CENTER AND REPRESENTATIONS OF INFINITESIMAL HECKE ALGEBRAS OF $\mathfrak{sl}_2$

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ABSTRACT. In this paper, we compute the center of the infinitesimal Hecke algebras H_z associated to $\mathfrak{sl}_2$; then using nontriviality of the center, we study representations of these algebras in the framework of the BGG category $\mathcal{O}$. We also discuss central elements in infinitesimal Hecke algebras over $\mathfrak{gl}_n$ and $\mathfrak{sp}(2n)$ for all n. We end by proving an analogue of Duflo's theorem for H_z.

1. INTRODUCTION

1.1. **Background.** In the paper [EGG], the authors introduce new families of algebras which they call continuous Hecke algebras and infinitesimal Hecke algebras (the latter being subalgebras of the former). They do this as a way to provide a unifying treatment of the representation theories of various algebras such as Drinfeld-Lusztig degenerate affine Hecke algebras, and symplectic reflection algebras of [EG] (which include rational Cherednik algebras). We briefly recall their definition.

We fix once and for all a ground field k (which will be assumed to be algebraically closed of characteristic zero), and let G be a reductive algebraic group over k (not necessarily connected), and $\rho : G \to GL(V)$ a finite-dimensional representation. Then one can form the semi-direct product algebra $TV \rtimes \mathcal{O}(G)^*$, where TV is the tensor algebra of V and $\mathcal{O}(G)^*$ is the algebra of algebraic distributions on G.

Now given a skew-symmetric G-equivariant k-linear pairing $\gamma : V \times V \to \mathcal{O}(G)^*$, the authors define in [EGG] an algebra $H_\gamma(G)$, as a quotient of $TV \rtimes \mathcal{O}(G)^*$ by the relations: $[x, y] = \gamma(x, y)$ for all $x, y \in V$.

One has an algebra filtration on $H_\gamma(G)$ obtained by assigning to V the filtration degree 1, and 0 to $\mathcal{O}(G)^*$. Hence we get a natural map : $H_0(G) \twoheadrightarrow \mathrm{gr}(H_\gamma(G))$, and $H_\gamma(G)$ is called a *continuous Hecke algebra* if and only if this map is an isomorphism (the *PBW property*).

If one takes distributions supported on $1 \in G$, instead of $\mathcal{O}(G)^*$, the resulting algebra is called an *infinitesimal Hecke algebra* if the corresponding PBW property is satisfied. Hence this algebra is a quotient of $TV \rtimes \mathfrak{Ug}$ by

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a $\mathfrak{g}$-invariant relation: $[x,y] = \gamma(x,y)$, where $\gamma : V \times V \to \mathfrak{U}\mathfrak{g}$. It is also a deformation of $\mathfrak{U}\mathfrak{g} \ltimes \mathrm{Sym}(V) = \mathfrak{U}(\mathfrak{g} \ltimes V)$.

If G is connected, one gets a continuous Hecke algebra if and only if the corresponding algebra is an infinitesimal Hecke algebra. When G is a discrete group, one recovers the symplectic reflection algebras of [EG] in this way. So in a sense, symplectic reflection algebras and infinitesimal Hecke algebras lie on opposite sides of the spectrum.

In this paper, we will mainly be concerned with the question of computing the center of the infinitesimal Hecke algebras of SL_2, and the spectral decomposition for the analogue of the BGG category $\mathcal{O}$ for these, over the center. It is well-known ([BG]) that the center of symplectic reflection algebras is either trivial, or the whole algebra is a finitely generated module over its center (when the one-dimensional parameter is 0).

It seems to us that one has a completely opposite picture for infinitesimal Hecke algebras. Namely, infinitesimal Hecke algebras of SL_2 and GL_2 have nontrivial (but not "large") centers, so the category $\mathcal{O}$ has a spectral decomposition. We expect similar phenomena for infinitesimal Hecke algebras of higher rank as well.

1.2. **Results.** We now describe (some of) the concrete results of the paper.

For the most part, we will work with $\mathfrak{g} = \mathfrak{sl}_2$ and $V = k^2$, the standard representation with basis vectors x, y. In this case we have $H_z = (TV \rtimes \mathfrak{U}\mathfrak{g})/([x,y] - z)$, where z is a central element of $\mathfrak{U}\mathfrak{g}$.

- We prove (Theorem 2.1) that the center of H_z is freely generated by a nontrivial quadratic element for any value of z (quadratic with respect to the filtration that assigns degree 1 to V and 0 to $\mathfrak{g}$). This central element also exists for $\mathfrak{g} = \mathfrak{sp}(2n)$ and $V = k^{2n}$, at least when the deformation parameter is trivial.
- Moreover, it is shown (also in Theorem 2.1) that this algebra has no outer derivations for nonzero z, and if $z = 0$, then the Euler derivation generates the outer derivations.
- The commutator quotient of H_z turns out to be finitely generated over the center (Theorem 3.1); it is generated by $\deg(z)$ elements (where we look at z as a polynomial in the Casimir element).

We also briefly consider the infinitesimal Hecke algebra associated with $\mathfrak{g} = \mathfrak{gl}_n$ and $V = \mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h} = k^n$ is the standard representation. In this case (at least when $\beta \equiv 0$), the center of H_β contains at least two (algebraically independent) quadratic elements. Moreover, we prove that for any β, the center of H_β is nontrivial (see Proposition 4.2).

We then consider some consequences of the nontriviality of the center of H_z, such as the spectral decomposition of the BGG category $\mathcal{O}$, the Harish-Chandra homomorphism, and so on. We also describe the multiplicities of irreducible modules in Verma modules when the parameter is a scalar.

Finally, we prove an analogue of Duflo's theorem on primitive ideals for the infinitesimal Hecke algebra H_z, by utilizing a theorem of Ginzburg [Gi].

2. The center

Let us start by recalling the exact definition of infinitesimal Hecke algebras for $\mathfrak{g} = \mathfrak{sp}(2n)$ and $V = k^{2n}$. Denote by ω the symplectic form on V; one then identifies $\mathfrak{g}$ with $\mathfrak{g}^*$ via the pairing $\mathfrak{g} \times \mathfrak{g} \to k,\ (A, B) \mapsto \mathrm{Tr}(AB)$, and $\mathrm{Sym}\,\mathfrak{g}$ with $\mathfrak{U}\mathfrak{g}$ via the symmetrization map. Then for any $x, y \in V,\ A \in \mathfrak{g}$, one writes

$$\omega(x, (1 - T^2A^2)^{-1}y) \det(1 - TA)^{-1} = l_0(x, y)(A) + l_2(x, y)(A)T^2 + \dots$$

where $l_i(x, y) \in \mathrm{Sym}\,\mathfrak{g} \cong \mathfrak{U}\mathfrak{g}$ is a polynomial in $\mathfrak{g}$ for each i.

For each polynomial $\beta = \beta_0 + \beta_2T^2 + \beta_4T^4 + \dots \in k[T]$, in [EGG] the authors define the algebra H_β to be the quotient of $TV \rtimes \mathfrak{U}\mathfrak{g}$ by the relations

$$[x, y] = \beta_0 l_0(x, y) + \beta_2 l_2(x, y) + \dots$$

for all $x, y \in V$. It is proved in [EGG] that this yields an infinitesimal Hecke algebra (i.e., the PBW property holds). Also note that setting $\beta \equiv 0$ yields the "undeformed" case: $H_0(\mathfrak{sp}(2n)) = \mathfrak{U}(\mathfrak{sp}(2n) \ltimes k^{2n})$.

We will restrict ourselves to the case $n = 1$. Let us describe more explicitly a presentation (via generators and relations) of this algebra (e.g., see [EGG, Example 4.12]). We have $V = kx \oplus ky$, with $[h, x] = x, [h, y] = -y$ (where e, f, h form the standard basis for $\mathfrak{sl}_2$, with standard relations $[h, e] = 2e, [h, f] = -2f$, and $[e, f] = h$). Then this algebra is a quotient of $TV \rtimes \mathfrak{U}\mathfrak{g}$ by the relation $[x, y] = z$, where z is a central element of $\mathfrak{U}\mathfrak{g}$. We will denote this algebra by H_z.

A few years before the paper [EGG] appeared, the representation theory of H_z was studied in great detail by A. Khare in [Kh]. In particular, he proved the PBW property there, the proof being completely different from the one in [EGG].

We start by determining the center and derivations of the algebra H_z. We have the following

Theorem 2.1.

(1) *The center of H_z is a polynomial algebra in one variable, and the generating central element has filtration degree* 2.
(2) *If $z = 0$, then $H^1(H_0, H_0)$ (Hochschild cohomology) is a rank one free module over the center, and if $z \neq 0$, then every derivation of H_z is inner.*

We prove the theorem in several steps, showing several small results along the way. It is noteworthy that if we replace H_z by its natural quantization, then (if $z \neq 0$) the center becomes trivial; see [GK, Theorem 11.1].

2.1. **An anti-involution and a central element.** First, recall an (algebra) anti-isomorphism of H_z, called j, defined in [Kh]:

$$j(x) = y, \quad j(y) = x, \quad j(h) = h, \quad j(e) = -f, \quad j(f) = -e.$$

More generally, let us also write down a basis for $\mathfrak{sp}(2n)$:

(1) $u_{jk} := e_{jk} - e_{k+n,j+n}, \quad v_{jk} := e_{j,k+n} + e_{k,j+n}, \quad w_{jk} := e_{j+n,k} + e_{k+n,j}$

We now claim

Lemma 2.1. *Let $\Delta = h^2 + 4ef - 2h$ be a multiple of the Casimir element of $\mathfrak{sl}_2$.*

1. *The map j, taking $u_{jk} \leftrightarrow u_{kj}, v_{jk} \leftrightarrow -w_{jk}$, and $e_i \leftrightarrow e_{i+n}$ (in $V = k^{2n}$) for all $1 \leq i \leq n$, is an anti-involution of $\mathfrak{U}(\mathfrak{sp}(2n) \ltimes TV)$.*
2. *It also factors through an anti-involution of $H_\beta(\mathfrak{sp}(2n))$ for scalar parameters β_0, as well as for all H_z (here, $n = 1$ and z is any central element in $\mathfrak{Ug}$).*
3. *For $n = 1$ and any z, the map j fixes the following elements in H_z: $h, \ \Delta, \ z, \ t := ey^2 + hxy - fx^2$.*
4. *Moreover, the element $t - \frac{1}{2}hz$ commutes with e, f, h in H_z.*

Proof.

1. Consider $\mathfrak{sp}(2n) \hookrightarrow \mathfrak{gl}(2n)$. Then on $\mathfrak{sp}(2n)$, j is the map $j(X) := -\tau X \tau^{-1}$, where $\tau = \tau^{-1} = \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$. On V, j is the map $v \mapsto \tau \cdot v$. One now easily checks that this yields an anti-involution of $\mathfrak{Ug} \ltimes TV$.
2. For a scalar parameter β_0, the added relations we have to quotient $\mathfrak{U}(\mathfrak{sp}(2n)) \ltimes TV$ by, are: $[e_i, e_k] = \beta_0 \delta_{|i-k|,n}(i-k)/n$. These are clearly preserved by j. Similarly, j preserves $[x, y]$ as well as $z = z(\Delta)$.
3. That j fixes h and Δ (and hence z) is easy. Now applying j to t, we get
$$j(t) = -x^2 f + xyh + y^2 e = hxy + ey^2 - fx^2 - [e, y^2] - (-[f, x^2]).$$
But the last two terms cancel each other, since
$$[e, y^2] = [e, y]y + y[e, y] = xy + yx = x[f, x] + [f, x]x = [f, x^2],$$
so this element is indeed fixed by j.
4. Note that
$$[e, t] = e(xy + yx) - 2exy + hx^2 - hx^2 = eyx - exy = -ez,$$
so we see that $[e, t - \frac{1}{2}hz] = 0$. Moreover, $t - \frac{1}{2}hz$ also commutes with h. Finally, applying $-j$ to $et = te$, we get $tf = ft$.

$\square$

Though we do not use it in this manuscript, we now generalize the above central element (note that $t \in \mathfrak{Z}(H_0)$) for all n:

Proposition 2.1. *For any n, the "undeformed" algebra $H_0(\mathfrak{sp}(2n))$ has at least one central element, namely:*

$$t_n := \sum_{1 \leq r,s \leq n} (v_{rs} e_{r+n} e_{s+n} + u_{rs} e_s e_{r+n} + u_{sr} e_r e_{s+n} - w_{rs} e_r e_s)$$

where $\{e_i : 1 \leq i \leq 2n\}$ is the standard basis of $V = k^{2n}$.

Note that if $n = 1$, then $t_n = 2t$.

Proof. We outline the steps of this long-winded but straightforward (and heavily computational) proof. Define $a_{rs} := v_{rs} e_{r+n} e_{s+n} - w_{rs} e_r e_s$, and $b_{rs} := u_{rs} e_s e_{r+n} + u_{s,r} e_r e_{s+n}$ for all r, s. The steps of the verification are:

(1) The anti-involution j (in Lemma 2.1) preserves a_{rs}, b_{rs} for all r, s; hence it preserves t_n too.
(2) $[e_i, a_{rs} + b_{rs}] = 0$ for all r, s and $1 \leq i \leq n$; hence the same holds by replacing e_i by e_{i+n}, using j.
(3) $\left[u_{pq}, \sum_{r,s=1}^n a_{rs}\right] = \left[u_{pq}, \sum_{r,s=1}^n b_{rs}\right] = 0\ \forall p, q$.
(4) $[v_{pq}, t_n] = 0\ \forall p, q$, whence $[w_{pq}, t_n] = 0$ using j.

□

2.2. **Commutators of powers of the Casimir element.** By Lemma 2.1, j fixes the subalgebra generated by the elements t, h, and $\mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ (the center of $\mathfrak{U}\mathfrak{g}$). Hence our goal now is to exhibit an element from this algebra which will commute with e, x, h (and hence with y, f, applying j), and therefore will lie in the center of H_z.

We now compute that

$$\begin{aligned} [x, t] &= e(zy + yz) - x^2 y + hxz + yx^2 \\ &= 2ezy - e[z, y] + hzx - h[z, x] - (2zx - [z, x]) \\ &= 2ezy - 2zx + [z, x] - e[z, y] + hzx - h[z, x], \end{aligned}$$

and

$$[x, \frac{1}{2}hz] = -\frac{1}{2}xz + \frac{1}{2}h[x, z] = -\frac{1}{2}zx + \frac{1}{2}[z, x] + \frac{1}{2}h[x, z],$$

so we get that

$$[x, t - \frac{1}{2}hz] = 2ezy - \frac{3}{2}zx + \frac{1}{2}[z, x] - e[z, y] - \frac{1}{2}h[z, x] + hzx. \tag{2}$$

Denote this element by ω. We now want to produce an element q_z in the center of $\mathfrak{U}\mathfrak{g}$ such that $[x, q_z] = \omega$, for then $t - \dfrac{1}{2}hz - q_z$ will be a central element in H_z.

To show this, we will analyze $\mathfrak{sl}_2$-maximal vectors in $\mathfrak{U}\mathfrak{g}$ (i.e., vectors annihilated by the adjoint action of e) and in H_z, of various weights. A first step in looking at such things is realizing that H_z is a direct sum of finite-dimensional $\mathfrak{g}$-modules (this is true for any infinitesimal Hecke algebra):

Lemma 2.2. *Given Lie algebras $\mathfrak{g} \neq 0, \mathfrak{h}'$ that are semisimple and abelian respectively, define $\mathfrak{h} := \mathfrak{h}_{\mathfrak{g}} \oplus \mathfrak{h}'$, the Cartan subalgebra of the reductive Lie algebra $\widetilde{\mathfrak{g}} := \mathfrak{g} \oplus \mathfrak{h}'$. If V is an $\mathfrak{h}$-semisimple completely reducible $\widetilde{\mathfrak{g}}$-module, then so is $\mathcal{A} := \mathfrak{U}\widetilde{\mathfrak{g}} \ltimes TV$.*

Corollary 2.1. *Every infinitesimal Hecke algebra is such a direct sum, and of finite-dimensional $\widetilde{\mathfrak{g}}$-modules.*

The corollary is obvious since such algebras are quotients of $\mathcal{A}$ for some finite-dimensional V (so that all "highest weights" of summands in $\mathcal{A}$ are sums of two dominant integral weights for $\mathfrak{g}$, one from each tensor factor $\mathfrak{U}\widetilde{\mathfrak{g}}, TV$).

Proof of Lemma 2.2. The $\mathfrak{h}$-semisimplicity is obvious. It is also easy to check that $\mathcal{A}$ is graded: $\mathcal{A} = \bigoplus_{n,I} \mathcal{A}_{n,I}$. Here, $\mathcal{A}_{n,I} := \mathfrak{U}\mathfrak{g} \otimes (k \cdot I) \otimes T^n V$, where $n \geq 0$ and I runs over some fixed basis of $\operatorname{Sym} \mathfrak{h}'$. Moreover, each summand has an increasing filtration by finite-dimensional $\widetilde{\mathfrak{g}}$-modules, using the standard filtration on $\mathfrak{U}\mathfrak{g}$:

$$\mathcal{A}_{n,I} = \mathcal{A}^{\bullet}_{n,I} = (F^{\bullet}\mathfrak{U}\mathfrak{g}) \otimes (k \cdot I) \otimes T^n V.$$

Using Zorn's lemma, one easily shows that a union of finite-dimensional (and hence completely reducible) $\mathfrak{h}$-semisimple $\widetilde{\mathfrak{g}}$-modules is itself completely reducible. But then, so is $\mathcal{A} = \bigoplus_{n,I} \mathcal{A}_{n,I}$. $\square$

Next, we have

Lemma 2.3. *The map $\varphi : k[X,Y] \to \mathfrak{U}\mathfrak{g}$, sending $X^m Y^n \mapsto \Delta^m e^n$, is a vector space isomorphism onto the set of maximal vectors in (the $\operatorname{ad}\mathfrak{g}$-module) $\mathfrak{U}\mathfrak{g}$.*

Proof. The injectivity is obvious. Now let α be such a maximal vector. We may assume without loss of generality, that α is in one weight space. We proceed by induction on the weight. If α is divisible by e and $\alpha = ge$ for some $g \in \mathfrak{U}\mathfrak{g}$, then we claim that $[g,e] = 0$ too. For we have

$$0 = [\alpha, e] = [ge, e] = [g,e]e \Rightarrow [g,e] = 0 \tag{3}$$

since $\mathfrak{U}\mathfrak{g}$ is an integral domain. (We will use this **dividing trick** later in this manuscript.)

Thus, we now assume that α is not divisible by e, so if we write it in the usual PBW basis, it will contain a monomial a containing no e. Thus a has non-positive weight, and since (by Lemma 2.2, with $\mathfrak{h}' = V = 0$) $\mathfrak{U}\mathfrak{g}$ is a direct sum of finite dimensional $\mathfrak{g}$-modules (under the adjoint action), it has no maximal vectors of negative weight. Therefore a has weight 0, and hence is annihilated by $\mathfrak{g}$ (from the structure theory of finite-dimensional $\mathfrak{sl}_2$-modules; see [Hu]). Hence, a is a central element. $\square$

Remark 2.1. Using the anti-involution j, we can get a similar description of elements which commute with f, as an algebra generated by f, Δ.

Recall that $\Delta = h^2 + 4ef - 2h$ is the Casimir element. Our next step will be to compute the commutators of powers of Δ, with x and y. For $n = 1$, we have

$$\begin{aligned}[\Delta, x] &= hx + xh + 4ey - 2x = (2h - 3)x + 4ey,\\ [\Delta, y] &= -hy - yh + 4xf + 2y = -(2hy + y) + 4(fx - y) + 2y\\ &= (-2h - 3)y + 4fx.\end{aligned}$$

(Note that $(\operatorname{ad} x)^3(\Delta) = (\operatorname{ad} y)^3(\Delta) = 0$ in H_0.) We next extract information about these commutators.

Proposition 2.2. *There exist polynomials $f_n, g_n \in \mathbb{Z}[T] \subset k[T]$ for all n, such that*

$$[\Delta^n, x] = (f_n(\Delta)h + g_n(\Delta))x + 2f_n(\Delta)ey$$

and for y, we have

$$[\Delta^n, y] = 2f_n(\Delta)fx + (g_n(\Delta) - f_n(\Delta)h)y.$$

The polynomials f_n, g_n are inductively defined as follows:

$$f_1(T) = 2, \qquad f_{n+1}(T) = 2T^n + (T-1)f_n(T) - 2g_n(T), \tag{4}$$

$$g_1(T) = -3, \qquad g_{n+1}(T) = -3T^n + (T+3)g_n(T) - 2Tf_n(T). \tag{5}$$

Proof. We show the various assertions made above.

(1) Note for any $g \in \mathfrak{Ug}$ that $[g, x]$ is, first, a $\mathfrak{Ug}$-linear combination of x and y only. Next, $[\Delta^n, x]$ is also a maximal vector for $\mathfrak{g}$, of weight 1. Thus, if it equals $\alpha x + \beta y$, then α has weight 0 and β has weight 2. We now write $[e, [\Delta^n, x]] = 0$ to get

$$0 = [e, \alpha x + \beta y] = ([e, \alpha] + \beta)x + [e, \beta]y.$$

By the PBW theorem, the coefficients of x, y therefore vanish. Thus, $\beta \in \mathfrak{Ug}$ is maximal of weight 2, hence is a central element times e (by Lemma 2.3).

Suppose we write $\beta = 2f_n(\Delta)e$ for some polynomial f_n in Δ. Then we get $[e, \alpha] + 2f_n(\Delta)e = 0$, whence we get that $\operatorname{ad} e(\alpha) = -2f_n(\Delta)e$.

Since $[e, f_n(\Delta)h] = f_n(\Delta) \cdot (-2e)$, hence we see that $\alpha - f_n(\Delta)h$ is killed by e. Moreover, it is a weight vector of weight 0, so it equals $g_n(\Delta)$ for some polynomial g_n.

Finally, the given initial values of f_1, g_1 do indeed satisfy the commutation relations that we verified above.

Remark 2.2. We will sometimes omit Δ from $f_n(\Delta)$, but this should not cause any confusion.

(2) We now compute the polynomials f_n, g_n inductively. We have

$$\begin{aligned}
[\Delta^{n+1}, x] &= \Delta^n[\Delta, x] + [\Delta^n, x]\Delta \\
&= \Delta^n(2h-3)x + \Delta^n 4ey \\
&\quad +(f_n(\Delta)h + g_n(\Delta))x\Delta + 2f_n(\Delta)ey\Delta \\
&= (\Delta^n(2h-3) + f_n(\Delta)\Delta h + g_n(\Delta)\Delta)x \\
&\quad +(2f_n(\Delta)\Delta e + \Delta^n 4e)y \\
&\quad -(f_n(\Delta)h + g_n(\Delta))(2h-3)x - 2f_n(\Delta)e4fx \\
&\quad -(f_n(\Delta)h + g_n(\Delta))4ey - 2f_n(\Delta)e(-2h-3)y.
\end{aligned}$$

Grouping all elements containing y, we get the coefficient of y to be

$$\begin{aligned}
&4\Delta^n e + 2f_n(\Delta)\Delta e - 4(f_n(\Delta)h + g_n(\Delta))e + 4f_n(\Delta)eh + 6f_n(\Delta)e \\
=&2(2\Delta^n - 2g_n(\Delta))e + 2f_n(\Delta)(\Delta e - 2he + 2eh + 3e) \\
=&2(2\Delta^n + f_n(\Delta)(\Delta-1) - 2g_n(\Delta))e,
\end{aligned}$$

whence we get that the coefficient of y is

$$2f_{n+1}(\Delta)e = 2(2\Delta^n + f_n(\Delta)(\Delta-1) - 2g_n(\Delta))e.$$

This proves the relation for f_{n+1}. Similarly, grouping all elements containing x, we get the coefficient of x to be

$$\begin{aligned}
&\Delta^n(2h-3) + f_n(\Delta)\Delta h + g_n(\Delta)\Delta - g_n(\Delta)2h \\
+\ &3g_n(\Delta) - 2f_n(\Delta)h^2 + 3f_n(\Delta)h - 8f_n(\Delta)ef.
\end{aligned}$$

Note that the sum of the last three terms is $-f_n(\Delta)h - 2f_n(\Delta)\Delta$. Hence we get that the coefficient is

$$\begin{aligned}
&f_{n+1}(\Delta)h + g_{n+1}(\Delta) \\
=\ &\Delta^n(2h-3) + f_n(\Delta)(\Delta h - h - 2\Delta) + g_n(\Delta)(\Delta - 2h + 3).
\end{aligned}$$

Subtracting $f_{n+1}(\Delta)h$ from both sides (and using the formula above), we conclude that

$$\begin{aligned}
g_{n+1}(\Delta) =&\Delta^n(2h-3) + f_n(\Delta)(\Delta h - h - 2\Delta) + g_n(\Delta)(\Delta - 2h + 3) \\
&- \Delta^n 2h - f_n(\Delta)(\Delta h - h) + 2g_n(\Delta)h \\
=&-3\Delta^n - 2f_n(\Delta)\Delta + g_n(\Delta)(\Delta + 3).
\end{aligned}$$

Thus, we have shown the inductive formulae.

(3) Computations with y are directly analogous to the ones above.

□

As a corollary of these calculations, we have

Corollary 2.2.

(1) *f_n and g_n are polynomials of degree $n-1$, with top coefficients $2n$ and $-n(2n+1)$ respectively.*
(2) *The f_n's (or g_n's) form a basis of $\mathfrak{Z}(\mathfrak{Ug})$.*
(3) *The only elements from $\mathfrak{Z}(\mathfrak{Ug})$ that commute with x or y are scalars.*

Proof.

(1) (At first, recall that $\mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ is generated by Δ; see [Hu].) All these facts are proved simultaneously by induction on n; they clearly hold for $n = 1$. Suppose they now hold for n. The inductive definitions then show that f_{n+1} has leading term arising from $2T^n + T\cdot(2nT^{n-1} + \cdots)$. Hence $f_{n+1} = 2(n+1)T^n + \cdots$.

Similarly, the top coefficient of g_{n+1} is the coefficient of T^n (unless it vanishes), and this equals

$$-3 - n(2n+1) - 2\cdot 2n = -(3 + 2n^2 + n + 4n) = -(n+1)(2(n+1)+1)$$

as claimed. Hence we are done by induction.

(2) This is because both denote a unipotent change of basis from the usual $\{1, T, T^2, \dots\}$, and the map sending T to Δ is an isomorphism $: k[T] \to \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$.

(3) Note that an element from $\mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ commutes with x if and only if it commutes with y (applying the anti-involution j and noting that j fixes Δ). Thus, we need to show that if $\sum_{i>0} a_i\Delta^i$ commutes with x, it must be 0. But we have

$$\sum_i a_i[\Delta^i, x] = \left(\sum_i a_i(f_i(\Delta)h + g_i(\Delta))\right)\cdot x + 2\left(\sum_i a_i f_i(\Delta)\right) ey.$$

Both coefficients (i.e., of x and y) must therefore be zero. Since the associated graded of H_z is an integral domain, hence $\sum_i a_i f_i(\Delta) = 0$; since the f_i's form a basis of the center, we get each a_i to be zero, and we are done.

□

We have the following proposition, which will be used later.

Proposition 2.3. *Suppose $\psi, \eta, \alpha, \beta$ are central in $\mathfrak{U}\mathfrak{g}$. Then the following are equivalent:*

(1) $2\psi ey + (h\psi + \eta)x = [\alpha, x] + \beta x$.
(2) $2\psi fx + (\eta - h\psi)y = [\alpha, y] + \beta y$.
(3) $\psi = \sum_{i>0} a_i f_i(\Delta)$, $\alpha = \sum_{i\geqslant 0} a_i\Delta^i$, $\beta = \eta - \sum_i a_i g_i(\Delta)$ *for some scalars* $a_i \in k$.

Thus, either of the first two equations has a unique solution in α, β (modulo the constant term in α).

Proof. We first prove that the last statement implies the first two. Given ψ, η and α, β as in the last part, we compute:

$$\begin{aligned}[\alpha, x] + \beta x =& \sum_i [a_i(f_i(\Delta)h + g_i(\Delta))x + 2a_i f_i(\Delta)ey] + \eta x - \sum_i a_i g_i(\Delta)x \\ =& \sum_i a_i f_i(\Delta) \cdot hx + 2\sum_i a_i f_i(\Delta) \cdot ey + \eta x \\ =& 2\psi ey + (h\psi + \eta)x.\end{aligned}$$

Similarly,

$$\begin{aligned}[\alpha, y] + \beta y =& \sum_i [a_i(g_i(\Delta) - f_i(\Delta)h)y + 2a_i f_i(\Delta)fx] + \eta y - \sum_i a_i g_i(\Delta)x \\ =& -\sum_i a_i f_i(\Delta) \cdot hy + 2\sum_i a_i f_i(\Delta) \cdot fx + \eta y \\ =& 2\psi fx + (\eta - h\psi)y.\end{aligned}$$

To prove that the first two parts imply the last, we first note that the solution set α, β is "additive" in the variables ψ, η. Therefore it suffices to show that if $[\alpha, x] + \beta x = 0$ or $[\alpha, y] + \beta y = 0$, then $\alpha = \beta = 0$.

So suppose $\alpha = \sum_{i \geqslant 0} a_i f_i(\Delta)$. Computing the above expressions, we have

$$[\alpha, x] + \beta x = \sum_{i>0} [a_i(f_i(\Delta)h + g_i(\Delta))x + 2a_i f_i(\Delta)ey] + \beta x.$$

Equating the coefficient of y to zero, since the $f_i(\Delta)$'s form a basis of the center, and since H_z is an integral domain, we get that $a_i = 0\ \forall i$, so $\alpha = a_0 \in k$. But then we are left with $\beta x = 0$, whence $\beta = 0$ too.

A similar proof is for the other equation, using the computations:

$$[\alpha, y] + \beta y = \sum_{i>0} [a_i(g_i(\Delta) - f_i(\Delta)h)y + 2a_i f_i(\Delta)fx] + \eta y - \sum_{i>0} a_i g_i(\Delta) \cdot x.$$

□

Proposition 2.4. *The polynomials f_n, g_n satisfy the recursive relations*

$$\begin{aligned} f_1(T) &= 2, \qquad f_2(T) = 4(T+1), \\ f_{n+2}(T) &= (2T+2)f_{n+1}(T) - (T^2 - 2T - 3)f_n(T), \\ g_1(T) &= -3, \qquad g_2(T) = -10T - 9, \\ g_{n+2}(T) &= (2T+2)g_{n+1}(T) - (4T^{n+1} + 3T^n) - (T^2 - 2T - 3)g_n(T). \end{aligned}$$

Proof. The initial values of f_1, f_2, g_1, g_2 can be computed easily using Proposition 2.2 above. We now compute the expressions for f_n, g_n.

Multiplying the equation in Proposition 2.2 for f_n by $(T+3)$, and that for g_n by 2, and adding these up, the coefficients of g_n on the right cancel each other. Hence we get

$$\begin{aligned} & (T+3)f_{n+1}(T) + 2g_{n+1}(T) \\ = \; & 2T^{n+1} + 6T^n + (T-1)(T+3)f_n(T) - 6T^n - 4Tf_n(T) \\ = \; & 2T^{n+1} + f_n(T)(T^2 - 2T - 3). \end{aligned}$$

But equation (4) for f_n also gives us an expression for $2g_n(T)$ in terms of the f_n's. Hence

$$2g_{n+1}(T) = 2T^{n+1} - f_{n+2}(T) + (T-1)f_{n+1}(T).$$

Replacing this in the previous equation, we get

$$\begin{aligned} & (T+3)f_{n+1}(T) + 2T^{n+1} - f_{n+2}(T) + (T-1)f_{n+1}(T) \\ = \; & 2T^{n+1} + f_n(T)(T^2 - 2T - 3), \end{aligned}$$

from which the relevant equation follows.

We now show the analogous result for g_n. Multiply the equation in Proposition 2.2 for f_n by $2T$, and that for g_n by $(T-1)$. If we now add the two, the coefficients for f_n cancel each other, and we get

$$\begin{aligned} & 2Tf_{n+1}(T) + (T-1)g_{n+1}(T) \\ = \; & 4T^{n+1} - 4Tg_n(T) - 3T^{n+1} + 3T^n + (T+3)(T-1)g_n(T) \\ = \; & (T^{n+1} + 3T^n) + g_n(T)(T^2 - 2T - 3). \end{aligned}$$

Once again, equation (4) for g_n also gives us an expression for $2Tf_{n+1}(T)$ (after a change of variables), namely,

$$2Tf_{n+1}(T) = -3T^{n+1} + (T+3)g_{n+1}(T) - g_{n+2}(T).$$

Substituting in the previous equation, and rearranging terms, we obtain

$$\begin{aligned} g_{n+2}(T) &= -3T^{n+1} - (T^{n+1} + 3T^n) + g_{n+1}(T)((T-1) + (T+3)) \\ &\quad - g_n(T)(T^2 - 2T - 3) \\ &= -(4T^{n+1} + 3T^n) + (2T+2)g_{n+1}(T) - g_n(T)(T^2 - 2T - 3). \end{aligned}$$

□

We end this subsection by explicitly computing f_n and g_n, though we will not use this anywhere else in the paper.

Lemma 2.4. *For all $n \geq 0$, we have*

$$\begin{aligned} f_n(T) &= \frac{1}{2}(T+1)^{\frac{n-1}{2}}[x_+^n - x_-^n], \\ g_n(T) &= T^n - \frac{1}{2}(T+1)^{\frac{n-1}{2}}\left[(\sqrt{T+1}+1)y_+^n + (\sqrt{T+1}-1)y_-^n\right], \end{aligned}$$

where $x_\pm := \sqrt{T+1} \pm 1$, and $y_\pm := \sqrt{T+1} \pm 2$.

Proof. The claim is verified for the f_n's by induction (using: $P(n-2), P(n-1) \Rightarrow P(n)$). Similarly, to verify the claim for the g_n's, we first define $h_n(T) = (g_n(T) - T^n)/T^{n-2} \in \mathbb{Z}[T, T^{-1}]$; one now shows that the equation for the g_n's in Proposition 2.4 is equivalent to: $h_1 = -T(T+3)$, $h_2 = -(T+1)(T+9)$, and

$$h_{n+2} = 2\left(\frac{T+1}{T}\right) h_{n+1} - \frac{(T+1)(T-3)}{T^2} h_n.$$

One checks by induction, that the given formula solves this system. □

2.3. **A central element that generates the center.** Recall that we wanted to write $\omega = [x, t - \frac{1}{2}hz]$ as a commutator of x with a central element of $\mathfrak{U}\mathfrak{g}$ (see the remarks after equation (2)). We first claim that ω can be rewritten as $z[\frac{1}{2}\Delta, x] - (e[z,y] + \frac{1}{2}h[z,x]) + \frac{1}{2}[z,x]$. Indeed, we can simplify this expression to get

$$\begin{aligned} & \frac{1}{2}z((2h-3)x + 4ey) - (e[z,y] + \frac{1}{2}h[z,x]) + \frac{1}{2}[z,x] \\ = \;& (2ezy - \frac{3}{2}zx + hzx) + \frac{1}{2}[z,x] - e[z,y] - \frac{1}{2}h[z,x], \end{aligned}$$

which equals the expression used to define ω. We therefore work with this new expression, and further rewrite it as

$$\begin{aligned} \omega &= z[\frac{1}{2}\Delta, x] - ([z, ey] + [z, \frac{1}{2}hx]) + \frac{1}{2}[z,x] \\ &= z[\frac{1}{2}\Delta, x] - \frac{1}{2}[z, 2ey + hx - x] \\ &= z[\frac{1}{2}\Delta, x] - \frac{1}{4}[z, 4ey + (2h-3)x + x] \\ &= z[\frac{1}{2}\Delta, x] - \frac{1}{4}[z, [\Delta, x] + x] \\ &= \frac{1}{4}(2z[\Delta, x] - z[\Delta, x] + [\Delta, x]z - [z,x]) \\ &= \frac{1}{4}(z[\Delta, x] + [\Delta, x]z - [z,x]) \\ &= \frac{1}{4}[x, z - \Delta z] + \frac{1}{4}(z[\Delta, x] - \Delta[z,x]). \end{aligned}$$

We would like to show that $\omega = [x, q_z]$ for some $q_z \in \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$. If we can now show that there exists $z_0 \in \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ such that $[z_0, x] = z[\Delta, x] - \Delta[z,x] = \Delta xz - zx\Delta$, then $t - \frac{1}{2}hz - q_z$ would be central, where

$$q_z = \frac{1}{4}z - \frac{1}{4}\Delta z - \frac{1}{4}z_0. \tag{6}$$

The existence of z_0 follows from the following result, setting $z' = \Delta$:

Proposition 2.5. *Given $z, z' \in \mathfrak{Z}(\mathfrak{Ug})$, we can find $z_0 = z_0(z, z') \in \mathfrak{Z}(\mathfrak{Ug})$ such that*

$$[z_0, x] = zxz' - z'xz = z'[z, x] - z[z', x].$$

and $z_0(c_1\Delta^m + l.o.t.,\ c_2\Delta^n + l.o.t.) = c_1c_2 \left(\dfrac{m-n}{m+n}\right) \Delta^{m+n} + l.o.t.$

(Here, lower order terms are smaller powers of Δ.)

Proof. First, it is easy to see that the "solution" z_0 is "bilinear" in z, z', in that $z_0(z+r, z'+s) = z_0(z, z') + z_0(z, s) + z_0(r, z') + z_0(r, s)$ for all z, r, z', s central in $\mathfrak{Ug}$. It therefore suffices to prove the result for $z = \Delta^m, z' = \Delta^n$ for some $m, n \geq 0$. But then we have

$$\begin{aligned} zxz' - z'xz &= \Delta^n[\Delta^m, x] - \Delta^m[\Delta^n, x] \\ &= \Delta^n[(f_m(\Delta)h + g_m(\Delta))x + 2f_m(\Delta)ey] \\ &\quad -\Delta^m[(f_n(\Delta)h + g_n(\Delta))x + 2f_n(\Delta)ey] \\ &= 2\psi ey + (h\psi + \eta)x, \end{aligned}$$

where

$$\psi = \Delta^n f_m(\Delta) - \Delta^m f_n(\Delta), \qquad \eta = \Delta^n g_m(\Delta) - \Delta^m g_n(\Delta).$$

In particular, the top degree and coefficient of ψ can be computed from Corollary 2.2:

$$\psi = (2m - 2n)\Delta^{m+n-1} + l.o.t.. \tag{7}$$

By Proposition 2.3 above, $zxz' - z'xz = [\alpha, x] + \beta x$ for some central $\alpha, \beta \in \mathfrak{Ug}$.

Let us also evaluate $\Delta^m y \Delta^n - \Delta^n y \Delta^m$. We get

$$\begin{aligned} zyz' - z'yz &= \Delta^n[\Delta^m, y] - \Delta^m[\Delta^n, y] \\ &= \Delta^n[(g_m(\Delta) - f_m(\Delta)h)y + 2f_m(\Delta)fx] \\ &\quad -\Delta^m[(g_m(\Delta) - f_n(\Delta)h)y + 2f_n(\Delta)fx] \\ &= 2\psi' fx + (\eta' - h\psi')y, \end{aligned}$$

where

$$\psi' = \Delta^n f_m(\Delta) - \Delta^m f_n(\Delta) = \psi, \qquad \eta' = \Delta^n g_m(\Delta) - \Delta^m g_n(\Delta) = \eta.$$

By Proposition 2.3, this equals $[\alpha, y] + \beta y$ for the same α, β as above. Thus,

$$zxz' - z'xz = [\alpha, x] + \beta x, \qquad zyz' - z'yz = [\alpha, y] + \beta y,$$

where $z, z' \in \mathfrak{Z}(\mathfrak{Ug})$. We now prove that $\beta = 0$, as desired. Applying the anti-involution j to the second of the equations, and noting that j preserves z, z' (since it preserves Δ) and sends y to x, we get

$$z'xz - zxz' = [x, \alpha] + x\beta.$$

Comparing with the (negative of the) first equation, we see that

$$[x, \alpha] - \beta x = [x, \alpha] + x\beta,$$

whence we conclude that

$$0 = \beta x + x\beta = 2\beta x - [\beta, x],$$

and the uniqueness result in Proposition 2.3 implies $\beta = 0$, as claimed. Hence $zxz' - z'xz = [\alpha, x]$, as desired.

Moreover, to show the last equation, it suffices by (bi)linearity of z_0 to show that $z_0(\Delta^m, \Delta^n)$ is of the desired form (with $c_1 = c_2 = 1$). But $z_0(\Delta^m, \Delta^n) = \alpha$ above, so we need to compute the top degree and coefficient of α. This comes from ψ (equation (7)) and the "unipotent" change of basis from the f_n's to the Δ^{n-1}'s (Corollary 2.2). Thus, $\psi = \frac{2m-2n}{2(m+n)} f_{m+n} + l.o.t.$. Now use Proposition 2.3 to get that

$$z_0(\Delta^m, \Delta^n) = \alpha = \frac{2m - 2n}{2(m+n)} \Delta^{m+n} + l.o.t..$$

□

As a consequence, we have information about q_z (see equation (6) above):

Corollary 2.3. *For any $z = c\Delta^m + l.o.t.$, $q_z = \frac{-cm}{2(m+1)} \Delta^{m+1} + l.o.t.$.*

Proof. From equation (6), the top term of q_z comes from the last two terms, since $z_0 = z_0(z, \Delta)$ here. If $z = c\Delta^m + l.o.t.$ here, then by Proposition 2.5, the top term is

$$-\frac{1}{4} c\Delta^{m+1} - \frac{1}{4} c \left(\frac{m-1}{m+1} \right) \Delta^{m+1}$$

and this simplifies to the desired form. □

This shows us that the center of H_z is nonempty and contains an element of the form

$$t_z = t - \frac{1}{2} hz - \frac{1}{4} z + \frac{1}{4} \Delta z + \frac{1}{4} z_0,$$

where $z_0 = z_0(\Delta, [x, y])$ as in the above results.

2.4. **Various centralizers and the center.** It just remains to prove that this element t_z generates the whole center of H_z. We do this in steps. First, we describe the elements of H_z which commute with various sets.

Proposition 2.6.

(1) *The centralizer in H_z of $\mathfrak{Ug}$ is freely generated by Δ, t_z.*
(2) (a) *The centralizer of e (i.e., the set of $\mathfrak{sl}_2$-maximal vectors) in H_z is the subalgebra generated by Δ, t_z, e, x.*
    (b) *The centralizer of e and x (together) in H_z is freely generated by t_z, e, x.*
(3) *The centralizer of V in H (for $z = 0$) is freely generated by t, x, y.*

Using the anti-involution j, we get similar results involving f, y.

Proof. In all but the last part, it is enough to show that the prescribed elements generate the centralizer (call it B for this paragraph and the next) in H (i.e., when $z = 0$). This is because all "claimed generators" (Δ, t_z, e, x) in H have lifts to H_z, and any $b \in B$ has a principal symbol in H, a lift of which can be subtracted from b to get $b' \in B$ of "smaller filtration degree" in V. Now proceed by induction.

Moreover, that the prescribed elements freely generate $B \subset H_z$ (except possibly for H_z^e) would follow from the corresponding statement for $z = 0$, since any relation among the lifts in H_z gives a relation in H. Let us start by showing that various elements are algebraically independent in H.

We first note that $t = t_0, \Delta$ are algebraically independent in H, for if $\sum a_{ij} t^i \Delta^j = 0$, then checking the coefficients of x, y (via (8) below) gives the result.

Next, we claim that t, e, x are algebraically independent in H. Indeed, if $\sum a_{q,r,s} e^q t^r x^s = 0$, then consider the highest power of y that occurs (i.e., $2r$ for the highest r); then for this r, consider the highest power of e. Now for these, the highest power of x must have coefficient $a_{qrs} = 0$.

Finally, t, x, y are algebraically independent, for if $\sum a_{qrs} t^q x^r y^s = 0$, then writing this element in terms of the ordered PBW-basis (e, f, h, x, y), we can conclude that $a_{qrs} \equiv 0$.

(1) By passing to the associated graded, it is enough to show the proposition for $z = 0$; thus, we assume that $H_z = H$.

Let a be an element in H which is in the centralizer of $\mathfrak{U}\mathfrak{g}$; without loss of generality, we may assume a to be a weight vector for $\operatorname{ad} h$ and to be homogeneous in x and y, by decomposing it into such components (since $\operatorname{ad} \mathfrak{g}$ preserves this grading degree).

Writing a as a polynomial in the PBW basis above, let n be the smallest power of x appearing in this polynomial. Thus, $a = bx^n$ for some $b \in H$.

Since H is an integral domain, and a, x commute with e, so does b, by the "dividing trick" (3). Since $[h, a] = 0$, hence a is in the 0 weight space, whence the weight of b is $-n$. But no maximal vectors in H may have a negative weight (by $\mathfrak{sl}_2$-theory and Lemma 2.2), whence $n = 0$.

Now let us look at the monomial term of a with the highest power of y. Since a is not divisible by x and is homogeneous, this term must be of the form cy^m, for some $c \in \mathfrak{U}\mathfrak{g}$. We claim that $[e, c] = 0$. This is because $[e, a] = 0$, and upon applying $\operatorname{ad} e$, the power of x in a monomial cannot decrease, and the power of y cannot increase. So we have

$$0 = [e, a] = [e, c]y^m + c[e, y^m] + [e, \dots],$$

and the only monomial with no x's and m y's in it, is $[e, c]y^m$.

We thus get that c is maximal in $\mathfrak{U}\mathfrak{g}$, of weight m. Thus m is even, and c is of the form $e^{m/2}\alpha$ for some central α, by Lemma 2.3.

Let us now consider $a - \alpha t^{m/2}$. By (8) below, the monomials in either term of highest y-degree, are $\alpha e^{m/2} y^m$. Therefore $a - \alpha t^{m/2}$ has highest power of y (without any power of x) in a monomial, strictly less than m. Arguing inductively, we get down to when $m = 0$, leaving us with a vector in $\mathfrak{Ug}$. This commutes with $\mathfrak{Ug}$, so it is central in $\mathfrak{Ug}$, and we are done.

(2) Once again, we may assume that $z = 0$. Let a be a *weight* vector that commutes with e; we may assume that it is also homogeneous (in V, say of degree k, on which we will do induction) and not divisible by x from the right (by the "dividing trick" (3)). Hence it may be written as $a = \sum_{0 \leq i \leq k} c_i y^i x^{k-i}$, where $c_i \in \mathfrak{Ug}\ \forall i$ (and $c_k \neq 0$). Moreover, $[e, a] = 0$ yields: $c_i = -[e, c_{i-1}/i]$, whence $c_i = (\mathrm{ad}(-e))^i (c_0)/i!$. In particular, $c_0 \neq 0$ as well.

Now consider c_k; we claim that c_k is $\mathfrak{g}$-maximal too, since $[e, c_k]$ is the coefficient of y^k in $[e, a]$. By Lemma 2.3, $c_k = \alpha e^n$ (since a is a weight vector), with $\alpha \in \mathfrak{Z}(\mathfrak{Ug})$.

(a) We now prove this part by induction on k. The base case of $k = 0$ follows from Lemma 2.3. Now continue with the above analysis. Note that $\alpha e^n \in \mathfrak{Ug}$ is $\mathfrak{sl}_2$-maximal of weight $2n$, whence $(-1)^k/k! \cdot (\mathrm{ad}\, e)^k (c_0) = \alpha e^n \notin (\mathrm{ad}\, e)^{2n+1}(\mathfrak{Ug})$. In particular, $k \leq 2n$, whence $n \geq \lceil k/2 \rceil$. We now have two cases:

- If k is even, we define $b := a - \alpha e^{n-(k/2)} t^{k/2}$.
- If k is odd, we use $[\Delta, x] =$ (up to scaling) $[4fe + h^2 + 2h, x] = 4ey + 2hx - x$. In this case, we define
$$b := a - \frac{1}{4} \alpha e^{n - \lceil k/2 \rceil} t^{\lfloor k/2 \rfloor} \cdot [\Delta, x].$$

In both cases, $b \in H \cdot x$ by (8) below, and by the "dividing trick" (3), the quotient is a weight vector with *smaller* degree of homogeneity (in V), so we are done by induction.

(b) We continue from where we had stopped before the previous sub-part. Now suppose that a commutes with x as well. Then $\alpha_y = 0$, where $[\alpha, x] = \alpha_x x + \alpha_y y$ (looking at the coefficient of y^{k+1}). But by Proposition 2.2, this can only happen if α is a constant; let us suppose it is 1. Thus, we have $c_k = e^n = (-1)^k/k! \cdot (\mathrm{ad}\, e)^k c_0$, whence c_0 has weight $2(n-k)$.
Next, note that if $[c_0, x] = rx + sy$ with $r, s \in \mathfrak{Ug}$, then $r = 0$ by considering the coefficient of x^{k+1} in $[a, x] = 0$. Now suppose that we write $c_0 = \sum_i e^{n-k+i} f^i p_i(h)$ for polynomials p_i. We claim that the p_i's are constant, for otherwise

$$\begin{aligned} [c_0, x] &= \sum_i e^{n-k+i} \left(i f^{i-1} y \cdot p_i(h) + f^i [p_i(h), x] \right) \\ &= \sum_i e^{n-k+i} \left(i f^{i-1} p_i(h+1) y + f^i (p_i(h) - p_i(h-1)) x \right), \end{aligned}$$

and this is not in $\mathfrak{U}\mathfrak{g} \cdot y$ as claimed above.

Thus, we have $c_0 = \sum_{i=0}^{N} \alpha_i e^{n-k+i} f^i$, say. Now consider a general situation in $\mathfrak{U}(\mathfrak{sl}_2)$: repeatedly applying $\operatorname{ad} e$ to f^i for any i can only lead to e^j (up to a scalar) if $j = i$; and then $\operatorname{ad} e(e^i) = 0$. On the other hand, if $(\operatorname{ad} e)^{2i}(f^i) \in k^\times \cdot e^i$, then $(\operatorname{ad} e)^j(f^i)$ is not a power of e if $j < 2i$ (else $(\operatorname{ad} e)^{j+1}(f^i) = 0$), and vanishes if $j > 2i$.

Thus, if we now consider the "last" summand in c_0, $(\operatorname{ad} e)^k$ must send f^N to e^{k-N}, in order that we get $e^{n-k+N+k-N} = e^n$. But then $k = 2N$, and we get that

$$c_0 = \alpha_{k/2} e^{n-k/2} f^{k/2} + \cdots + \alpha_0 e^{n-k} f^0$$

and $c_i = \operatorname{ad}(-e)^i(c_0)/i!$ is also divisible by e^{n-k} for all i. Hence taking e^{n-k} common on the left, we get that

$$a = e^{n-k}\left(e^k y^k + \cdots + (\alpha_{k/2} e^{k/2} f^{k/2} + \cdots + \alpha_0) x^k\right).$$

In particular, by the "dividing trick" (3), the terms in the parentheses commute with e, x. We can divide by e^{n-k} and then subtract $e^{k/2} t^{k/2}$.

Now note (as an aside) that $t = ey^2 + (hy + fx)x$, so that $t^n - (ey^2)^n \in H \cdot x \ \forall n$. It is also easy to check that $(ey^2)^n - e^n y^{2n} \in H \cdot x$ (e.g., by induction on n). Thus,

(8) $$t^n - e^n y^{2n} \in H \cdot x \ \forall n.$$

In particular, $e^{k/2} t^{k/2} - e^k y^k \in H \cdot x$. Using the "dividing trick" (3), dividing this by x yields a maximal vector a' that commutes with e, x, is a weight vector, and is homogeneous of *smaller* degree than k, whence we are done by induction.

It remains to check the base case; but $k = 0$ would mean the centralizer of e, x in $\mathfrak{U}\mathfrak{g}$, and by Lemma 2.3 and properties of $[\Delta^n, x]$, the only such elements are polynomials in e.

(3) Since both sides of the desired equality are (ad)$\mathfrak{g}$-submodules of H (and H is a direct sum of finite-dimensional $\mathfrak{g}$-modules by Corollary 2.1), it would suffice to show that any $\mathfrak{g}$-maximal vector from H^V belongs to $\mathfrak{Z}(H) \operatorname{Sym} V$. By the previous part, this consists of the y-centralizer of $H^{\{e,x\}} = k[t, e, x]$. Since t, x are in this centralizer, say $\sum_i r_i(t, x) e^i$ commutes with y. Thus,

$$\sum_i r_i(t, x) i e^{i-1} x = \left[\sum_i r_i(t, x) e^i, y\right] = 0$$

and by the algebraic independence of t, e, x, we are done.

□

We can finally conclude the proof of the first part of Theorem 2.1 above.

Proof of the first part. Let a be a central element of H_z. In particular, it commutes with $\mathfrak{g}$, so by Proposition 2.6, it can be written as a polynomial in t_z with coefficients in $\mathfrak{Z}(\mathfrak{U}\mathfrak{g})$. Let κt_z^n be a monomial of top degree. Since $[x, a] = 0$, passing to the associated graded ring (with respect to the filtration), we get that $[x, \kappa] = 0$.

By Corollary 2.2 above, κ is a scalar; so we may disregard the top term of a. Continuing by induction, we see that all coefficients of a are scalars. Hence the center of H_z is generated by t_z, and it is transcendental over k if t is transcendental in $H = H_0$. But this follows by the PBW property. □

We conclude our discussion of the center by giving an explicit formula for the central element when z is (at most) linear. Suppose $[x, y] = a\Delta + b$ for scalars a, b. We therefore want to produce z_0 central in $\mathfrak{U}(\mathfrak{sl}_2)$, such that

$$[z_0, x] = \Delta x(a\Delta + b) - (a\Delta + b)x\Delta = b(\Delta x - x\Delta) = b[\Delta, x].$$

Therefore $z_0 = b\Delta$ works, and we have the central element

$$t'_z = ey^2 + hxy - fx^2 - \frac{1}{2}h(a\Delta + b) + \frac{1}{4}(\Delta(a\Delta + b) - (a\Delta + b) + b\Delta).$$

Removing the scalar $-5b/4$, we get the desired generating central element to be (up to adding a scalar)

$$t_z = ey^2 + hxy - fx^2 - \frac{1}{2}h(a\Delta + b) + \frac{1}{4}(a\Delta^2 + (2b - a)\Delta).$$

3. Derivations and commutator quotient

3.1. **Derivations.** We now compute the space of derivations. Note that if D is a derivation of H, then we may assume, modulo an inner derivation, that it vanishes on $\mathfrak{U}\mathfrak{g}$, since $\mathfrak{g}$ is simple. Thus, D is a $\mathfrak{g}$-module map, so $D(x)$ is a maximal vector of weight 1. By Proposition 2.6, it is of the form

$$D(x) = \sum_{i \geq 0} b_i(t_z)r_i + c_i(t_z)s_i,$$

where $r_i := \Delta^i x,\ s_i := [\Delta^i, x]\ \forall i$. But since we can rewrite the sum of half of these terms as

$$\sum_i c_i(t_z)s_i = \sum_i c_i(t_z)[\Delta^i, x] = \left[\sum_i c_i(t_z)\Delta^i, x\right],$$

hence by subtracting another inner derivation, we may assume that $D(x) = \sum_i b_i(t_z)\Delta^i \cdot x$. (Note that this change does not affect the fact that $D \equiv 0$ on $\mathfrak{U}\mathfrak{g}$.) Let us also denote $\sum_i b_i(t_z)\Delta^i$ by ω.

We now compute $D(y)$: we claim that $D(y) = \omega y$. To see this, apply D to the relation $[e, y] = x$. Then

$$[e, D(y)] = D(x) = \sum_i b_i(t_z)\Delta^i \cdot x,$$

whence it is easy to see that $[e, D(y) - \omega y] = 0$. Since D is now a $\mathfrak{g}$-module map, hence $D(y)$, and thus $D(y) - \omega y$, are both weight vectors of weight -1. But the last is also maximal, from above. Hence it vanishes, i.e., $D(y) = \omega y$.

We also carry out a key computation, that we shall need later. Recall the polynomials f_n, g_n that came up while computing $[\Delta^n, x]$.

Lemma 3.1. *For all n, we have*

$$[\Delta^n, x]y - [\Delta^n, y]x = 2f_n(\Delta)(ey^2 + hxy - fx^2 - \frac{1}{2}hz) + g_n(\Delta)z.$$

Proof. In what follows, we omit the (Δ), and refer to the polynomials merely as f_n, g_n.

$$\begin{aligned}
[\Delta^n, x]y - [\Delta^n, y]x &= [2f_n ey + (f_n h + g_n)x]y - [2f_n fx + (g_n - f_n h)y]x \\
&= f_n(2ey^2 + h(xy + yx) - 2fx^2) + g_n(xy - yx) \\
&= f_n(2ey^2 + h(2xy - z) - 2fx^2) + g_n z,
\end{aligned}$$

and hence we are done. □

We are now ready to finish the proof of the second part of Theorem 2.1.

Proposition 3.1. *If $z = 0$ (and $H = H_0$), then $Der(H)/Inn(H)$ is a rank one free module over the center of H.*

Proof. Recall that $D(x) = \omega x$, $D(y) = \omega y$ and $\omega = \sum_i b_i(t)\Delta^i$. We first claim that $b_i(t) = 0$ for $i > 0$. Indeed, note that

$$\begin{aligned}
D(xy) &= \sum_i b_i(t)(\Delta^i xy + x\Delta^i y) = \sum_i b_i(t)(2\Delta^i xy - [\Delta^i, x]y), \\
D(yx) &= \sum_i b_i(t)(\Delta^i yx + y\Delta^i x) = \sum_i b_i(t)(2\Delta^i yx - [\Delta^i, y]x)
\end{aligned}$$

and since $xy = yx$, hence one of the summands cancels throughout, to give: $[\omega, x]y = [\omega, y]x$. Rewriting ω into another different summation for convenience, we get an equation of the form

$$\sum_{i=0}^m t^i [h_i(\Delta), x]y = \sum_{i=0}^m t^i [h_i(\Delta), y]x.$$

Let m be the highest index such that $h_m(\Delta)$ is not a constant. We claim that this equation can not hold if $m > 0$, since if we look at the coefficient of y^{2m+2}, then the coefficient on the left side is nonzero, whereas on the right side it is zero. This is a contradiction.

Thus we get $\omega = b(t) \in \mathfrak{Z}(H)$, and $D(x) = \omega x$. We now know the values of D on generators, so using the Leibnitz rule, we can now compute this map on all of H. Let us denote this map by D_ω. Since we have the PBW property (i.e., that $\mathfrak{U}(\mathfrak{g} \ltimes V) \cong \mathfrak{U}\mathfrak{g} \otimes \operatorname{Sym} V$ as vector spaces), we observe that the map D_ω is given by

$$D_\omega(-) = n\omega \cdot -, \text{ on } \mathfrak{U}\mathfrak{g} \otimes \operatorname{Sym}^n V \ \forall n \geq 0.$$

Moreover, it is not hard to verify that this defines a derivation, using the PBW property again.

Finally, we verify that the map : $\mathfrak{Z}(H) \to Der(H)/Inn(H)$, sending $\omega \mapsto D_\omega$, is a vector space isomorphism, by looking at $D_\omega(x)$, say (to verify linear independence). Hence $H^1(H,H) = Der(H)/Inn(H) \cong \mathfrak{Z}(H)$ as $\mathfrak{Z}(H)$-modules, if $[x,y]=0$. □

Finally, we have the following proposition.

Proposition 3.2. *If $z \neq 0$, then every derivation of H_z is inner.*

Proof. Note again that since we are working modulo $Inn(H_z)$, so that given a derivation D, we assume D kills $\mathfrak{Ug}$ and $D(x) = \omega x,\ D(y) = \omega y$ as above.

Let us write $D(x) = t_z^m h_m(\Delta)x + \sum_{0 \leq i < m} t_z^i h_i(\Delta)x$. If we now pass to the associated graded algebra $\operatorname{gr} H_z$ (under the usual filtration that assigns V degree 1 and $\mathfrak{g}$ degree 0), then we get a derivation of $\operatorname{gr} H_z = \mathfrak{U}(\mathfrak{g} \ltimes V)$, that sends x to $t_z^m h_m(\Delta)x$. By the previous case, we may assume without loss of generality that $h_m = 1$.

Similarly, $D(y) = t_z^m y + \sum_{i=0}^{m-1} t_z^i h_i(\Delta)y$. Applying D to z, we get

$$\begin{aligned} 0 &= D(z) = [Dx, y] + [x, Dy] = [\omega x, y] + [x, \omega y] \\ &= 2\omega[x,y] + [\omega, y]x - [\omega, x]y. \end{aligned}$$

We rearrange this to get

$$2\omega z = [\omega, x]y - [\omega, y]x.$$

Let us rewrite $\omega = \sum_i b_i(t_z)\Delta^i$. Then using Lemma 3.1, we get

$$\begin{aligned} 2\omega z &= \sum_i b_i(t_z)([\Delta^i, y]x - [\Delta^i, x]y) \\ &= \sum_i b_i(t_z)\left(2f_i(\Delta)(t - \frac{1}{2}hz) + g_i(\Delta)z\right). \end{aligned}$$

Also note, that $t_z = (t - \frac{1}{2}hz) + \frac{1}{4}(\Delta z - z + z_0)$, where $[z_0, x] = z[\Delta, x] - \Delta[z, x]$. Hence we rewrite the above equation as

$$2\omega z = 2\sum_i b_i(t_z)\left(f_i(\Delta)(t_z - \frac{1}{4}(\Delta z - z + z_0)) + \frac{1}{2}g_i(\Delta)z\right). \tag{9}$$

Now look at the highest power of t_z (or of y) in the equation, and say the corresponding summand on the left side is $t_z^n \sum_j \beta_j \Delta^j$, with $\beta_j \in k$. Then the corresponding expression on the right side yields

$$t_z^n \sum_j \beta_j \left(f_j(\Delta)(t_z - \frac{1}{4}(\Delta z - z + z_0)) + \frac{1}{2}g_j(\Delta)z\right).$$

Now note that there is an extra power of t_z in this latter expression. Therefore if we look at the highest power of y that occurs in the right side of equation (9), namely y^{2n+2}, then its coefficient must be zero (since the corresponding coefficient on the left side is zero). Since t_z is central, this means that $\sum_j \beta_j f_j(\Delta) = 0$. But the f_j's form a basis of the center of $\mathfrak{Ug}$. Hence $\beta_j = 0$ for all j, whence ω must equal zero too. We conclude that $D(x) = D(y) = D(\mathfrak{g}) = 0$, and so $D = 0$ modulo $Inn(H_z)$, as claimed. □

This concludes the proof of Theorem 2.1.

3.2. **Commutator quotient.** Next, we would like to determine the commutator quotient (or abelianization) $H_z/[H_z, H_z]$ as a module over the center of H_z. At first, let us consider the case $z = 0$.

Proposition 3.3. *The natural map from $\mathfrak{Z}(\mathfrak{Ug})$ to $H/[H, H]$ is an isomorphism, and the action of the center of H on its commutator quotient is trivial.*

We need a small lemma for this, which is also used later.

Lemma 3.2. *Inside any H_z, we have $\mathfrak{Ug} \cdot V = [\mathfrak{Ug}, V]$. More precisely, in terms of the standard filtration on $\mathfrak{Ug}$, $F^n\mathfrak{Ug} \cdot V = [F^{n+1}\mathfrak{Ug}, V]$ $\forall n \geq 0$.*

Proof. The second statement (for all n) implies the first; we will show both inclusions for the latter claim. One way is easy: $[F^{n+1}\mathfrak{Ug}, V] \subset F^n\mathfrak{Ug} \cdot V$ using induction on n.

For the other inclusion, we proceed by induction on n. Let $\alpha \in F^n\mathfrak{Ug}$; we want to show that $\alpha \otimes V \in [F^{n+1}\mathfrak{Ug}, V]$. When $n = 0$, we are done since $[\mathfrak{g}, V] = V$, so it suffices to show that

$$\alpha \otimes x \in [F^{n+1}\mathfrak{Ug}, V] \mod F^{n-1}\mathfrak{Ug} \otimes V.$$

But we have

$$\begin{aligned} [h^n, x] &\equiv nh^{n-1}x \mod F^{n-1}\mathfrak{Ug} \otimes V, \\ [h^n, y] &\equiv -nh^{n-1}y \mod F^{n-1}\mathfrak{Ug} \otimes V, \\ [f^n, x] &\equiv nf^{n-1}y \mod F^{n-1}\mathfrak{Ug} \otimes V, \\ [e^n, y] &\equiv ne^{n-1}x \mod F^{n-1}\mathfrak{Ug} \otimes V, \end{aligned}$$

so

$$\begin{aligned} [e^i h^j f^k, x] &\equiv je^i h^{j-1} f^k x + ke^i h^j f^{k-1} y \mod F^{n-1}\mathfrak{Ug} \otimes V, \\ [e^i h^j f^k, y] &\equiv ie^{i-1} h^j f^k x - je^i h^{j-1} f^k y \mod F^{n-1}\mathfrak{Ug} \otimes V. \end{aligned}$$

Now assume without loss of generality that $\alpha = e^i h^j f^k$, with $i+j+k = n$. Then

$$\begin{aligned} \alpha \otimes x &\equiv \frac{1}{j+1}[e^i h^{j+1} f^k, x] - \frac{k}{j+1} e^i h^{j+1} f^{k-1} y \mod F^{n-1}\mathfrak{Ug} \otimes V, \\ \alpha \otimes y &\equiv \frac{-1}{j+1}[e^i h^{j+1} f^k, y] + \frac{i}{j+1} e^{i-1} h^{j+1} f^k x \mod F^{n-1}\mathfrak{Ug} \otimes V. \end{aligned}$$

We thus repeatedly (alternately) apply these two identities to assume that either i or k becomes zero (in α). Applying (possibly both of) them once more, we are done. □

Proof of Proposition 3.3. Since $H = \mathfrak{U}(\mathfrak{g} \ltimes V)$, from the relation between Lie algebra homology and Hochschild homology, we get that

$$H/[H,H] = H/[H, \mathfrak{g} \ltimes V] = (H/[H,V])^{\mathfrak{g}}. \tag{10}$$

We now claim that $H/[H,V] = \mathfrak{U}\mathfrak{g}$, which would imply that $H/[H,H] = (\mathfrak{U}\mathfrak{g})^{\mathfrak{g}} = \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$, as desired. Indeed, obviously $\mathfrak{U}\mathfrak{g}$ injects into $H/[H,V]$, so we just need to demonstrate that $HV \subset [H,V]$.

Clearly, $[H,V]$ is a right module over $\operatorname{Sym} V$, so it suffices to show that $[\mathfrak{U}\mathfrak{g}, V] \supset \mathfrak{U}\mathfrak{g} \otimes V$. But this was shown in Lemma 3.2 above.

Now, since the generating central element of H lies in $HV^2 (\subset [H,V])$, it must act trivially on $H/[H,H]$, which concludes the proof. $\square$

Corollary 3.1. *For any z, $\mathfrak{Z}(\mathfrak{U}\mathfrak{g}) \cong \mathfrak{U}\mathfrak{g}/[\mathfrak{U}\mathfrak{g}, \mathfrak{U}\mathfrak{g}]$ surjects onto $H_z/[H_z, H_z]$. Every $X \in F^n\mathfrak{U}\mathfrak{g}$ is equivalent to some $X' \in F^n\mathfrak{U}\mathfrak{g} \cap \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ modulo $[\mathfrak{U}\mathfrak{g}, \mathfrak{U}\mathfrak{g}]$ or $[H_z, H_z]$.*

Proof. We make many statements here. The first equality comes from the fact that $\mathfrak{U}\mathfrak{g}$ is a direct sum of finite-dimensional $\mathfrak{sl}_2$-modules (e.g., by Lemma 2.2 with $V = \mathfrak{h}' = 0$), whence the images of $\operatorname{ad} e$ and $\operatorname{ad} f$ span a complement to the center (using weight vectors). Moreover, no polynomial in the Casimir is in the commutator, since one can always find a finite-dimensional $\mathfrak{U}\mathfrak{g}$-module on which it has nonzero trace.

Now for the surjection: we first claim that $\mathfrak{U}\mathfrak{g}$ surjects onto the abelianization of H_z. Indeed, the main step in showing this is the $z = 0$ case, which is the proposition above: $\mathfrak{U}\mathfrak{g} \twoheadrightarrow \mathfrak{Z}(\mathfrak{U}\mathfrak{g}) \xrightarrow{\sim} H/[H,H]$. But this implies that $\mathfrak{U}\mathfrak{g}$ surjects onto the associated graded of the abelianization of H_z, since $H/[H,H] \twoheadrightarrow \operatorname{gr}(H_z/[H_z,H_z])$.

So we just need to show that this can be "lifted" to a surjection as desired. Now given $a \in F^n H_z$ (for the usual filtration on H_z), we can find $c \in \mathfrak{U}\mathfrak{g}$ and $a_i, b_i \in H_z$ such that the filtration degrees of a_i, b_i always add up to at most n, and $\overline{a} = \overline{c} + \sum_i [\overline{a_i}, \overline{b_i}]$ in the associated graded, from above. But then $a - c - \sum_i [a_i, b_i] \in F^{n-1} H_z$, and we can proceed by induction.

Finally, $\mathfrak{U}\mathfrak{g} \hookrightarrow H_z$, so $[\mathfrak{U}\mathfrak{g}, \mathfrak{U}\mathfrak{g}]$ is killed by the map : $\mathfrak{U}\mathfrak{g} \twoheadrightarrow H_z/[H_z,H_z]$. Hence $\mathfrak{Z}(\mathfrak{U}\mathfrak{g})$ surjects onto $H_z/[H_z,H_z]$.

Next, we show the last statement. Consider the finite-dimensional $\mathfrak{U}\mathfrak{g}$-submodule $M_n := F^n\mathfrak{U}\mathfrak{g} \subset \mathfrak{U}\mathfrak{g}$, and its submodule $[\mathfrak{g}, M_n] \subset M_n$. Clearly, $M_n/[\mathfrak{g}, M_n]$ surjects onto the image of M_n modulo $[\mathfrak{U}\mathfrak{g}, \mathfrak{U}\mathfrak{g}]$ or $[H_z, H_z]$; on the other hand, $M_n/[\mathfrak{g}, M_n]$ is isomorphic to $\mathfrak{Z}(\mathfrak{U}\mathfrak{g}) \cap M_n$ by complete reducibility. We are done. $\square$

Thus we need to compute the kernel (which obviously contains at least z).

As an aside, we note that equation (10) holds for general z:

Lemma 3.3. $H_z/[H_z, H_z] = (H_z/[H_z, V])^{\mathfrak{g}}$.

Proof. Consider the following sequence of H_z-bimodules:

$$H_z \otimes (V \oplus \mathfrak{g}) \otimes H_z \to H_z \otimes H_z \to H_z \to 0$$

where the last map is the multiplication map, and the first map is given by $w \mapsto 1 \otimes w - w \otimes 1$ for any $w \in V \oplus \mathfrak{g}$. We claim that this sequence is right exact. Indeed, we only need to verify exactness of the middle term. But all terms of this sequence are naturally filtered, and after passing to the associated graded picture, we will get an analogous sequence for $H = \mathfrak{U}(\mathfrak{g} \ltimes V)$, for which the sequence is well known to be exact. But since $H_z/[H_z, H_z] = Tor_0(H_z, H_z)$ in the category of H_z-bimodules, after tensoring our sequence with H_z we get that

$$H_z/[H_z, H_z] = H_z/[H_z, V \oplus \mathfrak{g}] = (H_z/[H_z, V])^{\mathfrak{g}}$$

and we are done. □

We finally have the following theorem.

Theorem 3.1. *Let the parameter z be nonzero, say $z = c\Delta^m + l.o.t.$. If $m = 0$ (i.e., z is a constant), then the commutator quotient of H_z is trivial. Otherwise, if $\deg z = m \geq 1$, then $1, \Delta, ..., \Delta^{m-1}$ are linearly independent in $H_z/[H_z, H_z]$, and generate it as a module over the center of H_z. (In particular, $(H_z/[H_z, H_z])/(t_z)$ is a vector space of dimension m over k.)*

An important first step in showing this, is the following proposition.

Proposition 3.4. *For all $a, b \geq 0$, $t_z^a \Delta^b$ equals a (nonzero) polynomial in Δ of degree $a(m+1) + b$, modulo $[H_z, H_z]$.*

Proof. The case $a = 0$ is obvious; we will show the $a = 1$ case below. The case of higher a is then proved by induction on a: for a fixed b, if $t_z^a \Delta^b - p_{ab}(\Delta) = \sum_i [r_i, s_i] \in [H_z, H_z]$, then

$$t_z^{a+1} \Delta^b = t_z p_{ab}(\Delta) + t_z \sum_i [r_i, s_i] = t_z p_{ab}(\Delta) + \sum_i [t_z r_i, s_i]$$

and $t_z p_{ab}(\Delta)$ can be rewritten appropriately, using the $a = 1$ statement (for various b).

It remains to show the hypothesis for $a = 1$ and all b. In the rest of the proof, we will use the following result several times.

Lemma 3.4.

(1) *Let $d = [\alpha, x] + [\beta, y]$, with $\alpha, \beta \in \mathfrak{U}\mathfrak{g}$. Then modulo $[H_z, H_z]$, $dx \equiv -\beta z$, $dy \equiv \alpha z$.*
(2) *For any $z' \in \mathfrak{Z}(\mathfrak{U}\mathfrak{g})$, $z' e y^2 \equiv z' h x y \equiv -z' f x^2 \mod [H_z, H_z]$.*

Proof.

(1) We have $dx = [\alpha x, x] + [\beta x, y] - \beta z$, and $dy = [\alpha y, x] + [\beta y, y] + \alpha z$. Both claims now follow.
(2) Note that $[f, z' e x y] = -z' h x y + z' e y^2$, which proves the first equality; for the second, apply the anti-involution j. We note that j fixes the Casimir element, and hence the whole center. Applying j to the

above equation, $-xyhz' - x^2fz' = A \in [H_z, H_z]$, say. Hence we make the following reductions:

$$\begin{aligned} -z'fx^2 &= -fz'x^2 = -x^2fz' + [x^2, fz'] = [x^2, fz'] + A + xyhz', \\ xyhz' &= z'xyh + [xyh, z'] = z'hxy + [xyh, z'] \text{ (since } xy \text{ has weight 0).} \end{aligned}$$

Thus, $-z'fx^2 \equiv z'hxy \mod [H_z, H_z]$, as claimed.

□

We now prove the result for $t_z\Delta^n$ $(n \geq 0)$. Since $t_z = (ey^2 + hxy - fx^2) - \frac{1}{2}hz - q_z$ (see equation (6)), and since $hz = [e, fz] \in [\mathfrak{g}, H_z]$, we have

$$t_z\Delta^n \equiv \Delta^n t_z \equiv \Delta^n(3hxy - q_z) \mod [H_z, H_z]. \tag{11}$$

By Lemma 3.2, $\Delta^n hx \in \mathfrak{Ug} \cdot V = [\mathfrak{Ug}, V]$ is of the form $[a_n, x] + [b_n, y]$ for some $a_n, b_n \in \mathfrak{Ug}$. By Corollary 3.1, we may assume that $a_n, b_n \in \mathfrak{Z}(\mathfrak{Ug}) \cap F^{2n+1}\mathfrak{Ug}$ (modulo the commutator). By Lemmma 3.4, $3\Delta^n hxy \equiv 3a_nz \mod [H_z, H_z]$.

We thus have to prove (using equation (11)) that $3a_nz - \Delta^n q_z$ is a polynomial of degree $n + m + 1$ in Δ. In light of Corollary 2.3, it suffices to show that a_n is a polynomial of degree $n+1$ with positive (rational) top coefficient (in fact, it turns out to be $1/6(n+1)$).

To do this, consider the formula for $[\Delta^n, x]$, which yields: $f_n \cdot (hx+2ey) = [\Delta^n, x] - g_nx$. Again using Lemma 3.2 and Corollary 3.1, write $g_nx = [c_n, x] + [c'_n, y]$ for c_n, c'_n polynomials in Δ. Moreover, since $\deg(g_n(T)) = n - 1$, $c_n, c'_n \in F^{2n-1}\mathfrak{Ug}$; thus, $\deg(c_n) < n$ (as a polynomial in Δ).

But then Lemma 3.4 implies that on the one hand,

$$f_n(hx + 2ey)y \equiv f_n(hxy + 2ey^2) \equiv f_n(3hxy) \mod [H_z, H_z]$$

and on the other (modulo the commutator),

$$f_n(hx + 2ey)y \equiv (\Delta^n - c_n)z \equiv c\Delta^{m+n} + l.o.t..$$

We thus get: $f_n(3hxy) \equiv c\Delta^{m+n} + l.o.t.$ for all n. Using the "unipotent" (with positive coefficient $1/(2n)$) change of basis from f_n to Δ^n,we get

$$\Delta^n(3hxy) = \left(\frac{1}{2n+2}f_{n+1} + \text{“}l.o.t.\text{”}\right)(3hxy) = \frac{c}{2n+2}\Delta^{m+n+1} + l.o.t.$$

where "$l.o.t.$" stands for "lower-degree" f_i's. Now compare this to what we had above:

$$\Delta^n(3hxy) \equiv 3a_nz = a_n(3c\Delta^m + l.o.t.),$$

and we are done. □

Proof of Theorem 3.1. First of all we have $[H_z, H_z] \cap \mathfrak{Ug} \subseteq z \cdot \mathfrak{Ug} + [\mathfrak{Ug}, \mathfrak{Ug}]$ (since any time the filtration degree in x, y goes down in a commutator expression, a multiple of z appears). Since $\mathfrak{Ug}/(\mathfrak{Ug} \cap [H_z, H_z]) \subseteq H_z/[H_z, H_z]$, and $\mathfrak{Ug}/(\mathfrak{Ug} \cap [H_z, H_z])$ surjects onto $\mathfrak{Ug}/(z \cdot \mathfrak{Ug} + [\mathfrak{Ug}, \mathfrak{Ug}]) = \mathfrak{Z}(\mathfrak{Ug})/z \cdot \mathfrak{Z}(\mathfrak{Ug})$, hence the elements $1, ..., \Delta^{m-1}$ are linearly independent in $H_z/[H_z, H_z]$.

It remains to show that the following elements span $H_z/[H_z,H_z]$ - or in light of Corollary 3.1, the center of $\mathfrak{Ug}$: $\{t_z^a\Delta^b : a \geq 0,\ 0 \leq b < m\}$. We now show that all Δ^n lie in this span, modulo $[H_z,H_z]$. Clearly, $1,\dots,\Delta^{m-1}$ as well as $\Delta^m = (1/c)z - l.o.t.$ are in this span, since $z/c = [x/c, y]$. Next, $\Delta^{m+1},\dots,\Delta^{2m}$ are in the span: just consider $t_z, t_z\Delta,\dots,t_z\Delta^{m-1}$. As for Δ^{2m+1}, we have

$$\Delta^{2m+1} \equiv t_z z + l.o.t. \equiv [x/c, t_z y] + l.o.t. \mod [H_z,H_z]$$

similar to above. Keep repeating this procedure. □

We expect that a stronger statement is true: namely, that the commutator quotient is actually a free module over the center, with basis $1,\Delta,\dots,\Delta^{m-1}$. This would imply (via Hochschild cohomology considerations) that the algebras $H_{z_1}/(t_{z_1}-a), H_{z_2}/(t_{z_2}-b)$ are not Morita equivalent if $\deg z_1 \neq \deg z_2$, where $a,b \in k$.

4. Infinitesimal Hecke algebra of $\mathfrak{gl}_n$

We now recall the definition of an infinitesimal Hecke algebra of $\mathfrak{g} = \mathfrak{gl}_n$ and $V = \mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h} = k^n$ and $\mathfrak{h}^*$ is its dual representation. We (again) identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the pairing $\mathfrak{g}\times\mathfrak{g} \to k :\ (A,B) \mapsto \mathrm{Tr}(AB)$, and identify $\mathfrak{Ug}$ with $\mathrm{Sym}\,\mathfrak{g}$ via the symmetrization map.

Then for any $x \in \mathfrak{h}^*,\ y \in \mathfrak{h},\ A \in \mathfrak{g}$, one writes

$$(x,(1-TA)^{-1}y)\det(1-TA)^{-1} = r_0(x,y)(A) + r_1(x,y)(A)T + \dots$$

where $r_i(x,y)$ is a polynomial function on $\mathfrak{g}$, for all i.

Now for each polynomial $\beta = \beta_0 + \beta_1 T + \beta_2 T^2 + \cdots \in k[T]$, the authors define in [EGG] the algebra H_β as a quotient of $T(\mathfrak{h}\oplus\mathfrak{h}^*) \rtimes \mathfrak{Ug}$ by the relations

$$[x,x'] = 0, \qquad [y,y'] = 0, \qquad [y,x] = \beta_0 r_0(x,y) + \beta_1 r_1(x,y) + \dots$$

for all $x,x' \in \mathfrak{h}^*,\ y,y' \in \mathfrak{h}$. It is proved in [EGG] that these algebras are infinitesimal Hecke algebras. Also note that if $\beta \equiv 0$, then $H_0 = \mathfrak{U}(\mathfrak{gl}_n \ltimes (\mathfrak{h}\oplus\mathfrak{h}^*))$.

4.1. **Relations and anti-involution.** We start with an explicit presentation of H_β: it is generated by $\mathfrak{gl}_n = \bigoplus_{i,j} k e_{ij}$ and $\mathfrak{h} = \bigoplus_i k v_i,\ \mathfrak{h}^* = \bigoplus_i k v_i^*$, where $\{v_i\},\{v_i^*\}$ form dual bases of $\mathfrak{h},\mathfrak{h}^*$ respectively. We have the relations:

$$e_{ij}\cdot v_k := \delta_{jk} v_i, \qquad e_{ij}\cdot v_k^* := -\delta_{ik} v_j^*, \qquad v_i^*(v_j) = \delta_{ij}.$$

We next describe an anti-involution of H_β, for (at most) linear β. Suppose we have j sending $e_\alpha \leftrightarrow f_\alpha$ and $h \leftrightarrow h$ for all positive simple roots α for a reductive Lie algebra $\mathfrak{g}$ (and Cartan subalgebra elements h). One then checks that this gives an anti-involution j of $\mathfrak{g}$ (and hence of $\mathfrak{Ug}$).

Now let $\mathfrak{g} = \mathfrak{gl}_n$; then $j(X) = X^T$ in $\mathfrak{g}$. We now mention the anti-involution.

Lemma 4.1. *The map $j : (X, v_i) \leftrightarrow (X^T, -v_i^*)$ extends to an anti-involution of $\mathfrak{U}\mathfrak{g} \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$. Moreover, j factors to an anti-involution of H_β when β is at most linear.*

Proof. For the first part, we only need to check that j preserves (actually, permutes) the following relations:

$$[e_{ij}, e_{kl}] = \delta_{kj} e_{il} - \delta_{il} e_{kj}, \quad [e_{ij}, v_k] = \delta_{jk} v_i, \quad [e_{ji}, v_k^*] = -\delta_{jk} v_i^* \ \forall i, j, k, l.$$

This is easy to do. Next, for H_β with β at most linear, we refer to [EGG, Examples 4.6, 4.7]; thus, H_β is the quotient of the above algebra, by the relations

$$[v_i, v_j] = [v_i^*, v_j^*] = 0, \qquad [v_i, v_j^*] = \delta_{ij}(\beta_0 + \beta_1 \tau) + \beta_1 e_{ij} \ \forall i, j,$$

where $\tau = \mathrm{Id}_n \in \mathfrak{gl}_n$. That j preserves these relations, is is also easy to verify. □

4.2. **Central elements.** We now mention discuss central elements for various β (and general n). We first have a result for $\beta \equiv 0$, which can be verified using a strategy similar to the proof of Proposition 2.1.

Proposition 4.1. *The center of $H_0(\mathfrak{gl}_n)$ contains at least two algebraically independent elements, both fixed by j:*

$$r_n := \sum_{i=1}^n v_i v_i^*, \ \ s_n := \sum_{1 \leq p < q \leq n} (e_{pq} v_q v_p^* + e_{qp} v_p v_q^*) - (e_{pp} v_q v_q^* + e_{qq} v_p v_p^*). \quad □$$

Next, we prove that in general, H_β (over $\mathfrak{gl}_n$) has nontrivial center, by providing a lift r_β of r_n; clearly, r_β is transcendental in H_β since r_n is thus in H_0.

Proposition 4.2. *For any n, β, H_β contains the central element $r_\beta := \mathbf{h} + \tau$ (which is transcendental in H_β).*

Here, $\tau = \mathrm{Id}_n$, and $\mathbf{h}$ is the *Euler element* in [EGG, §5.2], given by

$$\mathbf{h} = \sum_i v_i^* v_i + \frac{n}{2} + c,$$

where $c \in \mathcal{O}(G)^*$ is defined via the following equation (see [EGG, §3.4]), with $t \in k$:

$$\kappa(x, y) := [x, y] = (y, x)t + (y, (1 - g)x)c, \text{ for all } x \in \mathfrak{h}^*, y \in \mathfrak{h}.$$

Proof. (Note that k is algebraically closed, of characteristic zero.) As mentioned in [EGG, §4.1], the infinitesimal Hecke algebra H_β only exists when $\mathrm{im}(\kappa) \subset \mathfrak{U}\mathfrak{g}$; thus, $f_{xy} \cdot c \in \mathfrak{U}\mathfrak{g}$ for all $f_{xy} := (y, (1 - g)x) \in \mathcal{O}(G)$ (with $x \in \mathfrak{h}^*, y \in \mathfrak{h}$). By the Nullstellensatz, $c \in \mathfrak{U}\mathfrak{g}$, so $\mathbf{h} \in \sum_i v_i v_i^* + \mathfrak{U}\mathfrak{g}$ now; therefore r_β is indeed a lift of r_n to H_β. That it is central follows from [EGG, Proposition 5.3], and because $\mathbf{h}, \tau$ commute with $\mathfrak{gl}_n$. □

5. Category $\mathcal{O}$ for Infinitesimal Hecke algebras

At first, let us discuss an analogue of the BGG category $\mathcal{O}$, for a class of algebras equipped with the following structure:

Let $A \supset k$ be an associative algebra, endowed with the following additional structure:

- A has an increasing filtration by k-subspaces $F^n A, n \geq 0$, that satisfy $F^n A \cdot F^m A \subseteq F^{n+m} A$;
- There are three finite-dimensional k-subspaces $\mathfrak{n}^+, \mathfrak{n}^-, \mathfrak{h} \subseteq F^1 A$, such that $\mathfrak{n}^+ + \mathfrak{n}^- + \mathfrak{h} = \mathfrak{n}^+ \oplus \mathfrak{n}^- \oplus \mathfrak{h}$.

From these data we require that

- A is generated as an algebra over k by $\mathfrak{n}^+ \oplus \mathfrak{n}^- \oplus \mathfrak{h}$; each summand is a Lie (sub)algebra, and
$$[\mathfrak{h}, \mathfrak{h}] = 0, \ [\mathfrak{h}, \mathfrak{n}^+] = \mathfrak{n}^+, \ [\mathfrak{h}, \mathfrak{n}^-] = \mathfrak{n}^-.$$
- There is a (fixed) subspace $\mathfrak{h}_0 \subset \mathfrak{h}$, and both $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are diagonally acted upon by the adjoint action of $\mathfrak{h}$, and the eigenvalues occurring in these decompositions have images in opposite non-intersecting cones in $\mathfrak{h}_0^* (\twoheadleftarrow \mathfrak{h}^*)$.
- The multiplication map : $B_1 \otimes B_2 \otimes B_3 \to \operatorname{gr}(F^\bullet A)$ is a vector space isomorphism, where $\{B_1, B_2, B_3\} = \{\mathfrak{U}\mathfrak{n}^-, \mathfrak{U}\mathfrak{n}^+, \operatorname{Sym}(\mathfrak{h})\}$ (i.e., in every possible order). Moreover, $\operatorname{Sym} \mathfrak{h} \subset F^0 A$.
- In addition, we require that $\operatorname{gr}(F^\bullet A)$ is equipped with a filtration consisting of finite-dimensional subspaces G^n ($n \geq 0$), such that $\mathfrak{n}^+ \oplus \mathfrak{n}^- \oplus \mathfrak{h} \oplus k = G^1 \operatorname{gr}(A)$, and $\operatorname{gr}(\operatorname{gr}(F^\bullet A))$ is a polynomial algebra, i.e., $\operatorname{Sym}(\mathfrak{n}^+ \oplus \mathfrak{n}^- \oplus \mathfrak{h}) \to \operatorname{gr}(\operatorname{gr}(F^\bullet A))$ is an isomorphism.

Moreover, if A is such an algebra, then so are $\operatorname{gr}(F^\bullet A)$ and $\operatorname{gr}(G^\bullet(\operatorname{gr}(F^\bullet A)))$.

Of course, the main examples we have in mind are infinitesimal Hecke algebras (the axiomatics of category $\mathcal{O}$ in more general settings is considered in [Kh2]). The axiom about $\mathfrak{h}_0 \subset \mathfrak{h}$ is needed (later) for technical purposes: though we can choose $\mathfrak{h}_0 = \mathfrak{h}$ for H_z (over $\mathfrak{sl}_2$), we need to choose $\mathfrak{h}_0 = kh \subset \mathfrak{h} = kh \oplus k\tau$ in H_β (for $\mathfrak{gl}_2$). Moreover, for infinitesimal Hecke algebras, we clearly have $\operatorname{gr}(F^\bullet H_\beta) = H_0 = \mathfrak{U}(\mathfrak{g} \ltimes V)$ and $\operatorname{gr}(G^\bullet H_0) = \operatorname{Sym}(\mathfrak{g} \oplus V)$.

We now mimic some standard definitions.

Definition 5.1.

(1) The *category* $\mathcal{O}$ for the algebra A (as above), denoted by $\mathcal{O}_A$, is the full subcategory of finitely generated left A-modules, defined by: $M \in \mathcal{O}_A$ if and only if $\mathfrak{n}^+$ acts locally nilpotently on M, and $\mathfrak{h}$ acts on it diagonalizably with finite-dimensional eigenspaces. That is, $M = \bigoplus_{\chi \in \mathfrak{h}^*} M^\chi$, with $\dim M^\chi < \infty \ \forall \chi$.

(2) An element $v \in M$ is said to be a *maximal vector* if it is an eigenvector for the $\mathfrak{h}$-action, and $\mathfrak{n}^+ v = 0$.

(3) (**Definition-proposition.**) Let $\chi \in \mathfrak{h}^*$. Then there exists an object $M(\chi) \in \mathcal{O}_A$, characterized by the following uniqueness property: $M(\chi)^\chi = k$, and if $v \in M(\chi)^\chi$, then for every pair v_1, M_1 with $v_1 \in M_1^\chi$ a maximal vector, there exists a unique $f \in \operatorname{Hom}_A(M(\chi), M_1)$ such that $f(v) = v_1$. Such a module $M(\chi)$ is called a *Verma module* for the weight χ.

Proof. Let A_- be the subalgebra of A generated by $\mathfrak{h} \oplus \mathfrak{n}^-$; then there exists $\chi : A_- \to k$, such that $\chi|_{\mathfrak{h}} = \chi,\ \chi(\mathfrak{n}^-) = 0$. Indeed, we just need to check that $\mathfrak{h} \cap \mathfrak{n}^- A_- = 0$, which is immediate from weight space theory. But then, k turns into a left A_--module (which will be denoted by k_χ).

Now define $M(\chi) := A \otimes_{A_-} k_\chi$. It is clear that this module lies in $\mathcal{O}_A$ and $v = 1 \otimes 1$ is a maximal nonzero vector of weight χ. If $v_1 \in M^\chi$ is a maximal vector in an A-module (M), then we have a map of A_--modules $f : k_\chi \to M$ such that $f(1) = v_1$. Hence we get $f \otimes_{A_-} \operatorname{Id} : A \otimes_{A_-} k_\chi \to A \otimes_{A_-} M \to M$, such that v maps to v_1; obviously this map is unique. □

We have the following standard

Proposition 5.1. *For any $\chi \in \mathfrak{h}^*$, $M(\chi)$ has a unique maximal subobject and irreducible quotient (both in $\mathcal{O}_A$); call the latter $V(\chi)$. Then every irreducible object in $\mathcal{O}_A$ is of the form $V(\chi)$ for some $\chi \in \mathfrak{h}^*$.*

Proof. If $V \subset M(\chi)$ is a proper subobject, then $V^\chi = 0$. Hence the sum of all proper subobjects of $M(\chi)$ is still a proper submodule, which proves the first assertion. Now if V is an irreducible object, then it must have a maximal vector $v \in V^\chi$ for some χ. Hence $\operatorname{Hom}(M(\chi), V) \neq 0$, so $V = V(\chi)$. □

As is usual in representation theory, one would like to study (irreducible) finite-dimensional representations, compute the multiplicity of $V(\chi)$ in $M(\mu)$ (for all $\chi, \mu \in \mathfrak{h}^*$), and so on. One has the usual spectral decomposition of $\mathcal{O}_A$ with respect to its center: $\mathcal{O}_A = \bigoplus_{\phi \in \operatorname{Spec}(\mathfrak{Z}(A))} \mathcal{O}^\phi$, where $\mathcal{O}^\phi$ is the full subcategory consisting of objects on which $\phi(t) - t$ acts locally nilpotently for any $t \in \mathfrak{Z}(A)$. In particular, we have a Harish-Chandra map $\eta : \mathfrak{h}^* \to \operatorname{Spec} \mathfrak{Z}(A)$.

Let us compare $\mathcal{O}_A$ and $\mathcal{O}_{\operatorname{gr}(A)}$. If $M \in \mathcal{O}_A$, let $V \subset M$ be a finite-dimensional vector space generating M over A. Then M has the usual increasing filtration: $F^n M := (F^n A)V$, which makes $\operatorname{gr}(M)$ a $\operatorname{gr}(A)$-module (note that this construction depends on our choice of V).

Moreover, $\operatorname{gr}(M)$ belongs to $\mathcal{O}_{\operatorname{gr}(A)}$, and $\operatorname{ch}_{\mathcal{O}_A}(M) = \operatorname{ch}(\operatorname{gr}(M))$ (where $\operatorname{ch}(M) := \sum_{\chi \in \mathfrak{h}^*} (\dim(M^\chi)\chi)$ is the character of an $\mathfrak{h}$-semisimple module). Hence we see that $\mathcal{O}_{\operatorname{gr}(A)}$ provides an "upper bound" for $\mathcal{O}_A$ (i.e., $\operatorname{gr}(M) \in \mathcal{O}_{\operatorname{gr}(A)}\ \forall M \in \mathcal{O}_A$).

We also remark that if we start with a Verma module $M(\lambda)$ and $V = k \cdot v_\lambda$ (the highest weight space in it), then we will get a Verma module $\mathrm{gr}(M(\lambda))$ over $\mathrm{gr}(A)$ of weight λ. In particular,

$$\mathrm{gr}(\mathrm{Ann}(M(\lambda))) \subseteq \mathrm{Ann}(\mathrm{gr}(M(\lambda))). \tag{12}$$

This fact is used in the section about primitive ideals.

In the remaining part of this section, we focus on the category $\mathcal{O}$ for $A = H_z$ (which does fit into the above setup). This category was studied in great detail in [Kh]. We now reinterpret some of those results using the center of H_z. We have $\mathcal{O} = \bigoplus_{\lambda \in k} \mathcal{O}^\lambda$, where $(t_z - \lambda)$ acts nilpotently on $\mathcal{O}^\lambda$ (though as we see presently, $\mathcal{O} = \mathcal{O}^0$ if $z = 0$).

At first, let us compute the action of t_z on $M(\lambda)$. We have

$$\begin{aligned} & (ey^2 + hxy - fx^2 - \frac{1}{2}hz - q_z)v_\lambda \\ = \quad & (y^2e + 2yx + z + hyx + hz - \frac{1}{2}hz - q_z)v_\lambda = ((1 + \frac{1}{2}h)z - q_z)v_\lambda \\ = \quad & \left(\frac{1}{2}\lambda + 1\right) z(\lambda^2 + 2\lambda)v_\lambda - q_z(\lambda^2 + 2\lambda)v_\lambda. \end{aligned}$$

Let us denote by $\phi_z(t)$ the following polynomial in $k[t]$:

$$\phi_z(t) = \left(\frac{1}{2}t + 1\right) z(t^2 + 2t) - q_z(t^2 + 2t), \tag{13}$$

where as usual, we treat z as a polynomial of Δ (note that $\phi_0(t) \equiv 0$). As a corollary, $V(\lambda) \in \mathcal{O}^\mu$ only if $\phi_z(\lambda) = \mu$.

Now suppose $z \neq 0$. Then the degree of $\phi_z(t)$ equals $2(\deg(z) + 1)$, and the multiplicity of $V(\lambda)$ in M is at most $\dim_k M^\lambda$. Hence all Verma modules - and thus, all objects in category $\mathcal{O}$ - have finite length.

Moreover, every central character of H_z is of the form $\chi_\mu : t_z \mapsto \mu \in k$, and since k is algebraically closed, and $\deg \phi_z > 0$, we can find $\lambda \in k$ such that $\phi_z(\lambda) = \mu$. To summarize, we get the following result, most of which is contained in [Kh], but is proved there by a completely different approach.

Proposition 5.2. *Each module in $\mathcal{O}^\lambda$ (for any λ) has finite length, and $V(\mu) \in \mathcal{O}^\lambda$ if and only if $\mu \in \phi_z^{-1}(\lambda)$. In particular, the number of non-isomorphic irreducible objects in $\mathcal{O}^\lambda$ is at most $2(\deg(z) + 1)$. Furthermore, every central character for H_z is associated to some Verma module.*

As an aside, the algebra H_z has the following peculiar property:

Proposition 5.3. *If the parameter z is nonzero, then there are at most finitely many non-isomorphic irreducible finite-dimensional H_z-modules.*

Proof. For the proof, we are going to use a theorem proved by Khare in [Kh]. We need to recall some definitions from there. For any pair of integers

r, m, he considers the following expression:

$$\alpha_{rm} = \sum_{i=0}^{m-2} (r+1-i)(z(r+1-i)^2 - 1)$$

(where $z(-)$ is viewed as a polynomial in the Casimir element). Then his result ([Kh, Theorem 11]) says that $V(r)$ is finite-dimensional if and only if there exists a nonnegative integer $s \leq r$ such that $\alpha_{r,r-s+2} = 0$.

Let us explain why this can not happen as long as $z \neq 0$ and r is large enough. We may rewrite α_{rm} as follows:

$$\alpha_{rm} = \sum_{i=1}^{r+1} iz(i^2-1) - \sum_{i=1}^{r+2-m} iz(i^2-1),$$

Therefore if we denote $\sum_{i=1}^{j} iz(i^2-1)$ by $f(j)$ (thus f is a polynomial of some positive degree), then $\alpha_{rm} = f(r+1) - f(r+2-m)$. So if $V(r)$ is finite-dimensional, then $f(r+1) = f(r+2-(r-s+2)) = f(s)$ for some $0 \leq s \leq r$. It thus suffices to show that for a nonconstant polynomial $f \in k[T]$, the numbers $f(1), f(2), \dots$ are "eventually pairwise distinct"; we show this now, in Lemma 5.1. □

Lemma 5.1. *Suppose $f \in k[T]$ is a nonconstant polynomial with coefficients in a field of characteristic zero. Then beyond some $r_0 \gg 0$ (in $\mathbb{Q} \hookrightarrow k$), $f : [r_0, \infty) \cap \mathbb{Q} \to k$ is injective.*

This result does not generalize (much) more; consider $f(T) = T^2$ evaluated at $0, 1, -1, 2, -2, \dots$ in $\mathbb{Q}$.

Proof. Consider the coefficients $c_0, \dots, c_d \in k$ of $f(T) = c_0 + c_1 T + \cdots + c_d T^d$. Now choose any $\mathbb{Q}$-basis $\{b_1, \dots, b_s\}$ of the $\mathbb{Q}$-span of the c_i's, and rewrite $f(T) = f_1(T)b_1 + \cdots + f_s(T)b_s$, where $f_i(T) \in \mathbb{Q}[T]$. Then at least one polynomial is nonconstant, say f_1 (without loss of generality).

Now, the absolute value of $f_1(r)$ ($r \in \mathbb{Q}$) is a strictly increasing function of r for $r \gg 0$, and this proves the result (since the b_i's are $\mathbb{Q}$-linearly independent). □

6. Primitive ideals of H_z

Let us start with the following definition.

Definition 6.1. We say that a (unital) k-algebra A is *almost commutative (of order 1)* if it admits an increasing filtration $F^\bullet A$ such that the corresponding associated graded is a finitely generated commutative k-algebra.

For $n > 1$, we say that a k-algebra is *almost commutative of order n* if it admits an increasing filtration compatible with the algebra structure, such that the associated graded is an almost commutative algebra of order $n-1$.

We have the following direct generalization of Quillen's theorem [Q], whose proof goes through essentially word by word; we reproduce this proof for the reader's convenience. (In what follows, k is an arbitrary field.)

Theorem 6.1 (Quillen). *Let A be an almost commutative algebra of some order and let M be a simple module over A. If $\phi \in \mathrm{End}_A(M)$, then ϕ is algebraic over k.*

Proof. Note the following elementary facts: if a k-algebra B is filtered with associated graded algebra $C = \mathrm{gr}(F^\bullet B)$, then any finitely generated B-module M is automatically filtered as well: let V be the k-span of a (finite) set of generators for M, and define a filtration on M via:

$$F^i M = F^i B \cdot V.$$

Then $\mathrm{gr}\, F^\bullet M$ is automatically a finitely generated C-module. Moreover, $\mathrm{gr}(B[T]) = C[T]$. Finally, choose $\theta \in \mathrm{End}_B M$; then M is naturally a $B[T]$-module, via: $(b \otimes p(T))(m) := p(\theta)(b \cdot m) = b \cdot p(\theta)(m)$. Then $\mathrm{gr}\, F^\bullet M$ is a finitely generated module over $C[T]$ (as mentioned in [Q]; here, $T \mapsto \mathrm{gr}(\phi)$).

We now "rewrite" the proof from [Q]. Note that M is an $A[T]$-module as above (with $T \mapsto \phi$); taking the associated graded of this (successively), we get a finitely generated module N over $B[T]$, where B is almost commutative, and N is obtained from M by taking successive associated graded modules in a standard way. Then $\mathrm{gr}(N)$ is finitely generated over $\mathrm{gr}(B[T])$.

By the generic flatness lemma (see [Q]), there exists a nonzero polynomial $f \in k[T]$, such that $\mathrm{gr}(N)$ is free over $k[T]_f$. This implies that N is free over $k[T]_f$, whence we will get that so is M (with $T \mapsto \phi$ when acting on M). On the other hand, $\mathrm{End}_A(M)$ is a skew field, so M is a vector space over $k(\phi) \subset \mathrm{End}_A(M)$. This is a contradiction if ϕ is transcendental over k. □

Next, recall the following definition from [Gi].

Definition 6.2. Let $k \subset A$ be an associative algebra endowed with two (non-unital) finitely generated commutative subalgebras $A'_\pm$ and an element $\delta \in A$. One says that this data defines an *algebra with commutative triangular decomposition* if the following hold:

- $\mathrm{ad}\,\delta$ preserves both $A'_\pm$;
- $\mathrm{ad}\,\delta$ acts diagonalizably on A; the eigenvalues for the action on $A'_\pm$ lie in $\pm\mathbb{Z}_{>0}$; and
- the algebra A is finitely generated as an A_--A_+ bimodule, where $A_\pm := A'_\pm \oplus k \subset A$. (This differs from [Gi] in order to reconcile our notion of $\mathcal{O}$ to his.)

In this case, Ginzburg's "Generalized Duflo Theorem" [Gi, Theorem 2.3] (which actually concerns a wider class of algebras) says that primitive ideals are the same as prime ideals, and are annihilators of simple objects of the appropriately defined BGG category $\mathcal{O}$ (provided it has finitely many simple objects). Applying this to our algebra H_z, we get:

Theorem 6.2 (Analogue of Duflo's theorem). *Primitive ideals in H_z are the same as prime ideals, and are annihilators of simple objects in $\mathcal{O}$.*

Proof. Let $R_\lambda := H_z/(t_z - \lambda)H_z$. Given a primitive ideal $I \subset H_z$, we get a simple H_z-module M; since $k = \overline{k}$, Quillen's theorem says that M is a simple R_λ-module for some $\lambda \in k$.

Suppose we show that $A = R_\lambda$ is a finitely generated A_--A_+-bimodule, where $A_\pm$ are the images of $B_+ := k[e,x], B_- := k[f,y]$ (respectively) under the quotient map $(a \mapsto \overline{a}) : H_z \twoheadrightarrow R_\lambda$. Then Ginzburg's theorem holds for R_λ (using $\delta = \overline{h}$ and $A'_\pm$ to be the augmentation ideals in $A_\pm$). Moreover, the category $\mathcal{O}_{R_\lambda}$ is contained in $\mathcal{O}^\lambda_{H_z}$, the summand in the spectral decomposition mentioned in a previous section, and hence it contains only finitely many simples.

Thus, primitive ideals for H_z are indeed annihilators of simple objects in $\mathcal{O}_{H_z}$. Moreover, $\overline{I}$ is prime, hence so is I. Conversely, if I is prime, then so is $\overline{I}$, whence it annihilates a simple object in $\mathcal{O}_{R_\lambda}$. Thus, I annihilates some $V(\mu) \in \mathcal{O}_{H_z}$.

Therefore, it suffices to show that $H_z/(t_z - \lambda)H_z$ is finitely generated as an A_--A_+ bimodule for any $\lambda \in k$. In view of the PBW decomposition $H_z = B_- \otimes k[h] \otimes B_+$, it will suffice to show that $h^i \in B_- M B_+ \ \forall i$, for some finite-dimensional M.

We claim that we may take $M = k \oplus kh \oplus \cdots \oplus kh^{2\deg(z)+1}$. Indeed,

$$\lambda = t_z = ey^2 + hxy - fx^2 - \frac{1}{2}hz - q_z \equiv z + \frac{1}{2}hz - q_z \mod B_- M B_+.$$

Now note that $\Delta = 4ef + (h^2 - 2h)$, whence (abusing notation)

$$h^a \Delta^b \in B_- \cdot k[h]/(h^{a+2b+1}) \cdot B_+ \ \forall a, b \geq 0.$$

In particular, $z, hz \in B_- M B_+$, so that $q_z \in B_- M B_+$. On the other hand, since $\deg(q_z) = \deg(z) + 1$ and since $h^{2\deg(z)+2} \in kq_z + k[f]Mk[e]$, we get that $h^{2\deg(z)+2} \in B_- M B_+$. From this, it follows that for any i, $h^i \in B_- M B_+$. ∎

It is an interesting problem to determine for which pairs of weights λ, μ, one has $I_\lambda := \mathrm{Ann}(V(\lambda)) \subset I_\mu := \mathrm{Ann}(V(\mu))$. As a first step, we have the following

Theorem 6.3. *If the central element t_z acts on $M(\lambda)$ by multiplication by α, then* $\mathrm{Ann}(M(\lambda))$ *is a two sided ideal generated by $t_z - \alpha$ in H_z.*

Proof. For the proof, at first we assume that $z = 0$. In this case $t_z = t = ey^2 + hxy - fx^2$ always acts by 0 on all Verma modules, so there is only one block. Thus we need to show that $\mathrm{Ann}(M(\lambda)) = tH$. As both sides of the desired equality are $\mathrm{ad}\,\mathfrak{g}$-submodules of H, and since the annihilator obviously contains tH, it will suffice to prove that if we have any (h-weight vector) $g \in H$ such that $[f, g] = 0 = gM(\lambda)$, then $g \in tH$. (We are considering "lowest weight vectors" inside H, which is a direct sum of finite-dimensional $\mathfrak{g}$-modules.)

Write g as $\sum g_{ijl} h^l e^i x^j$ where $g_{ijl} \in k[f, y]$. Since by assumption $[f, g] = 0$ then $gM(\lambda) = 0$ if and only if

$$gv_\lambda = 0 = gy^n v_\lambda = \sum g_{ijl}[h^l, y^n] e^i x^j v_\lambda = \sum g_{00l}[h^l, y^n] v_\lambda \ \forall n,$$

where the penultimate equality follows because $[x, y] = 0$ and $[e, y]v_\lambda = 0$. But $hy^n = y^n h - y^n$, so we get

$$gy^n v_\lambda = \sum_l g_{00l} y^n (h^l - (h - n)^l) v_\lambda = 0,$$

whence (cancelling y^n on the left in $M(\lambda) \cong B_- = k[f, y]$, an integral domain) we get that $f(n) = 0$ for all n, where

$$f(T) = \sum_{l>0} g_{00l}(\lambda - T)^l - \sum_{l>0} g_{00l} \lambda^l \in k[T]$$

By Lemma 5.1 (and induction on l), we conclude that

$$g_{00l} = 0 \ \forall l > 0 \tag{14}$$

Next, rewrite g as $\sum_{n=0}^N \sum_{i=0}^n a_{in} x^{n-i} y^i$, where $a_{in} \in \mathfrak{U}\mathfrak{g}$. Using the "dividing trick" (3), we may assume that g is not divisible by y from the right, so some $a_{0n} \neq 0$. Now, we have

$$0 = [f, g] = \sum_{i,n} (n - i) a_{in} x^{n-i-1} y^{i+1} + \sum_{i,n} [f, a_{in}] x^{n-i} y^i,$$

so $[f, a_{i+1,n}] = (n - i) a_{in}$ for all i, n. In particular, $[f, a_{0n}] = 0 \ \forall n$. Since H is a direct sum of finite-dimensional $\mathfrak{g}$-modules, $\mathrm{wt}(g)$ must be nonpositive, so $\mathrm{wt}(a_{0n}) \leq -n$.

There are only two steps remaining. First, we claim that $N > 1$ if $g \neq 0$, and second, if so, then we can find $a \in tH$ such that $g - a$ has "smaller N-value"; this finishes the proof, by induction on N.

Suppose $N = 0$ first. Then by a result similar to Lemma 2.3, $g = a_{00} = p(\Delta) \cdot f^l$ for some $l \geq 0$ and $p \in k[T]$. If this kills $y^n v_\lambda \ \forall \lambda$, then

$$p\left((\lambda - 2l - n)^2 + 2(\lambda - 2l - n)\right) = 0 \ \forall n$$

and this would imply that p is a constant, by Lemma 5.1. This contradicts that $p \cdot f^l$ annihilates $M(\lambda)$, unless $p = 0$.

Next, suppose $N = 1$ and $g = a_0 x + a_1 y + a_2$ (with all $a_i \in \mathfrak{U}\mathfrak{g}$), so that $a_0, a_2 \in k[f, \Delta]$. (Then $a_2 = 0$ by considering the parity of the possible weights.) Moreover, $a_1 = [e, a_0] + b$, where $[f, b] = 0$; therefore $a_1 y$ will contain a PBW monomial not containing e, x and containing h. But this contradicts equation (14) above.

This proves the first step; moreover, $f | a_{0N}$, since $\mathrm{wt}(a_{0N}) < N$ and $a_{0N} \in k[f, \Delta]$. Now consider a_{0N}/f; as in the proof of Proposition 2.6, there exists an element $g' = (a_{0N}/f) x^{N-2} + \sum_{i=1}^{N-2} c_i x^{N-2-i} y^i$ which commutes with f. Thus, $g + g' t \in \mathrm{Ann}(M(\lambda))$ commutes with f, and it is divisible by y from the right, hence we may divide by it. Proceeding by induction on N, the result is proved when $z = 0$.

Now let z be arbitrary. Given $\lambda \in k$, recall the inclusion in equation (12): $\mathrm{gr}(\mathrm{Ann}(M(\lambda))) \subseteq \mathrm{Ann}(\mathrm{gr}(M(\lambda))$. Moreover, $\mathrm{gr}(M(\lambda))$ is just a Verma module over H. Therefore if $g \in \mathrm{Ann}(M(\lambda))$, then $g = (t_z - \alpha)g' + g''$, where g'' has lower filtration degree than g (since $g'' \in \mathrm{Ann}(M(\lambda))$). Proceeding by induction on the filtration degree of g, we are done. □

We conclude by considering the constant parameter case: $z = 1$. The following theorem describes the primitive spectrum of H_z, as well as the multiplicities of irreducible modules in Verma modules.

Theorem 6.4. *For $\lambda \neq \mu$, $V(\lambda), V(\mu)$ lie in the same block if and only if $\lambda + \mu = -3$, and $M(\lambda)$ is irreducible if and only if $\frac{3}{2} + \lambda$ is not a positive integer. Otherwise we have $0 \to V(-3-\lambda) \to M(\lambda) \to V(\lambda) \to 0$, and $I_{-3-\lambda} \subsetneq I_\lambda$.*

In particular, (primitive) annihilator ideals for $\lambda \neq \mu$ are either not comparable ($\lambda \neq -\mu - 3$), or equal ($\lambda = -\mu - 3 \notin \dfrac{1}{2} + \mathbb{Z}$), or strictly comparable (otherwise).

Proof. Recall that in this case, the central element is equal to $t_1 := ey^2 + hxy - fx^2 - \dfrac{1}{2}h + \dfrac{1}{2}\Delta$, so it acts on $V(\lambda)$ by the scalar $1 + \dfrac{1}{2}(\lambda + ((\lambda+1)^2 - 1))$, hence $V(\lambda), V(\mu)$ lie in the same block if and only if $\lambda = \mu$, or $\lambda + \mu = -3$.

Next, note that $[x, y^2 + 2f] = 2y - 2y = 0$, therefore $x(y^2 + 2f)^n v_\lambda = 0$. We now determine when $(y^2 + 2f)^n v_\lambda$ is annihilated by e. Using that $[x, y^2 + 2f] = 0$, we have

$$\begin{aligned} 0 &= e(y^2 + 2f)^n v_\lambda = [e, (y^2 + 2f)^n] v_\lambda \\ &= \sum_{l<n} (y^2 + 2f)^l (2(yx + h) + 1)(y^2 + 2f)^{n-l-1} \\ &= n(2\lambda + 3 - 2n)(y^2 + 2f)^{n-1} v_\lambda. \end{aligned}$$

Hence if n is minimal among those for which $(y^2 + 2f)^n v_\lambda$ is a maximal vector, we must have $\lambda = n - \frac{3}{2}$. Now assume that $g = \sum_i a_i f^i y^{n-2i} v_\lambda$ is a maximal vector; then $x \cdot g$ must vanish. In other words,

$$0 = \sum a_i [x, f^i y^{n-2i}] v_\lambda = \sum (n - 2i) a_i f^i y^{n-2i-1} v_\lambda - \sum i a_i f^{i-1} y^{n-2i+1} v_\lambda.$$

This implies that $(n - 2i)a_i = (i + 1)a_{i+1}$ for all i. Hence, n is even and this system of equalities has exactly one solution up to multiplication by a constant; therefore $g = (y^2 + 2f)^{n/2} v_\lambda$.

To conclude, we have shown that $M(\lambda)$ is irreducible if $\frac{3}{2} + \lambda$ is not a positive integer, and otherwise we have the desired short exact sequence. Finally, since $(y^2 + 2f)^n \in \mathrm{Ann}(V(\lambda))$, therefore

$$\mathrm{Ann}(V(\mu)) = \mathrm{Ann}(M(\mu)) = \mathrm{Ann}(M(\lambda)) \subsetneq \mathrm{Ann}(V(\lambda)),$$

where $\lambda + \mu = -3$. □

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