes \Gamma$-modules, with simple objects $\{M_x : x \in X\}$ in $\mathcal{C}$. If $\Gamma = 1$, then $X = G$.

**Theorem 1.** *$\mathcal{C}$ is completely reducible.*

**Remarks.** $M_x$ contains weight vectors only of weights $\{\gamma(\lambda_x) : \gamma \in \Gamma\}$ for some $\lambda_x \in G$. Each $\lambda \in G$ equals $\lambda_x$ for some $x \in X$.

**Definitions.**

(1) Define a *partial order* on $X$, via: $x \geq x'$ if $x = x'$, or there is an $\alpha \in \mathbb{N}_0\Delta \setminus \{0\}$ and $\gamma \in \Gamma$, so that $\alpha * \gamma(\lambda'_x) = \lambda_x$. Setting $\Gamma = 1$, we get the usual partial order on $G$.

(2) Define *Verma modules* $Z(x)$ for $x \in X$: give $M_x$ an $(H \otimes B_+) \rtimes \Gamma$-module structure, via $n_+ M_x = 0$ for $n_+ \in N_+$. Now define $Z(x) := \operatorname{Ind}_{(H\otimes B_+)\rtimes\Gamma}^{A\rtimes\Gamma} M_x$.

**Lemma.** $Z(x)$ is indecomposable, and has a unique simple quotient $V(x)$ in $\mathcal{O}$, for all $x$. The $V(x)$'s are pairwise nonisomorphic, have no self-extensions in $\mathcal{O}$, and exhaust all simple objects in $\mathcal{O}$.

**Proposition 2.** *Suppose $v_{\lambda_x}$ is any maximal vector in $V(x)$. Then $k[\Gamma]v_{\lambda_x} \cong M_x$. Now let $v_{\lambda_1}, \dots, v_{\lambda_n}$ be a weight basis of $M_x$. Then $B_- v_{\lambda_i} \cong V_A(\lambda_i)$ $\forall i$, where $V_A(\lambda_i)$ is the simple module in $\mathcal{O}_A$ (setting $\Gamma = 1$); moreover, $V(x) \cong \oplus_{i=1}^n B_- v_{\lambda_i}$ in $\mathcal{O}_A$.*

**Proposition 3.** *Two simple modules have a nonsplit extension if and only if they (or their "duals") are the first two subquotients in a filtration for a Verma module.*

0.5. **Condition (S).** In the absence of central characters in general, we introduce the following condition to help us carry out block decomposition.

**Definitions.**

1. Given $x \in X$, let $S(x)$ denote the symmetric and transitive closure of $\{x\}$ in $X$, under the two relations
$$x \to x' \text{ iff } [Z(x') : V(x)] > 0, \qquad x \to_F x' \text{ iff } x' = F(x)$$
2. *Condition (S)* holds for $A \rtimes \Gamma$ if every $S(x)$ is finite.
3. The *block* $\mathcal{O}(x)$ consists of all finite length objects in $\mathcal{O}$, each of whose Jordan-Holder factors is in $\{V(x') : x' \in S(x)\}$.

If $\Gamma = 1$, then we only look at the relation $\mu \to \lambda$ iff $[Z(\lambda) : V(\mu)] > 0$. For example, $A = \mathfrak{U}\mathfrak{g}$ (over $\mathbb{C}$) satisfies Condition (S), since $S(\lambda) \subset W \bullet \lambda$ by Harish-Chandra's Theorem.

**Assumption.** Condition (S) holds for $A$.

**Theorem 2.** *Condition (S) holds for $A \rtimes \Gamma$. Thus $\mathcal{O}$ is abelian, self-dual, and of finite length. Moreover, $\mathcal{O} = \bigoplus_x \mathcal{O}(x)$, where each block is a highest weight category, and contains finitely many nonisomorphic simple modules, all with the same central character. Morphisms and extensions between different blocks are trivial.*

In particular, each block has enough projectives - e.g. the projective cover $P(x)$ of $V(x)$. If $P_x$ is a progenerator for $\mathcal{O}(x)$, then $\mathcal{O}(x)$ is equivalent to the module category $(\text{mod-}B_x)^{fg}$ of finitely generated right-modules over the finite-dimensional algebra $B_x := \text{End}_{\mathcal{O}} P_x$.

0.6. **BGG Reciprocity.** $[P(x) : Z(x')] = [Z(F(x')) : V(F(x))]$

where $x, x' \in X$, and given $x$, $F(V(x))$ is a simple module $V(x'')$; we denote $x''$ by $F(x)$.

**Definitions.** Order the elements in a block $S(x)$ as $\{x_1, \ldots, x_N\}$, and define the *modified Cartan matrix* (resp. the *duality, decomposition* matrix) $C'_x$ (resp. $F_x, D_x$) by

$$\begin{aligned}
(C'_x)_{ij} &= [P(x_i) : V(F(x_j))] \\
(F_x)_{ij} &= \delta_{x_i, F(x_j)} \\
(D_x)_{ij} &= [Z(x_i) : V(x_j)]
\end{aligned}$$

($F_x$ is a symmetric permutation matrix of order 1 or 2; the order is 1 if $\Gamma = 1$.)

**Theorem 3.** *$C'_x$ is symmetric; more precisely, $C'_x = F_x D_x^T F_x D_x F_x$.*

0.7. **Complete reducibility and functoriality.**

**Theorem 4.** *If $A$ satisfies Condition (S), and no block $\mathcal{O}(x)$ has two finite-dimensional simple modules, then every finite-dimensional $M \in \mathcal{O}$ is completely reducible.*
*Moreover, complete reducibility holds in $\mathcal{O}$ if and only if it holds in $\mathcal{O}_A$.*

For example, $A = \mathfrak{U}\mathfrak{g}$ and $\Gamma = 1$.

**Functoriality**: Given HRTAs $A_i$, define $A = \otimes_{i=1}^n A_i$.

**Theorem 5.** *Each simple $V_A(\lambda) \in \mathcal{O}$ is of the form $\otimes_{i=1}^n V_{A_i}(\lambda_i)$, where $\lambda = (\lambda_1, \ldots, \lambda_n) \in G$, and $V_{A_i}(\lambda_i)$ is simple in $\mathcal{O}_{A_i}$ for all $i$.*
*Condition (S) (and assuming it, complete reducibility for finite-dimensional modules) holds in $\mathcal{O}$ if and only if it holds in every $\mathcal{O}_i$.*