TOPIC PROPOSAL : SEMISIMPLE LIE ALGEBRAS AND THEIR REPRESENTATIONS

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1. Abstract

In this topic proposal, we discuss the representation theory of semisimple Lie algebras, including results on the structure of these algebras and their modules. In particular, the theory of root systems and Verma modules are very important in this context. We classify all simple Lie algebras, and obtain a presentation for any semisimple Lie algebra. We also encounter several striking formulas that help us learn more about the structure of the modules. We describe certain lattices in the modules, and related groups. Mostly we work over algebraically closed fields of characteristic zero, but some constructions extend to arbitrary fields as well.

2. Basics

Let L be a finite dimensional Lie algebra over a ground field F. We define $\text{ad} : L \rightarrow \text{Der } L$ by $(\text{ad } x)(y) = [x, y]$.We say $x \in L$ is ad-nilpotent if $(\text{ad } x)^n = 0$ for some $n \geq 0$. We have *Engel's Theorem*, which says that L is nilpotent iff every x in L is ad-nilpotent. We also have the following two results, the first of which implies Engel's Theorem.

Proposition 1. *Let V be a nonzero finite dimensional vector space over F, and $L \subset \mathfrak{gl}(V)$.*

1. *Suppose L consists of nilpotent endomorphisms. Then $\exists v \in V, v \neq 0$, such that $L \cdot v = 0$.*
2. *Suppose L is solvable, and further, F is algebraically closed, of characteristic zero. Then V contains a common eigenvector for all of L, i.e. $\exists v \in V, v \neq 0$ such that $L \cdot v \subset F \cdot v$.*

- From now on, we take F to be algebraically closed, and $\text{char } F = 0$.

From part (2), we get *Lie's Theorem* : Under the setup of part (2) of the proposition, all matrices in L are upper triangular with respect to a suitable basis. In other words, L stabilizes some flag of subspaces in V.

Proposition 2. *Suppose dim V is finite. Then $\forall x \in$ End $V, \exists!$ x_n, x_s in End V, such that x_s semisimple (i.e. diagonalizable), x_n nilpotent, $x = x_s + x_n$, and $[x_s, x_n] = 0$. We call $x_s(x_n)$ the semisimple (nilpotent) part of x.*

Call a subalgebra of $\mathfrak{gl}(V)$ *good* if it contains the semisimple and nilpotent parts of its elements. Thus, End V is good. For any finite dimensional F-algebra U, Der U is good as well.

Proposition 3 (Cartan's Criterion)**.** *dim V finite, $L \subset \mathfrak{gl}(V)$. Suppose tr$(x.y) = 0$ $\forall x \in L, y \in [L, L]$. Then L is solvable.*

3. The Killing Form, Complete Reducibility, Weight Decomposition, and $\mathfrak{sl}_2$

Now we define the *Killing form* κ on L. For all $x, y \in L$, (ad x.ad y) is in End L, hence has a trace independent of basis. So we define $\kappa(x,y) = \mathrm{tr}(\text{ad } x.\text{ad } y)$. The Killing form has *radical* Rad $\kappa = \{x \in L | \kappa(x, L) = 0\}$, and is *non-degenerate* if this is zero. Also, L is *semisimple* if it has no nonzero solvable (or abelian) ideals.

Proposition 4 (Cartan's Criterion for Semisimplicity)**.** *L is semisimple iff κ is non-degenerate.*

Proposition 5. *If L is semisimple, L is a (finite) direct sum of simple ideals L_i. Further, every simple ideal of L must be some L_i, and the Killing form on L_i is the restriction of κ to L_i.*

Hence all ideals and homomorphic images of semisimple L are also semisimple.
Any semisimple L also satisfies $L \cong \text{ad } L = \text{Der } L$, so ad L is good. So for any $x \in L$, $(\text{ad } x)_s$ and $(\text{ad } x)_n \in \text{ad } L \cong L$, so $\exists!\ x_s, x_n \in L$ satisfying the properties listed in Prop.(2). These give us the *abstract Jordan decomposition* of x, which satisfies ad $x_s = (\text{ad } x)_s$ etc.

- From now on, we take L to be semisimple, and dim V to be finite.

Theorem 1 (Weyl's Theorem on Complete Reducibility)**.** *Given L semisimple, any finite dimensional L-module V is completely reducible.*

Corollary 1. *The abstract and usual Jordan decompositions coincide in any semisimple $L \subset \mathfrak{gl}(V)$.*

(Remark) A *toral* subalgebra is one that consists entirely of semisimple elements, e.g. the diagonal matrices in $\mathfrak{sl}(l+1, F)$. Then every toral subalgebra T is abelian. Now say V is an L-module, i.e. $L \subset \mathfrak{gl}(V)$. Then T is abelian, hence simultaneously diagonalizable, and therefore by Corollary (1), T acts diagonally on any L-module V. Hence we can split V into eigenspaces $V_\lambda = \{v \in V | hv = \lambda(h)v\}$. When T is maximal toral (denoted H here), the V_λ's are called *weight spaces*, and consist of *weight vectors*. If $V_\lambda \neq 0$, the corresponding $\lambda \in H^*$ is a *weight*. Finally, if $V = L$ under the adjoint representation, then we replace the V_λ's by the *root spaces* L_α, and weights λ by *roots* $\alpha \in H^* \setminus \{0\}$.

For the rest of this section, we fix $L = \mathfrak{sl}_2(F)$, the traceless 2×2 matrices over F, with basis $\{x, h, y\}$. Now say V is an L-module. Then $H = Fh$ is maximal toral, hence (Remark) applies. Any weight vector killed by x is a *maximal* weight vector. We can now explicitly describe all (irreducible) finite dimensional L-modules V.

Theorem 2. *If V is a finite dimensional irreducible L-module of dimension $\lambda + 1$, then V has a maximal weight vector v_0 of weight λ. Define $v_i = (y^i v_0)/i!$. Then $V = \bigoplus_{i=0}^{\lambda} F v_i$, and the v_i are weight vectors satisfying : $hv_i = (\lambda - 2i)v_i$; $yv_i = (i+1)v_{i+1}$; $xv_i = (\lambda - i + 1)v_{i-1}$ (where $v_{-1} = v_{\lambda+1} = 0$).*

4. Root Systems, Bases, and Weights

- Now we take L to denote any semisimple Lie algebra.

We now choose (and fix) a maximal toral subalgebra H in L. Then (Remark) applies, and we can split L into $\bigoplus L_\alpha$. Define a *root* to be a nonzero α with $L_\alpha \neq 0, \Phi$ to be the set of roots. Then Φ is nonempty and finite, and we have $L = C_L(H) \bigoplus \coprod_{\alpha \in \Phi} L_\alpha$. Then the set of roots Φ turns out to be a root system, as below. In some sense. L is made up of copies of $\mathfrak{sl}_2$, because given nonzero $x_\alpha \in L_\alpha$ $(\alpha \in \Phi)$, $\exists! y_\alpha \in L_{-\alpha}$, such that $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ span a 3 dimensional subalgebra of L, isomorphic to $\mathfrak{sl}_2(F)$.

Given a vector α in Euclidean space $E = \mathbb{R}^n$, we have the hyperplane P_α perpendicular to it, and also the reflection σ_α, that sends α to $-\alpha$, and acts as the identity on P_α. Explicitly, $\sigma_\alpha(\beta) = \beta - \langle\beta, \alpha\rangle\alpha$, where $\langle\beta, \alpha\rangle \overset{\text{def}}{=} 2(\beta, \alpha)/(\alpha, \alpha)$.

We now define a *root system* to be a subset Φ of E satisfying (a) Φ is finite, spans E, and $0 \notin \Phi$, (b) If $\alpha, c\alpha \in \Phi$ for some scalar c, then $c = \pm 1$, (c) $\alpha \in \Phi \Rightarrow \sigma_\alpha(\Phi) = \Phi$, and (d) $\langle\beta, \alpha\rangle \in \mathbb{Z}\ \forall \alpha, \beta \in \Phi$.

The *Weyl group* W is the subgroup of $\mathrm{Aut}(E)$ that is generated by all reflections $\{\sigma_\alpha | \alpha \in \Phi\}$. We say $\Delta \subset \Phi$ is a *base* if it is a basis of E, and if every root β can be written as $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$, where k_α's are integers, all ≥ 0 or all ≤ 0. Then Φ has a base, and W acts simply transitively on bases. Define an ordering on Φ by $\beta \succ \gamma$ if $\beta - \gamma = \sum_{\alpha\in\Delta} k_\alpha \alpha$, where all $k_\alpha \geq 0$. The roots in Δ are said to be *simple*.

- Henceforth while talking of a root system Φ, we will have a base $\Delta = \{\alpha_1, \ldots, \alpha_l\}$.

Given Φ, Δ as above, we define the *Cartan matrix* C (of Φ) by $C = ((c_{ij})) = ((\ \langle\alpha_i, \alpha_j\rangle\ ))$. It determines Φ up to isomorphism. We can classify all irreducible root systems, based on their Dynkin diagrams. They are of the type $A_l (l \geq 1), B_l (l \geq 2), C_l (l \geq 3), D_l (l \geq 4)$ (*classical*), and E_6, E_7, E_8, F_4, G_2 (*exceptional*). Explicit constructions of these root systems as vectors in Euclidean space are known.

Now we define the *weight lattice* Λ to be $\{\lambda \in E | \langle\lambda, \alpha\rangle \in \mathbb{Z}\ \forall \alpha \in \Phi\}$, and the *root lattice* Λ_r to be the subgroup of Λ generated by Φ. The dual basis λ_i to α_i satisfy $\langle\lambda_i, \alpha_j\rangle = \delta_{ij}$, and are called the *fundamental dominant weights* (relative to Δ). They form a $\mathbb{Z}$-basis of Λ. Call $\lambda \in \Lambda$ *dominant* if $\langle\lambda, \alpha_i\rangle \geq 0\ \forall i$, and define Λ^+ to be the set of dominant weights. One important weight is $\delta = \frac{1}{2}\sum_{\alpha\succ 0} \alpha = \sum_i \lambda_i$. It is actually strongly dominant.

We say a set of weights Π is *saturated* if $\forall \lambda \in \Pi, \forall \alpha \in \Phi$, and $\forall i$ between 0 and $\langle\lambda, \alpha\rangle$, we have $\lambda - i\alpha \in \Pi$. If $\exists \lambda \in \Pi$ saturated, with $\lambda \succ \mu\ \forall \mu \in \Pi$, then Π is saturated *of highest weight* λ.

5. Isomorphism Theorem, Cartan Subalgebras

Now we return to the semisimple Lie algebra case. Say we have L, L' semisimple, with H, H' and Φ, Φ' respectively. Say we have an isomorphism $\pi_0 : \Phi \to \Phi'$, inducing $\pi_0 : H \to H'$. Fix a base Δ of Φ, so that $\pi_0(\Delta) = \Delta'$ is a base of Φ'. Choose nonzero $x_\alpha \in L_\alpha, x'_\alpha \in L'_\alpha$ for all simple α, and let $\pi_\alpha : x_\alpha \to x'_\alpha$. Then $\exists!$ isomorphism $\pi : L \to L'$ extending π_0 and all the π'_αs.

This result is called the **Isomorphism Theorem**. It helps us to construct various automorphisms of L, including for instance, $\tau_\alpha = \exp(\mathrm{ad}\ x_\alpha)\exp(\mathrm{ad}(-y_\alpha))\exp(\mathrm{ad}\ x_\alpha)$. This has the property that it sends L_β to $L_{\sigma_\alpha(\beta)}$ for all roots, h_α to $-h_\alpha$, and h to itself for every h in $\mathrm{Ker}(\alpha)$. τ_α also permutes weight spaces of L-modules ; thus it is an automorphism of L-modules as well. The τ_α's generate a subgroup of Aut L or Aut V (V an L-module) that is ″very similar″ to the Weyl group W.

Another application is that we now know all simple Lie algebras, cf. the previous section. They are the five *Exceptional* Lie algebras, and the four series of *Classical* Lie algebras, namely $A_l = \mathfrak{sl}(l+1,F)$; $B_l = \mathfrak{o}(2l+1,F)$; $C_l = \mathfrak{sp}(2l,F)$; $D_l = \mathfrak{o}(2l,F)$.

We define a *Cartan Subalgebra (CSA)* to be a self-normalizing nilpotent subalgebra. If L is semisimple, the CSA's are precisely the maximal toral subalgebras. We define a *Borel subalgebra* to be a maximal solvable subalgebra of L.

Theorem 3. *All Borel subalgebras of L are conjugate. The same holds for all CSA's of L.*

Thus, we now see that the root system Φ is independent of H, hence depends only on L.

6. Universal Enveloping Algebras ; the PBW and Serre Theorems

We now obtain a presentation for our semisimple Lie algebra L in terms of generators and relations. However, we first introduce some more machinery, that is valid for *any* Lie algebra. We say a map j from L into an associative algebra $\mathfrak{A}$ is *good* if $j([x,y]) = j(x)j(y) - j(y)j(x)\ \forall x,y \in L$.

Now we define the *Universal Enveloping Algebra* $\mathfrak{U}(L)$of a Lie algebra L. This is just an associative algebra $\mathfrak{U}(L)$with 1, and a good map $i : L \to \mathfrak{U}(L)$, satisfying the following universal property : for any associative algebra $\mathfrak{A}$ with 1, and a good map $j : L \to \mathfrak{A}, \exists!$ map $\phi : \mathfrak{U}(L) \to \mathfrak{A}$ (sending 1 to 1), so that $\phi \circ i = j$. For example, putting $\mathfrak{A} = \mathfrak{gl}(V)$ for an L-module V, we see that every L-module is automatically a $\mathfrak{U}(L)$-module (for the converse, just compose ϕ with i).

Essentially, $\mathfrak{U}(L) = \mathfrak{T}(L)/J$, where $\mathfrak{T}(L)$ is the tensor algebra of L, and J the ideal generated in it by all $\{(x \otimes y - y \otimes x - [x,y]) | x,y \in L\}$. Thus it inherits a filtration from $\mathfrak{T}(L)$, making $\mathfrak{G} = \mathrm{Gr}(\mathfrak{U}(L))$ an associative graded algebra with 1. Now fix an ordered basis $\{x_1, x_2, \ \ldots\}$ of L. Let $\mathfrak{S}$ be the polynomial algebra over this basis. Then we have

Theorem 4 (Poincaré-Birkhoff-Witt)**.** $\mathfrak{S} \cong \mathfrak{G}$ *as algebras. In other words, if* $\pi : \mathfrak{T}(L) \to \mathfrak{U}(L)$ *is the natural map, then all* $x_{i(1)} \ldots x_{i(m)} = \pi(x_{i(1)} \otimes \cdots \otimes x_{i(m)})$ *(where* $m > 0, i(1) \leq \cdots \leq i(m)$ *), together with 1, form a basis of* $\mathfrak{U}(L)$.

Theorem 5 (Serre)**.** *Fix a root system Φ and a base $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. Let L be the Lie algebra with $3l$ generators $\{x_i, y_i, h_i : 1 \leq i \leq l\}$, and the following relations* $(\forall\ i,j)$:
$(S1)$ $[h_i, h_j] = 0$ *;*
$(S2)$ $[x_i, y_j] = \delta_{ij} h_j$ *;*
$(S3)$ *Define* $c_{ij} = \langle \alpha_j, \alpha_i \rangle$. *Then* $[h_i, x_j] = c_{ij} x_j$, $[h_i, y_j] = -c_{ij} y_j$
(S^+_{ij}) $(ad\ x_i)^{1-c_{ij}}(x_j) = 0$ *;* (S^-_{ij}) $(ad\ y_i)^{1-c_{ij}}(y_j) = 0$.
Then L is a finite dimensional semisimple Lie algebra, with CSA spanned by the h_i's, and root system Φ.

Therefore given a root system Φ, there exists a semisimple Lie algebra L having Φ as its root system.

7. Verma Modules and Representation Theory ; Introduction to Quantized Enveloping Algebras

- From now on, L is semisimple once more, except for the second half of this section.

If V is an L-module, then (Remark) applies. As in §3§, we define a *maximal (weight) vector* to be any nonzero v^+ killed by L_α, $\forall \alpha \succ 0$. Of course, $L_\alpha V_\mu \subset V_{\alpha+\mu}$.

We have the concept of a *standard cyclic module (of weight λ)*. This is just $V = \mathfrak{U}(L)v^+$ for a maximal vector v^+ of weight λ. We see then (by PBW) that V is spanned by $y_{\beta_1}^{i_1} \dots y_{\beta_m}^{i_m} v^+$, where $m \geq 0$ and $\beta_i \in \Phi^+$. The ″universal″ ones are called *Verma modules* $Z(\lambda)$. Also, there exists a unique irreducible standard cyclic module of highest weight λ up to isomorphism, denoted $V(\lambda)$. It is finite dimensional iff $\lambda \in \Lambda$ is dominant integral, i.e. $\lambda \in \Lambda^+$, cf. §4§. And then $\Pi(\lambda) = \Pi(V(\lambda))$ is a saturated set of weights.

For this part, we work over an arbitrary field k. Let $q \in k, q \neq 0, q^2 \neq 1$. We define the *Quantized Enveloping Algebra of* $\mathfrak{sl}_2$ to be an associative algebra $U = U_q(\mathfrak{sl}_2)$ with 1, generated over k by E, F, K, K^{-1}, that satisfy the relations (1) $KK^{-1} = K^{-1}K = 1$; (2) $KEK^{-1} = q^2 E,\ KFK^{-1} = q^{-2}F$; (3) $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$.

U has no zerodivisors, and it has a PBW-type basis, namely $\{F^s K^n E^r | r, s, n \in \mathbb{Z},\ r, s \geq 0\}$. We can look at U-modules, analogous to $\mathfrak{sl}_2$- or $\mathfrak{U}(\mathfrak{sl}_2)$- modules. We can also construct Verma modules here. These are $M(\lambda)$ $(\lambda \in k) \overset{\text{def}}{=} U/(UE + U(K - \lambda))$, with basis given by the weight vectors $m_i =$ coset of F^i.

Now assume q is not a root of unity. Then each simple U-module of dimension $n + 1$ is of the form $L(n, +)$ or $L(n, -)$. Let c denote $'+'$ or $'-'$. Then the relations on $L(n, c)$ are : (1) $Km_i = cq^{n-2i} m_i$; (2) $Fm_i = m_{i+1}$; (3) $Em_i = c[i][n - i + 1]m_{i-1}$, where $[i] = (q^i - q^{-i})/(q - q^{-1})$, and $m_{-1} = m_{n+1} = 0$.

Some definitions of U include $q = e^{-h/2}, K = e^{-hH/2}$. Using these in the limit $q \to 1$ gives us the corresponding relations in $\mathfrak{sl}_2$, cf. Theorem (2) etc.

8. Formulas of Freudenthal, Weyl, and Kostant ; Harish Chandra's Theorem

We can use the *universal Casimir element* $c_L \in \mathfrak{U}(L)$ to obtain the multiplicity of weight spaces in $V(\lambda)$ inductively. Say $m_\lambda(\mu) = \dim V(\lambda)_\mu$, where $\lambda \in \Lambda^+$. Then we have

Theorem 6 (Freudenthal's Formula)**.** $((\lambda+\delta, \lambda+\delta)-(\mu+\delta, \mu+\delta))m_\lambda(\mu) = 2\sum_{\alpha\succ 0}\sum_{i=1}^{\infty} m_\lambda(\mu+ i\alpha)(\mu + i\alpha, \alpha)$

This helps us compute multiplicities effectively, from higher ones. We also use the crucial fact that $m_\lambda(\mu) = m_\lambda(\sigma\mu)$ for $\sigma \in W$, because τ_α (cf. §5§) permutes weight spaces by taking V_μ to $V_{\sigma_\alpha \mu}$.

Let $\mathfrak{Z}$ be the center of $\mathfrak{U}(L)$. Then every $z \in \mathfrak{Z}$ acts on $Z(\lambda)$ as a scalar, denoted $\chi_\lambda(z)$. We call $\chi_\lambda \in \mathrm{Hom}_{F-alg}(\mathfrak{Z}, F)$ the *character determined by* λ. We say λ and μ are *linked* (denoted $\lambda \sim \mu$) if $\lambda + \delta$ and $\mu + \delta$ are W-conjugate.

Theorem 7 (Harish-Chandra)**.** $\chi_\lambda = \chi_\mu \Leftrightarrow \lambda \sim \mu$

We now introduce the notion of a *formal character*. First, the *group ring* of Λ over $\mathbb{Z}$ is $\mathbb{Z}[\Lambda] = \bigoplus_{\lambda\in\Lambda} \mathbb{Z}e(\lambda)$, where the $e(\lambda)'$s are a basis satisfying $e(\lambda)e(\mu) = e(\lambda+\mu)$. Then W acts naturally on $\mathbb{Z}[\Lambda]$ by $\sigma e(\lambda) = e(\sigma\lambda)$.

The *formal character* of any finite dimensional L-module V is simply $ch_V = \sum_{\mu\in\Pi(V)} \dim(V_\mu)e(\mu)$. Since $\Pi(V)$ is saturated (cf. §7§), we have $ch_V \in \mathbb{Z}[\Lambda]^W$. Then $ch : \mathcal{R}(L) \to \mathbb{Z}[\Lambda]^W$ is a homomorphism, where $\mathcal{R}(L)$ is the *representation ring* of L-modules, with $\oplus$ and $\otimes$ as its operations.) We thus have $ch_{V\otimes W} = ch_V ch_W$, etc.

Proposition 6. *The map $ch : \mathcal{R}(L) \to \mathbb{Z}[\Lambda]^W$ is an isomorphism. In fact, $\mathcal{R}(L) \overset{ch}{\cong} \mathbb{Z}[\Lambda]^W = \mathbb{Z}[\Gamma_1, \dots, \Gamma_l]$, where $\Gamma_i = ch_{\lambda_i}$ corresponds to the fundamental dominant weight λ_i.*

Also define the *Kostant function* $p(\lambda)$ to be the number of sets of non-negative integers $\{k_\alpha : \alpha \succ 0\}$ satisfying $\sum_{\alpha\succ 0} k_\alpha \alpha = -\lambda$. For each $\sigma \in W$, we define $\mathrm{sn}(\sigma)$ to be $(-1)^t$ if σ can be expressed as a product of t reflections σ_α. Thus $\mathrm{sn}(\sigma) = \det\sigma$ (viewed as an element of Aut H^*). Now define $\omega(\lambda) = \sum_{\sigma\in W} \mathrm{sn}(\sigma)e(\sigma\lambda)$.

Using Harish-Chandra's theorem, we know that $Z(\lambda)$ has a composition series, with each factor of the form $V(\mu)$ where $\mu \prec \lambda$ and $\mu \sim \lambda$. This can be used to express $ch_{V(\lambda)}$ as a $\mathbb{Z}$-linear combination of the $ch_{Z(\mu)}$s, where $q \cdot ch_{Z(\lambda)} = e(\lambda+\delta)$. In turn, this gives us several useful formulae for computing dimensions and multiplicities.

Theorem 8 (Kostant's Formula)**.** $m_\lambda(\mu) = \sum_{\sigma\in W} sn(\sigma)\ p(\mu+\delta-\sigma(\lambda+\delta))$

Theorem 9 (Weyl's Character Formula)**.** *Let $\lambda \in \Lambda^+$. Then $\omega(\delta)ch_\lambda = \omega(\lambda+\delta)$, or $ch_\lambda = \frac{\sum_\sigma sn(\sigma)e(\sigma(\lambda+\delta))}{\sum_\sigma sn(\sigma)e(\sigma\delta)}$*

Corollary 2 (Weyl's Dimension Formula)**.** *$deg\ \lambda\ (\overset{def}{=} dim\ V(\lambda)) = \prod_{\alpha\succ 0} \frac{(\lambda+\delta,\alpha)}{(\delta,\alpha)}$*

Theorem 10 (Steinberg's Formula)**.** *Let $\lambda', \lambda'' \in \Lambda^+$, and let $V(\lambda')\otimes V(\lambda'') = \sum_{\lambda\in\Lambda^+} c_\lambda V(\lambda)$. Then*

$$c_\lambda = \sum_{\sigma,\tau\in W} sn(\sigma\tau)p(\lambda + 2\delta - \sigma(\lambda'+\delta) - \tau(\lambda''+\delta))$$

9. Chevalley Bases, Admissible Lattices, and Chevalley Groups

We return to (L, Φ). Here we can show that there exists a *Chevalley basis* of L, which by definition is a basis $\{x_\alpha, h_i : \alpha \in \Phi, 1 \le i \le l\}$, satisfying (1) $[x_\alpha, x_{-\alpha}] = h_\alpha$ ($\in$ the $\mathbb{Z}$-span of the h_i's), and (2) if $\alpha, \beta, \alpha+\beta \in \Phi$, and $[x_\alpha, x_\beta] = c_{\alpha\beta}x_{\alpha+\beta}$, then $c_{\alpha\beta} = -c_{-\alpha,-\beta}$. Such a basis automatically gives $c_{\alpha\beta} \in \mathbb{Z}$. In fact, the structure constants of a Chevalley basis are all integers.

- We now fix a Chevalley basis. The $\mathbb{Z}$-span of this basis, denoted $L(\mathbb{Z})$, is an admissible lattice, as below.

Now we display an important lattice (that is, the $\mathbb{Z}$-span of a basis) in $\mathfrak{U}(L)$. Fix some ordering $\{\alpha_1, \dots, \alpha_m\}$ of Φ^+, and let $A = (a_1, \dots, a_m), B = (b_1, \dots, b_l), C = (c_1, \dots, c_m)$ be sets of non-negative integers. We then define

$$f_A = \frac{x_{-\alpha_1}^{a_1}}{a_1!} \cdots \frac{x_{-\alpha_m}^{a_m}}{a_m!},\ h_B = \binom{h_1}{b_1} \cdots \binom{h_l}{b_l},\ e_C = \frac{x_{\alpha_1}^{c_1}}{c_1!} \cdots \frac{x_{\alpha_m}^{c_m}}{c_m!}$$

Theorem 11 (Kostant)**.** *Let* $\mathfrak{U}(L)_{\mathbb{Z}}$ *be the subring of* $\mathfrak{U}(L)$ *with 1, generated by all* $x_\alpha^t/t!, \alpha \in \Phi, t \geq 0$*. Let* $\mathfrak{B}$ *be the lattice (thanks to PBW) with* $\mathbb{Z}$*-basis consisting of the set* $\{f_A h_B e_C : A, B, C\}$*. Then* $\mathfrak{B} = \mathfrak{U}(L)_{\mathbb{Z}}$*.*

We now look at finite dimensional L-modules V, and lattices in them which we can take into arbitrary fields via tensoring. A lattice M in V is said to be *admissible* if it is invariant under $\mathfrak{U}(L)_{\mathbb{Z}}$. Such a lattice always exists in V, and $M_\mu \overset{\text{def}}{=} M \cap V_\mu$ is a lattice in V_μ. Hence $M = \oplus_{\mu \in \Pi(V)} M_\mu$.

- Henceforth we assume V is a *faithful* L-module.

We can then show that the stabilizer of M in L is also a lattice, denoted by L_V. In fact, L_V depends only on V, and not on M. Further, $\exists M_{max}$ and M_{min} (the maximal and minimal admissible lattices) so that every admissible lattice M satisfies : $M_{min} \subset M \subset M_{max}$.

We can generalize this notion to an arbitrary field K by tensoring : $L_V(K) = L_V \otimes K,\ V(K) = M \otimes K$. Now we construct certain matrix groups here. Denote the representation of L on V by ϕ. Now, the endomorphism $x_{\alpha,t}$ $(= \phi(x_\alpha)^t/t!)$ leaves M, and hence $V(K)$, invariant. More generally, if T is an indeterminate, then look at $\theta_\alpha(T) = \sum_{t=0}^{\infty} T^t x_{\alpha,t}$, and specialize T to any $c \in K$. We then get an automorphism $\theta_\alpha(c)$ of $V(K)$. The group $G_V(K)$ $(\subset SL(V(K)))$, generated by $\theta_\alpha(c)$ $(c \in K, \alpha \in \Phi)$ is called the *Chevalley Group of type* $\Lambda(V)$, *adjoint* if $\Lambda(V) = \Lambda_r$, *universal* if $\Lambda(V) = \Lambda$. Since here $V(K)$, and hence the matrices $x_{\alpha,t}$ actually do depend on the choice of M in V, hence so also does $G_V(K)$.

Comments : Jacobson, in [4], uses the concept of a *splitting Cartan subalgebra (s.c.s.)* to develop the theory. An s.c.s. is a CSA H so that the roots of the characteristic polynomial of $\text{ad}_L h$ are in F, for every $h \in H$. This is used as an alternative to working in algebraically closed fields. He also extends several of the results to arbitrary fields of characteristic zero. Further, he uses CSA$'$s and weight spaces of L-modules to prove Cartan's Criteria, whereas the approach above, taken from [1], assumes F to be algebraically closed, of characteristic 0, and then uses less machinery.

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