

# INFINITESIMAL HECKE ALGEBRAS

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## References:

- *Category  $\mathcal{O}$  over a deformation of the symplectic oscillator algebra*, Journal of Pure and Applied Algebra (2005).
- *Quantized symplectic oscillator algebras of rank one* (joint with W.L. Gan), Journal of Algebra (2007).
- *Functoriality of the BGG Category  $\mathcal{O}$* , to appear in Communications in Algebra.
- *Center and representations of infinitesimal Hecke algebras of  $\mathfrak{sl}_2$*  (joint with A. Tikaradze), to appear in Communications in Algebra.

## 0.1. The setup.

- We will always work over  $\mathbb{C}$ .
- Fix a reductive Lie algebra  $\mathfrak{g}$  and a finite-dimensional  $\mathfrak{g}$ -module  $V$ .
- Study deformations  $H_\beta$  of  $\mathfrak{U}(\mathfrak{g} \ltimes V)$ , and their representations.

## 0.2. The problems.

- Which deformations  $H_\beta$  are flat (PBW property)?
- What is the character of simple finite-dimensional modules?
- What is the center of  $H_\beta$ ? How about central characters?
- Study the primitive ideals of  $H_\beta$ .

## 1. Background

- Drinfeld (1986) defined degenerate affine Hecke algebras over a finite group  $G$ ; these deform  $\mathbb{C}G \rtimes \text{Sym } V$ .
- Etingof and Ginzburg (2002) defined symplectic reflection algebras; here  $G \subset Sp(V)$ .
- E., G., and Gan (2005) introduced
  - continuous Hecke algebras, which deform  $\mathcal{O}(G)^* \rtimes \text{Sym } V$  ( $G =$  reductive algebraic group), and
  - infinitesimal Hecke algebras, which deform  $\mathfrak{U}\mathfrak{g} \rtimes \text{Sym } V$ .
- K. (2005) studied infinitesimal Hecke algebras over  $\mathfrak{sl}_2$ .
- K. and Gan (2007) studied the quantum analogue of  $H_\beta(\mathfrak{sl}_2)$ .
- Tikaradze (2006) computed the center of  $H_\beta(\mathfrak{sl}_2)$ .
- Montarani (2006) obtained a Morita equivalence between continuous deformed preprojective algebras and continuous symplectic reflection algebras.

## 2. Introduction; flatness

2.1. **Desirable properties.** For semisimple  $\mathfrak{g}$ ,  $\mathcal{U}\mathfrak{g}$  has several “good” properties:

- Triangular decomposition.
- Complete reducibility.
- A large enough center (but not too large).
- A good category  $\mathcal{O}$  of modules associated to primitive ideals (Duflo’s Theorem) and central characters.
- A block decomposition of  $\mathcal{O}$  into highest weight categories (using central characters).
- A quantized analogue  $U_q(\mathfrak{g})$  with similar representation theory.

Similar properties hold for  $H_\beta(\mathfrak{sl}_2)$ .

2.2. **Flatness.** A Drinfeld-Hecke algebra is generated by

- a (cocommutative) Hopf algebra  $A$ , and
- an  $A$ -module  $V$ ,
- subject to the relations:

$$a(v) = \sum a_{(1)}vS(a_{(2)}), \quad [v, v'] = \beta(v, v'),$$

where  $\beta : V \wedge V \rightarrow A$  is skew-symmetric.

**Examples:** Hopf algebras ( $V = 0$ ), infinitesimal Hecke algebras  $H_\beta(\mathfrak{g})$ , symplectic reflection algebras.

**Question:** For what  $\beta$  is  $H_\beta$  flat?

Drinfeld, Etingof-Ginzburg, E.-G.-Gan obtain conditions similar to Braverman-Gaitsgory (1996) - Conditions (I) and (J).

In general,

**Theorem 1** (K., 2007). *Given  $A, V, \beta, H_0 \twoheadrightarrow gr(H_\beta)$  is an isomorphism if and only if*

- $\beta : V \wedge V \rightarrow A$  is an  $A$ -module map.
- The Jacobi identity holds in  $A \otimes V$ :

$$\sum_{cyclic} \beta(v, v')v'' = \sum_{cyclic} v''\beta(v, v').$$

**Remark.**

- This holds even if  $A$  does not have a counit or an antipode.
- The Jacobi identity always holds if  $\dim V \leq 2$ .
- Bazlov-Berenstein (2007) call the the first condition, the *Yetter-Drinfeld condition*.

### 3. Symplectic oscillator algebras

3.1. **The classical case.** Now fix  $\mathfrak{g} = \mathfrak{sl}_2$ .

Then E.-G.-G. showed that if  $V = \mathbb{C}^2 = \mathbb{C}x \oplus \mathbb{C}y$ , then for flatness,

$[x, y]$  is central in  $H_\beta = H_z$ , where  $z = z(T)$  is a polynomial (and  $[x, y] = z(\Delta)$ ).

3.2. **The quantum case.**  $\text{char}(k) \neq 2$ .

$\text{Sym } k^2$  is a Hopf-module algebra over  $\mathfrak{U}(\mathfrak{sl}_2)$ .

For the quantum case,  $k_q[x, y]$  is a Hopf-module algebra over  $U_q(\mathfrak{sl}_2)$  (where  $q$  is not a root of unity).

$$H_{z,q} := \frac{U_q(\mathfrak{sl}_2) \rtimes k\langle x, y \rangle}{[x, y]_q - z(\Delta_q)}.$$

### 3.3. Center.

If  $z = 0$ , then  $\text{Sym}^2 V \cong \mathfrak{g} = \mathfrak{sl}_2$ , so  $t := ey^2 + hxy - fx^2$  is central.

**Remark.** This uses

**Lemma 1.** *Suppose  $A$  is a Hopf algebra, and an algebra  $R \supset A$  is an  $A$ -module algebra containing copies of finite-dimensional  $A$ -modules  $V, V^*$ . Then  $\sum_i v_i v_i^*$  commutes with  $A$ .*

This also helps construct central elements in higher rank cases, e.g.,:

- For  $H_0(\mathfrak{gl}_n)$  (and  $V = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$ ),  $\sum_i v_i v_i^*$  is central.
- For  $H_0(\mathfrak{sp}(2n))$ .

**Theorem 2** (Tikaradze). *The element  $t$  lifts to a central generator  $t_z$  of  $\mathfrak{Z}(H_z)$ .*

### 3.4. Duflo's Theorem and the BGG Category.

- $H_z$  has a triangular decomposition:  $H_z = B_- \otimes H \otimes B_+$ , where
  - $B_- = k[y, f]$  and  $B_+ = k[x, e]$  are commutative;
  - $H = k[h]$  is a commutative Hopf algebra.
- $\mathcal{O}$  contains all finitely generated  $H$ -semisimple  $H_z$ -modules, with finite-dimensional  $H$ -weight spaces and a locally finite  $B_+$ -action.

**Theorem 3** (K.-Tikaradze).

- (*Duflo, Ginzburg.*) *Primitive ideals, prime ideals, and annihilators of simple objects in  $\mathcal{O}$ , coincide for  $H_z$ .*
- *Every central character comes from a simple object in  $\mathcal{O}$ .*
- *If  $z \neq 0$ , there are at most finitely many nonisomorphic finite-dimensional simple  $H_z$ -modules  $L(\mu)$ .*
- *If  $z \neq 0$ , then every derivation of  $H_z$  is inner.*

## Consequences.

- $\mathcal{O} = \bigoplus \mathcal{O}_C(\mu)$ ; each block is an abelian category, that has
  - finite length.
  - finite cohomological dimension.
  - enough projectives (and injectives).
  - BGG Reciprocity.
- In fact, each block is a *highest weight category*.
- $\mathcal{O}$  contains *Verma modules*  $M(\mu)$ ; moreover, (K., 2005)
 
$$[M(\mu) : L(\mu')] \leq 2.$$
- The following are equivalent for any  $\mu, \mu' \in \mathbb{C}$  (Neidhardt):
  - $\text{Hom}(M(\mu), M(\mu')) \neq 0.$
  - $M(\mu) \hookrightarrow M(\mu').$
  - $[M(\mu') : L(\mu)] > 0.$

## 4. The quantum case, revisited

**Theorem 4** (Gan-K.).

- The quantized algebra  $H_{z,q}$  has trivial center, if  $z = [x, y]_q \neq 0$ .
- $H_{z,q}$  has classical limit  $H_z$  as  $q \rightarrow 1$ .

So the representation theories should be similar.

To see this, observe the common structure shared by  $A = \mathfrak{U}\mathfrak{g}$  ( $\mathfrak{g}$  reductive) and  $H_z, H_{z,q} \forall z$ :

- $A \cong B_- \otimes H \otimes B_+$ , with  $H$  a commutative Hopf algebra.
- $B_{\pm} = \bigoplus_{\mu \in \pm \mathbb{Z}_{\geq 0} \Delta} (B_{\pm})_{\mu}$  with respect to  $\text{ad } H$ .
- Each weight space is finite-dimensional and  $(B_{\pm})_0 = k$ .
- $A$  has an anti-involution  $i$ , such that  $i|_H \equiv \text{id}|_H$ .

**Definition.** Such an  $A$  is called a *Hopf regular triangular algebra (HRTA)*.

## 4.1. HRTA's - examples and first properties.

### Examples.

- $\mathfrak{U}\mathfrak{g}$  for  $\mathfrak{g}$  with a *regular triangular decomposition*, e.g.,
  - (centerless) Virasoro,
  - symmetrizable Kac-Moody,
  - extended (centerless) Heisenberg,
  - direct sums and split central extensions.
- $H_z, H_{z,q}$ .
- $U_q(\mathfrak{g})$  for  $\mathfrak{g}$  semisimple; can use “intermediate lattices” in  $\mathfrak{h}$ .

### Properties.

Now define  $\mathcal{O}_A$ .

- $\{\text{simples in } \mathcal{O}\} \leftrightarrow G := \text{group of weights} \leftrightarrow \{L(\mu) : \mu \in G\}$ .
- There exist *Verma modules*  $\{M(\mu) \twoheadrightarrow L(\mu) : \mu \in G\}$ .
- The center acts by a *character*  $\chi_\mu$  on  $M(\mu), L(\mu)$ .
- For  $\mu, \mu' \in G$ ,  $A_\mu A_{\mu'} \subset A_{\mu*\mu'}$ ,  $A_\mu M_{\mu'} \subset M_{\mu*\mu'}$ .

**How to use central characters.** Each of  $H_z, \mathfrak{U}\mathfrak{g}, U_q(\mathfrak{g})$  satisfies:

**Condition (S4).** For all  $\mu$ , the set  $S^4(\mu) := \{\mu' \in G : \chi_\mu = \chi_{\mu'}\} \subset G$  is finite.

Then  $\mathcal{O} = \bigoplus \mathcal{O}_C(\mu)$ , and  $\mathfrak{U}\mathfrak{g}$  satisfies (S4) because  $S^4(\mu) = W \bullet \mu$ .

Alternate conditions (e.g., for  $H_{z,q}$ ):

**Definitions.**

- $S^3(\mu) = S^2(\mu)$  is the equivalence closure of  $\mu$  under:  $\mu \rightarrow \mu'$  if  $[M(\mu') : L(\mu)] > 0$ .
- *Partial order:*  $\mu \geq \mu'$  if  $\mu * (\mu')^{-1} \in \mathbb{Z}_{\geq 0}\Delta$ .
- $S^1(\mu) := S^3(\mu)^{\leq \mu}$ .
- $A$  satisfies *Condition (S1)-(S3)* if the corresponding sets  $S^m(\mu) \subset G$  are all finite.
- The *block*  $\mathcal{O}(\mu) \subset \mathcal{O}$  contains all  $M \in \mathcal{O}$  whose subquotients are all in  $\{L(\mu') : \mu' \in S^3(\mu)\}$ .

## 4.2. Results without central characters.

**Theorem 5.** *A is any HRTA.*

- $(S4) \Rightarrow (S3) \Rightarrow (S2) \Rightarrow (S1)$ , since  $S^1(\mu) \subset \cdots \subset S^4(\mu) \forall \mu$ .
- $(S1) \Rightarrow \mathcal{O} = \bigoplus \mathcal{O}(\mu)$ ; each block is abelian and finite length.
- If  $A$  satisfies  $(S2)$ , then each  $\mathcal{O}(\mu)$  has enough projectives (and injectives), each with a Verma flag.
- If  $A$  satisfies  $(S3)$ , then each  $\mathcal{O}(\mu)$  is a highest weight category.
- If  $A$  satisfies  $(S3)$  and  $B_-$  is an integral domain, Neidhardt's theorem holds.

We can now prove that the quantum case is very “nice” too:

**Theorem 6** (Gan-K.).  $H_{z,q}$  satisfies  $(S3)$  (but not  $(S4)$ ) if  $z \neq 0$ .

Thus, most of the “desirable properties” also hold for  $H_{z,q}$ .

## 5. Functoriality

### 5.1. Tensor products.

Suppose we want to deform  $\mathfrak{U}(\oplus_i(\mathfrak{g}_i \times V_i))$ , e.g.,  $H_{z_1} \otimes \cdots \otimes H_{z_n}$ .  
 Or - study  $\mathfrak{U}\mathfrak{g} \otimes U_q(\mathfrak{g}') \otimes H_z \otimes H_{z'} \otimes H_{z'',q'}$ .

**Theorem 7.** *Fix a ground field  $k$ .*

- *If  $A_1, \dots, A_n$  are HRTA's, then so is  $A := \otimes_i A_i$  (with  $H = \otimes_i H_i, G = \times_i G_i$ , etc.).*
- *Simple and Verma objects satisfy: if  $\mu = (\mu_i)$ , then*

$$M(\mu) = \otimes_i M_i(\mu_i), \quad L(\mu) = \otimes_i L_i(\mu_i).$$
- *For all  $1 \leq m \leq 4$  and  $\mu = (\mu_i) \in G$ ,  $S^m(\mu) = \times_i S_i^m(\mu_i)$ .*
- *$A$  satisfies any of Conditions (S1)-(S4) if and only if each of the  $A_i$ 's do.*
- *Complete reducibility holds in  $\mathcal{O}_A$  if and only if it holds in all  $\mathcal{O}_{A_i}$ 's.*

## 5.2. Wreath products.

- Assume that  $k = \mathbb{C}$ ,  $A$  is an HRTA, and  $n \in \mathbb{N}$ .
- The *wreath product* is defined to be  $S_n \wr A := A^{\otimes n} \rtimes S_n$ .
- $\mathcal{O}$  over  $S_n \wr A$  is defined as:  $\mathcal{O} = (S_n \wr A)\text{-Mod} \cap \mathcal{O}_{A^{\otimes n}}$ .
- Define  $X$  to be the set of (isoclasses of)  $H^n$ -semisimple simple  $S_n \wr H$ -modules.
- Now,  $S^3(x), S^4(x) \subset X$ , while  $S^1(x), S^2(x) \subset G^n$ .

### Proposition 1.

- $X$  characterizes the simple objects  $\{L(x) : x \in X\}$  in  $\mathcal{O}$ .
- There exists a map  $\text{wt} : X \rightarrow G^n/S_n$ , such that:
  - $\text{wt}$  is a finite-to-one map.
  - $L(x) = \bigoplus_{\mu \in \text{wt}(x)} L'(\mu)^{\oplus \dim x_\mu}$  as  $A^{\otimes n}$ -modules.
  - $\dim_k L(x) = \dim L'(\text{wt}(x)) \cdot \dim x$  (abusing notation).
  - $M(x)$  has finite length if and only if  $M'(\text{wt}(x))$  does.

**Theorem 8.** *Suppose  $A$  is an HRTA as well as an integral domain (e.g.,  $A = \mathfrak{U}\mathfrak{g}, U_q(\mathfrak{g}), H_z, H_{q,z}$ ). Then:*

- *The center of  $S_n \wr A$  is  $(\mathfrak{Z}(A)^{\otimes n})^{S_n}$ .*
- *The center acts on each  $L(x), M(x)$  by:  $\chi_x = \chi_{\text{wt}(x)}$ .*
- *For  $\sigma \in S_n$ ,  $\mu = (\mu_i) \in G^n$ , and  $1 \leq m \leq 4$ ,*  

$$S^m(\sigma(\mu)) = \sigma(S^m(\mu)).$$
- *Any of Conditions (S1)-(S4) - as well as complete reducibility in  $\mathcal{O}$  - holds for  $S_n \wr A$ , if and only if it holds for  $A$ .*
- *Theorem 5 holds (except for Neidhardt's result).*

**Remark.** These results can be (suitably) generalized to any  $\Gamma \subset S_n$  acting on  $A^{\otimes n}$ , or on any finite  $\Gamma$  acting on an HRTA  $A'$  (e.g., on  $A' = A^{\otimes n}$ ).

## 6. Future work - higher rank cases

- E.-G.-G. classified the flat deformations of  $H_0(\mathfrak{gl}_n), H_0(\mathfrak{sp}(2n))$ .
- For all  $n, \beta$ ,  $H_\beta(\mathfrak{gl}_n)$  is an “HRTA” with nontrivial center.
- If  $A = \mathcal{U}\mathfrak{g}$  for reductive  $\mathfrak{g}$ , then  $H_\beta$  is a direct sum of finite-dimensional  $\mathfrak{g}$ -modules. (Useful in computing centralizers.)

### Questions.

- (1) Is  $\mathfrak{Z}(H_\beta(\mathfrak{g}))$  a polynomial algebra ( $\dim \mathfrak{h}$  generators)?
- (2) Does  $H_\beta(\mathfrak{g})$  admit a natural quantization? What is its center?
- (3) Does Duflo’s Theorem hold here?
- (4) Does complete reducibility hold for at least one value of  $\beta$ ?
- (5) Which Conditions (S) does  $H_\beta(\mathfrak{g})$  satisfy?
- (6) Given  $\chi$ , classify all irreducible Whittaker modules for  $H_\chi$ .
- (7) Bring in Bazlov-Berenstein’s theory of braided doubles?