DIRECT SUM AND TENSOR PRODUCT DECOMPOSITION OF MODULES

Contents

1. Classification of simple objects

Fix a field k. Suppose $\{R_i : 1 \leq i \leq n\}$ are unital k-algebras, and we define $R = \otimes_i R_i$, where we tensor over k. The goal is to prove that simple R-modules come from tensor products of simple R_i-modules:

Conjecture 1. *Let X_i (resp. X) denote the set of isoclasses of simple R_i- (resp. R-)modules. Then $\otimes : \times_i X_i \to [R\text{-Mod}]$, sending $\{M_i\} \mapsto \otimes_i M_i$ (or their isoclasses, more precisely), is a bijection onto X.*

Here are some examples of bijections of the above kind, or between "special" subsets:

(1) As in [Ser, Theorem 10, §3.2], if $R_i = \mathbb{C}G_i$ is the group algebra of a finite group over complex numbers (or any algebraically closed field of characteristic zero), then the result holds.

(2) Suppose $\mathfrak{g}_i \supset \mathfrak{h}_i$ is a complex semisimple Lie algebra containing (some choice of) its Cartan subalgebra, for each i. First set $R_i = \mathfrak{U}\mathfrak{h}_i$; then the set of *weights* (or simple objects in the BGG Category $\mathcal{O}$ over $\mathfrak{g}_i$) is parametrized by $\mathfrak{h}_i^* = \mathrm{Hom}_{alg}(R_i, \mathbb{C})$. But then the tensor products of these simple R_i-modules are in bijection with the set $\mathfrak{h}^* = \mathrm{Hom}_{alg}(R, \mathbb{C})$.

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(3) Next, in the same setup, let $R_i = \mathfrak{U}\mathfrak{g}_i$, and look only at simple objects in the BGG Category $\mathcal{O}$. Then a $\mathfrak{g}$-module V is simple if and only if it is of the form $\otimes_i V_i(\mu_i)$, with $\mu_i \in \mathfrak{h}_i^*$ for all i, and $V_i(\mu_i)$ in the BGG Category $\mathcal{O}_i$ over $\mathfrak{g}_i$. (Note that these modules are not finite-dimensional in general.)

Of course, this result is *false* in general; for example (cf. [Boy]), let $R_i = K$, a field extension of k; then $X_i = \{K\}$, but X is not necessarily a singleton set - otherwise $R = K^{\otimes_k^n}$ is simple over itself (and commutative), whence it is a field.

For instance, let $n = 2, K = k(t)$, the field of rational functions in one variable. Then R is the localization of the polynomial algebra $k[x, y]$ obtained by inverting all nonzero elements of $k[x]$ and all nonzero elements of $k[y]$. Thus, for instance, $x + y$ is not invertible in R.

So we need some assumptions. We will prove the result in two different setups, and in various steps in each setup. The first step is common:

Step 1.

Let us first show that if the map is defined, it is one-to-one. Note firstly that for any fixed i, $M := \otimes_j M_j$ is isomorphic to a direct sum of copies of the simple module M_i. (This is because R_i embeds into R via: $r_i \mapsto \otimes_{l \neq i} 1 \otimes r_i$.) Hence if $\otimes_j M_j \cong \otimes M'_j$, then let us look at any simple quotient of both sides, as R_i-modules. But by the following lemma, any simple quotient of the left side is M_i, and on the right side, it is M'_i; hence $M_i \cong M'_i$ for all i.

Lemma 1. *For any ring R, every simple (sub)quotient of a direct sum of R-modules, is automatically a simple (sub)quotient of some summand.*

2. Proof by Mitya Boyarchenko

The result here will be stronger; the idea is to apply it to the case of algebras R_i containing commutative unital subalgebras H_i (which are to be thought of as $\mathfrak{U}\mathfrak{h}_i$ in some sense), and to the theory of finite-dimensional modules which are simple over R_i and diagonalizable (semisimple) over H_i.

Theorem 1 (Boyarchenko)**.** *Let k be an algebraically closed field, X_i denote the set of finite-dimensional H_i-semisimple R_i-modules, and similarly define X over R (using $H := \otimes_i H_i$). Then $\times_i X_i = X$.*

For instance, to dispense with the H_i-semisimplicity, simply put $H_i = k\ \forall i$. We also remark (as said by Victor Ginzburg) that this follows from Jacobson's Density Theorem.

Proof. This proof is an exercise in Wedderburn Theory. The underlying philosophy is that it does not matter how small or large the algebras R_i's are, as long as the modules M_i are finite-dimensional - because then R_i

factors through its image in $\mathfrak{gl}_k(M_i)$, and hence may just be replaced by this finite-dimensional k-algebra, for which M_i is a faithful module.

We prove the theorem in three (more) stages; the first two stages are from [Boy]; we forget about the H_i's here (i.e. suppose $H_i = k$, say). The final stage will bring in the H_i's.

Step 2.
Thus, let M_i be simple R_i-modules for now; define R'_i to be the image of R_i in $\mathfrak{gl}_k(M_i)$. Then R'_i is an Artinian (since it is finite-dimensional) k-algebra with a faithful simple module M_i. By Wedderburn theory, it is simple (**why?** See [FD].) Hence it is isomorphic to $M_{n_i}(D_i)$ for some $n_i \geq 0$ and some division ring D_i.

Now D_i is a finite-dimensional division algebra over k (since R'_i is finite-dimensional); since k is algebraically closed, $D_i = k\ \forall i$.

Let us now look at the R-action on $M = \otimes_i M_i$. It clearly factors through its image R' in $\otimes_i R'_i = \otimes_i M_{n_i}(k)$; this image is, of course, all of $\otimes_i R'_i$. Moreover, the module now is $\otimes_i M_i = \otimes_i k^{n_i}$ - and this is a simple R'-module, i.e. a simple R-module.

Step 3.
Conversely, let $R = \otimes_i R_i$, and M a simple finite-dimensional R-module. Since the R_i's act on M, the actions factor through finite-dimensional quotients $R'_i \subset \mathfrak{gl}_k(M)$ respectively; M is a faithful R'_i-module for each i.

We now claim that each R'_i is a semisimple k-algebra. To see this, note that they are all finite-dimensional, hence Artinian; thus the obstruction to being semisimple is a nonzero Jacobson radical. But the Jacobson radical is now nilpotent.

Now, if $I \subset R'_i$ is any nilpotent two-sided ideal for some i, then $I \otimes \otimes_{j \neq i} R'_j$ is a nilpotent two-sided ideal of $R' := \otimes_i R'_i$, whence it lies in $J(R')$, and hence kills every simple R'-module. This is a contradiction, since $I \otimes \otimes_{j \neq i} k \subset R'_i \cap (I \otimes \otimes_{j \neq i} R'_j)$ acts faithfully on the simple R'-module M (the action is faithful as R'_i-modules).

Therefore each $R'_i \cong \bigoplus_j M_{n_{ij}}(D_{ij})$ as above (by Wedderburn Theory), where each D_{ij} is a finite-dimensional (associative) division ring over the algebraically closed field k (as above). Hence it equals k; thus, R is a direct sum of algebras of the type $\otimes_i M_{n_i}(k)$. But M is a simple R'-module, so R' acts only through one of these, and $M = \otimes_i k^{\oplus n_i}$. Hence M is isomorphic to $\otimes_i M_i$, where each M_i is a simple R'_i-submodule of M, i.e. a simple R_i-module.

Step 4.
Finally, we bring in the commutative unital subalgebras H_i. Then a *weight* of H_i is an algebra map or character $\mu_i : H_i \to k$, and the μ_i*-weight space* in a module M_i is the set $\{m \in M_i : h \cdot m = \mu_i(h)m\ \forall h \in H_i\}$.

The first task is to relate the weights of H and of the H_i's. Denote these sets by G, G_i respectively. Then we claim that $G = \times_i G_i$; indeed, algebra maps $\mu : \otimes_i H_i \to k$ determine, and themselves are determined by their "components" $\{\mu_i = \mu|_{H_i} : H_i \to k\}$. Note that one can do this because the H_i's are all unital, thereby embedding into H for all i.

Suppose first that $M_i \in X_i$. By above, $M := \otimes_i M_i$ is a simple R-module. Moreover, it is H-semisimple, since the $\mu = (\mu_1, \dots, \mu_n)$-weight space is precisely $\otimes_i (M_i)_{\mu_i}$. Hence $M \in X$.

Conversely, given $M \in X$, by the above theory, $M = \otimes_i M_i$. We first show that $M_\mu \subset \otimes_i (M_i)_{\mu_i}$ for each $\mu \in G = \times_i G_i$. But this is easy: if $m_\mu \in M_\mu$ can be written as $\sum_j \otimes_i m_{ij}$ (in terms of a suitable product of k-bases of the M_i's), then for each $h_i \in H_i$,

$$\sum_j \otimes_{l \neq i} m_{lj} \otimes (h_i \cdot m_{ij}) = h_i \cdot m_\mu = \mu(h_i) m_\mu = \mu_i(h_i) \sum_j \otimes_l m_{lj}$$

and by linear independence over k, $h_i m_{ij} = \mu_i(h_i) m_{ij}$ for all i, j.

Therefore we finally conclude the proof of this theorem, by noting that

$$M = \bigoplus_{\mu \in G = \times_i G_i} M_\mu \subset \bigoplus_{\mu \in G} \bigotimes_i (M_i)_{\mu_i} = \bigotimes_i \oplus_{\mu_i \in G_i} (M_i)_{\mu_i} \subset \bigotimes_i M_i = M$$

whence all inclusions are actually equalities, and $M_i = \bigoplus_{\mu_i \in G_i} (M_i)_{\mu_i}$ is H_i-semisimple for all i. So $X = \times_i X_i$. $\square$

3. Proof by Sebastian Zwicknagl

Let M_i be a simple R_i-module for each i; we look at $M := \otimes_i M_i$. To show that M is a simple R-module, it suffices to show that any nonzero $m \in M$ generates all of M. We show it as follows: clearly, the module $M' = R \cdot m$ is a proper (since each R_i is unital) nonzero R-submodule of M. We will try to produce a surjection : $M' \twoheadrightarrow M$, which will prove the result in all categories $\mathcal{C}$ of R-modules that satisfy:

Standing Assumption 1. No object in $\mathcal{C}$ has a submodule that surjects onto it.

For instance, if $\mathcal{C}$ is the category of finite-dimensional (over k) objects - or if $\mathcal{C}$ is the *Harish-Chandra* category, wherein every module has a *formal character ch* with finite-dimensional entries (and $N \subset N' \Rightarrow ch_N \leq ch_{N'}$) - then the result is true. So let us state (rather imprecisely!) what we will prove.

Theorem 2 (Zwicknagl)**.** *Let X_i (resp. X) denote the set of isoclasses of simple R_i- (resp. R-)modules, where R_i's are unital k-subalgebras over any field k. Then $\otimes_i : \times_i X_i \to R - \mathrm{Mod}$, sending $\{M_i\}$ to $\otimes_i M_i$, is a injection into X - provided the standing assumption above and below hold in the (sub)categories of modules that we work in.*

So here goes the proof now. There will be one more technical assumption, which we will remark on, *after* the proof.

Proof. The (start of the) proof is taken from [Zwi]. (We have yet to complete it!)

Step 2.

The next task is to prove that the image of the map is contained in X. To do so, we use the argument in the first paragraph in this section: define $M' = R \cdot m$ for any fixed $0 \neq m \in M$, and try to prove that $M' \twoheadrightarrow M$.

To do this, we look at the individual components of M'. By above remarks, $R_i \cdot m$ is a direct sum of copies of M_i for each i.

Standing Assumption 2. Suppose also that any cyclic submodule of $\oplus M_i$ is one copy M_i of the simple R_i-module. (This is a nontrivial assumption, as we will see after the end of the proof.)

We now use induction to prove *two statements together*:

$(I)_n$: The simple $\otimes_{i=1}^n R_i$-modules are precisely $\otimes_i M_i$, where M_i is R_i-simple for all i.

$(II)_n$: If M is a simple $\otimes_{i=1}^n R_i$-module, and m is a nonzero vector in a direct sum of copies of M, then $\otimes_{i=1}^n R_i \cdot m \cong M$.

We will prove the results for larger and larger collections from inside our fixed set $\{R_1, \dots, R_n\}$. Note that the base cases are already done: we know $(I)_1$ and assume $(II)_1$ for each R_i.

Thus we assume that $R_i \cdot m \cong M_i$ (continuing from the first paragraph in Step 2). We now prove the result by induction. We know it for $n = 1$; given the result for $n - 1$, define $S := \otimes_{i>1} R_i$. Then $N := \otimes_{i>1} M_i$ is a simple S-module by $(I)_{n-1}$.

We now use this to define a surjection $\varphi : R := R_1 \otimes S \to M_1 \otimes N = M$. Given $y \otimes s$ with $y \in R_1, s \in S$, define

$$\varphi(y \otimes s) = (y \cdot m) \otimes (s \cdot m)$$

and extend this action by linearity. The image lies in $(R_1 \cdot m) \otimes (S \cdot m) = M_1 \otimes N$ by $(II)_{n-1}$.

This map is clearly an R-module map (with R an R-module via left-multiplication), and a set-theoretical surjection. We now show that it factors through to M', which will complete the proof.

Now suppose $\sum y_i \otimes s_i$ kills m. We may take a k-basis of S that extends a basis of $\operatorname{Ann}_S(m)$, and write the above sum with each s_i in this basis. Thus some of the s_i's kill m, and this "part" of the definition of φ automatically is zero.

The rest of the $s_i \cdot m$ are linearly independent in N, so we write $M = M_1 \otimes N$ as a sum of copies of M_1, using the above basis for the vector space N. Then note that $s_i m$ and $y_i s_i m$ are in the same copy of M_1. Since their sum is zero across different copies, each $y_i s_i m = 0$, and since $s_i m \neq 0$ in the simple R_1-module M, hence $s_i : R_1 m \to s_i \cdot R_1 m$ is an R_1-module isomorphism. Hence if $s_i y_i m = 0$, then $y_i m = 0$ for all i.

In short, if $\sum y_i \otimes s_i$ kills m, then at least one of y_i, s_i is in the annihilator of m (in R_1, S respectively), whence φ is well-defined. This completes the proof of $(I)_n$, given $(I)_{n-1}$ and $(II)_{n-1}$.

It remains to show $(II)_n$ (equivalently, $(II)_2$). Suppose we have $R = \otimes_{i=1}^n R_i$ and an R-simple module M. Thus we have R_i-simple modules M_i by $(I)_n$, and $M = \otimes_i M_i$. Now choose any $0 \neq m \in \oplus M$. To consider $R \cdot m$, we first note that $R_i \cdot m$ is an R_i-submodule of $M = M_i \otimes W$ (where W is the vector space $\otimes_{j \neq i} M_j$). Hence by $(II)_1$, $R_i \cdot m \cong M_i$ for all i.

Moreover, let us use $(II)_{n-1}$ as follows: define S, N as above. Then $S \cdot m \subset M_1 \otimes N = \oplus N$ (as S-modules), hence it must be isomorphic to N.

We now produce an R-module isomorphism : $R \cdot m \to \otimes_i M_i$. Once again, start with $\varphi : R = R_1 \otimes S \to M = M_1 \otimes N$, via: $y \otimes s \mapsto (y \cdot m) \otimes (s \cdot m)$, and extend by linearity. This is an R-module map, and a set-theoretical surjection; one has to show that (a) it factors through $R \cdot m$, and (b) that this new map is injective.

The second step is the easier one, so we answer this first. Clearly, $R_1 \otimes S \twoheadrightarrow M_1 \otimes N$ (which is a simple module too); and the kernel of this surjection is clearly $\mathrm{Ann}_{R_1}(m) \otimes S + R_1 \otimes \mathrm{Ann}_S(m)$, where $1 \mapsto m$ in the above surjection. (To see this, just compute the quotient, using that the annihilators are maximal left ideals.) But it is also clear that

$$\mathrm{Ann}_{R_1}(m) \otimes S + R_1 \otimes \mathrm{Ann}_S(m) \subset \mathrm{Ann}_R(m) = \mathrm{Ann}_{R_1 \otimes S}(m)$$

since R_1 and S commute inside R. Thus, the new map factors through $\mathrm{Ann}_R(m)$, hence (by the above equation) through the "smaller" kernel (of the other map), and will be injective.

Finally, the proof of surjectivity is the same as the one above (for $\varphi : M' \twoheadrightarrow M_1 \otimes N$), so we do not write it again. □

Attempt at Step 3.

The last step is to prove that the above map is a surjection. Once again, we prove this by induction on n. We know the statement for $n = 1$; once again, assume it for $n - 1$. Now suppose we have a simple $R = R_1 \otimes S$-module M. Fix a nonzero $m \in M$; this $R \cdot m = M$. Now let $M_1 = R_1 \cdot m$ and $N = S \cdot m$.

The aim is now to show that $M = M_1 \otimes N$, for if we do this, then we automatically claim that M_1, N are simple R_1, S-modules respectively. This follows from the following obvious lemma:

Lemma 2. *If A_i are (unital) k-algebras over a field k, and M_i are A_i-modules, then any set of filtrations of M_i by A_i-submodules, of lengths l_i respectively, yield a filtration of $M := \otimes_i M_i$ by $A := \otimes_i A_i$-modules, of length $\prod_i l_i$. (Here, all $\otimes$'s are over k.)*

But if $M = M_1 \otimes N$, then we are done by the induction hypothesis for N as a simple S-module.

It thus remains to show that $M = M_1 \otimes N$. Let us define a surjection $\varphi : R \to M_1 \otimes N$ as above: $\varphi(y \otimes s) := (y \cdot m) \otimes (s \cdot m)$, and extend it by linearity.

(What next??)

Remark 1. The second Standing Assumption above (or Condition $(II)_1$) says:

R is a ring, such that if V is a simple R-module, and M is a cyclic submodule of a sum of copies of V, then $M \cong V$.

Cf. [Boy], the following hold: this condition is equivalent to:

Every maximal left ideal is a two-sided ideal.

Moreover, the image of $R \subset \mathfrak{gl}_k(V)$ is a division ring. (Note: this setup does not need V to be finite-dimensional, since we're trying to prove the theorem for all modules.)

So if R is a finitely generated division $\mathbb{C}$-algebra, for instance, then its image is also such an algebra, hence it is $\mathbb{C}$ (or the set of scalar matrices, since $1 \in R$). But then $V = \mathbb{C}$ too!

Moreover, here's a rather interesting fact, which says that the above condition is not true. It is taken from [CR, Corollary 5.7.3].

Proposition 1. *Suppose R is a simple ring, and not left-Artinian. Then every module of finite length is cyclic.*

So in this case, the above property is about as untrue, as possible!

4. Complete reducibility in the wreath product case

The idea behind this section comes from proving that if $\mathfrak{g}$ is any complex semisimple Lie algebra, then one can form $S_n \wr \mathfrak{U}\mathfrak{g} := (\mathfrak{U}\mathfrak{g})^{\otimes n} \rtimes S_n$, with S_n permuting the factors. Then since finite-dimensional modules over $\mathfrak{g}$ are semisimple, the same is true of the wreath product algebra $S_n \wr \mathfrak{U}\mathfrak{g}$.

To do this in general, we work in steps: the wreath product is obtained by first considering the tensor product of copies of $\mathfrak{U}\mathfrak{g}$, and then taking the semidirect product. This second construction is easier, so we approach it first.

In what follows, we will use homological methods. Since the categories in question are those of all finite-dimensional modules over various algebras, they are full abelian subcategories of the categories of all modules. So in order to prove that Ext^1 vanishes for all objects in the category, it is enough, by the long exact sequence of Ext's, to prove that all Hom's in the category - or (by fullness) in $R-\mathrm{Mod}$ for various R, are exact functors.

Occasionally, we will also refine the methods to say that Ext's vanish for all objects (we talk here of the vanishing of the functions

$$\mathrm{Hom}(M,-),\ \mathrm{Hom}(-,N),\ \mathrm{Ext}^1(M,-),\ \mathrm{Ext}^1(-,N)$$

for M,N finite-dimensional modules) if and only if it vanishes for all simple objects. This is because all finite-dimensional modules have finite length.

4.1. **Smash products and complete reducibility.**

Proposition 2. *For a finite group Γ and an algebra R over a field of characteristic zero, suppose $\mathcal{P} \subset R-\mathrm{Mod}$ and $\mathcal{D} \subset (R\times\Gamma)-\mathrm{Mod}$ are full abelian subcategories of finite-dimensional modules, with each $D \in \mathcal{D}$ satisfying: $\mathrm{Res}_{\mathrm{R}}^{\mathrm{R}\times\Gamma} D \in \mathcal{P}$. If $\mathcal{P}$ is now a semisimple category, then so is $\mathcal{D}$.*

Proof. It suffices to show that $\mathrm{Hom}_{\mathcal{D}}(M,-)$ is exact for each object M of $\mathcal{D}$. Since $\mathcal{D}$ is also full, we use [Mac, Equation (A.1), Appendix], and compute:

$$\mathrm{Hom}_{\mathcal{D}}(M,-) = \mathrm{Hom}_{R\times\Gamma}(M,-) = \Big(\mathrm{Hom}_R(\mathrm{Res}_{\mathrm{R}}^{\mathrm{R}\times\Gamma} M, \mathrm{Res}_{\mathrm{R}}^{\mathrm{R}\times\Gamma} -)\Big)^{\Gamma}$$

By Maschke's Theorem (over characteristic zero), taking Γ-invariants is an exact functor (since everything is finite-dimensional here), as is $\mathrm{Res}_{\mathrm{R}}^{\mathrm{R}\times\Gamma} - : \mathcal{D} \to \mathcal{P}$. By semisimplicity in the full subcategory $\mathcal{P}$, $\mathrm{Hom}_{\mathcal{P}}(\mathrm{Res}_{\mathrm{R}}^{\mathrm{R}\times\Gamma} M,-)$ is also exact. Thus their composite is exact as well. □

The result proves the above assertion, in fact, since $(\mathfrak{Ug})^{\otimes n} = \mathfrak{U}(\mathfrak{g}^{\oplus n})$, and we have Weyl's Theorem.

4.2. **Tensor products and complete reducibility.**

The result we wish to prove is as follows: suppose R_i are unital k-algebras; define $R = \otimes_i R_i$. Now fix full abelian subcategories $\mathcal{C}_i$ (resp. $\mathcal{C}$) of finite-dimensional modules over R_i (resp. R), with $\otimes_i \mathcal{C}_i = \mathcal{C}$. Then if each $\mathcal{C}_i$ is a semisimple category, then so is $\mathcal{C}$.

This is slightly trickier than the previous result. We present an approach by Akaki Tikaradze [Tik]. First recall that by above remarks, it suffices to show that $\mathrm{Hom}_{\mathcal{C}}(M,-)$ is exact for each simple $M \in \mathcal{C}$. Next, we of course wish to do this by relating $\mathrm{Hom}_{\mathcal{C}}(\otimes_i M_i,-)$ with $\otimes_i \mathrm{Hom}_{\mathcal{C}_i}(M_i,-)$. The result one needs to use, therefore, is a Künneth-type formula (in some setting). In order to use such an approach, we need k to be algebraically closed, so that by the above theorem (proved by Boyarchenko), proving such a relation is enough.

We will need the following preliminary results.

Lemma 3. *Suppose F is a free R-module with basis I, and M is any R-module - for a ring R. Then $\mathrm{Hom}_R(F,M) = M^I$ as vector spaces.*

Note that M^I is the direct product of I copies of M (also known as the free R-module of M-valued functions on I). This is *different* from the direct *sum* of I copies of M, unless I is a finite set.

Proof. Fix an R-basis $\{f_i : i \in I\}$ of F. By universality of F, each R-module map : $F \to M$ is uniquely determined by its values at each f_i. □

Lemma 4. *Suppose R_i are unital k-algebras, M_i are R_i-modules, and F_i are free R_i-modules of finite rank n_i (for each $1 \leq i \leq n$). Then $\otimes_i F_i$ is a free R-module of rank $n := \prod_i n_i$, and the obvious map $\omega : \otimes_i \operatorname{Hom}_{R_i}(F_i, M_i) \to \operatorname{Hom}_R(\otimes_i F_i, \otimes_i M_i)$ is a vector space isomorphism.*

Proof. The obvious map is k-linear, and it is injective too: suppose $\otimes_i \eta_i \neq 0$. Then no η_i is zero, so there is some $f_i \in F_i$ so that $\eta_i(f_i) \neq 0$. But then $\omega(\otimes_i \eta_i)$ does not kill $\otimes_i f_i$.

It remains to show that ω is onto; but for this it suffices to compute the dimensions of both sides, and show that both these (finite) numbers are equal. This follows from the previous lemma, since the left-side yields

$$\otimes_i M_i^{n_i}$$

and the right-hand side yields $(\otimes_i M_i)^n$, with n as above. Since we only deal with finite exponents, the two sides are equal. □

We now present the result and the proof.

Theorem 3 (Tikaradze)**.** *($R_i, R, \mathcal{C}_i, \mathcal{C}$ as above, $k = \overline{k}$.) Suppose each R_i is Noetherian. If $\mathcal{C}_i$ is semisimple for all i, then so is $\mathcal{C}$.*

Proof. Given any finite-dimensional module over a Noetherian ring, it is the quotient of a free module of finite rank; the kernel, being a submodule of a Noetherian module, is itself finitely generated, and hence is a quotient of another free module of finite rank - and so on.

The theorem also works if we only assume (instead of the Noetherian-ness property) that each simple finite-dimensional R_i-module (or the maximal left ideal annihilating it) is finitely presented, for all we are concerned with, is to show that $\operatorname{Ext}^1_{\mathcal{C}}$ vanishes, and we will obtain this via a "Künneth-type" formula, so we only need to use $\operatorname{Ext}^1_{\mathcal{C}_i}$'s.

We now prove the result. We have to show that $\operatorname{Ext}^1_{\mathcal{C}}(M, N)$ vanishes for each pair of simple objects M, N in $\mathcal{C}$, and by the theorem proved by Boyarchenko (see above), each simple M in $\mathcal{C}$ is of the form $\otimes_i M_i$, for M_i a simple object in $\mathcal{C}_i$.

Now take any free resolution $F_i^\bullet$ of M_i in $R_i - \operatorname{Mod}$, where each term is a free module of finite rank (this is possible by above remarks). Then one forms the "Künneth-type" complex $\otimes_i F_i^\bullet$ (using the "total complex" as an intermediate step). This is a free resolution of $M = \otimes_i M_i$ in $R - \operatorname{Mod}$, with the mth term of the form

$$\mathbf{F}^m := \sum_{S_m} \otimes_i F_i^{l_i}$$

where we sum over the set $S_m := \{(l_1, \dots, l_n) : 0 \leq l_i, \sum_i l_i = m\}$. This is also a free ($R$-)module of finite rank. Now given any $N = \otimes_i N_i \in \mathcal{C}$, we apply $\mathrm{Hom}_R(-, N) = \mathrm{Hom}_{\mathcal{C}}(-, N)$ to the free resolution

$$\cdots \to \mathbf{F}^2 \to \mathbf{F}^1 \to \mathbf{F}^0 \to M \to 0$$

to get the complex

$$0 \to \mathrm{Hom}_{\mathcal{C}}(M, N) \to \mathrm{Ext}^1_{\mathcal{C}}(M, N) \to \mathrm{Ext}^2_{\mathcal{C}}(M, N) \to \dots$$

Finally, apply the Künneth theorem to conclude that $\mathrm{Ext}^1_{\mathcal{C}}$ involves summing over terms involving $\mathrm{Ext}^1_{\mathcal{C}_i}$, each of which vanishes. This is because we have (e.g. cf. [May, Page 134]):

$$\alpha : \otimes_i H^*(\mathrm{Hom}_{R_i}(F_i^\bullet, N_i)) \to H^*(\otimes_i \mathrm{Hom}_R(\mathbf{F}^\bullet, \otimes_i N_i))$$

which gives

$$\omega \circ \alpha : \otimes_i H^*(M_i; N_i) \to H^*(\otimes_i M_i; \otimes_i N_i)$$

□

5. Block decomposition using central characters

The results in this section have to do with *central character block decompositions*. Fix a unital k-algebra A over a field k with a (unital) subalgebra $R \subset \mathfrak{Z}(A)$. Given an augmentation (or *central character* since R is commutative) $\varepsilon : R \to k$, we now define the category $\mathcal{C}(\varepsilon)$ as follows: Let M be an A-module, and define

$$M(\varepsilon) = \{m \in M : \forall z \in R,\ \exists n \in \mathbb{N} \text{ so that } (z - \varepsilon(z))^n m = 0\}$$

We can now define the subcategory $\mathcal{C}(\varepsilon)$ of $A - \mathrm{Mod}$, to consist of the objects $M(\varepsilon)$, for M an A-module. One verifies that for all $z \in R, a \in A$,

$$(z - \varepsilon(z))^n (am) = a(z - \varepsilon(z))^n m = a \cdot 0 = 0$$

whence $M(\varepsilon)$ is an A-submodule of M. The morphisms in $\mathcal{C}(\varepsilon)$ need to be A-module maps, of course.

5.1. **The main result and a corollary.** The question is: can we find a condition on M sufficient to ensure that $M = \bigoplus_{\varepsilon \in \widehat{R}} M(\varepsilon)$, where $\widehat{R}$ is the set of characters : $R \to k$? Here is one answer, motivated by the BGG Category $\mathcal{O}$ over complex semisimple Lie algebras. We need some notation for this.

Definition 1.

(1) Define $\mathcal{C}'$ to be the full subcategory of all A-modules M, so that for all $m \in M$, there exist $\varepsilon_1, \dots, \varepsilon_s \in \widehat{R}$ and $n_i \in \mathbb{N}$, with $\prod_{i=1}^s (z - \varepsilon_i(z))^{n_i} \cdot m = 0$ for all $z \in R$.
(2) Define $\mathcal{D}$ to be the full subcategory of all A-modules M for which there exist $\varepsilon_1, \dots, \varepsilon_s \in \widehat{R}$ and $n_i \in \mathbb{N}$, so that $\prod_{i=1}^s (z - \varepsilon_i(z))^{n_i}$ kills M, for all $z \in R$.

(3) Given $\varepsilon \in \widehat{R}$, define $\mathcal{D}(\varepsilon)$ to be the full subcategory of $A - \mathrm{Mod}$, consisting of all M for which there exists $n \in \mathbb{N}$ such that $(z - \varepsilon(z))^n$ kills M for all $z \in R$.

Theorem 4. *$\mathcal{D} = \bigoplus_{\varepsilon \in \widehat{R}} \mathcal{D}(\varepsilon)$ and $\mathcal{C}' = \bigoplus_{\varepsilon \in \widehat{R}} \mathcal{C}(\varepsilon)$.*

Note that $\mathcal{D}(\varepsilon) \subset \mathcal{C}(\varepsilon)$ for all ε.

For example, if $A = \mathfrak{U}\mathfrak{g}$ for a complex semisimple Lie algebra, then $\mathcal{D} \supset \mathcal{O}$, because every object of $\mathcal{O}$ has a filtration with standard cylic subquotients, and each of these is killed by a central character.

A corollary of this theorem is a Fitting-type result.

Corollary 1. *Fix $M \in \mathcal{D}, \varepsilon \in \widehat{R}, z \in R$, and define $\varphi := z - \varepsilon(z) \in \mathrm{End}_A(M)$. Then there is some $N \gg 0$ so that for $n \geq N$, $\mathrm{im}(\varphi^n) = \mathrm{im}(\varphi^N), \ker(\varphi^n) = \ker(\varphi^N)$, and $M = \ker(\varphi^N) \oplus \mathrm{im}(\varphi^N)$.*

To prove this result, we need a preliminary lemma, which we will use several times below.

Lemma 5. *Given $\varepsilon \neq \varepsilon'$ in $\widehat{R}$, $M \in \mathcal{C}(\varepsilon')$, and $z \notin \ker(\varepsilon - \varepsilon')$, the map $\varphi := z - \varepsilon(z)$ is an isomorphism on M.*

Proof. Acting by $z - \varepsilon(z)$ is clearly a module map on M, since $z - \varepsilon(z)$ is central. Now define $s = z - \varepsilon'(z)$ and $c = \varepsilon'(z) - \varepsilon(z) \in k^\times$ (since $\varepsilon \neq \varepsilon'$); then our map is $\varphi = s + c = c(1 + c^{-1}s)$.

We note that since $M \in \mathcal{C}(\varepsilon')$, hence for each m there is some $n = n(m) > 0$ so that $s^n \cdot m = 0$. Hence the operator $\varphi' := c^{-1}(1 - c^{-1}s + c^{-2}s^2 \pm \dots)$ is well-defined on each $m \in M$ (and "central", hence a module map). Moreover, $\varphi'(\varphi(m)) = \varphi(\varphi'(m)) = m$; thus we are done. □

We next prove the corollary.

Proof of Corollary 1. By Theorem 4 above, it is enough to show the result for $M \in \mathcal{D}(\varepsilon')$. There are now two cases. If $\varepsilon(z) = \varepsilon'(z)$, then φ is nilpotent on M, and the result follows (the kernels stabilize to all of M, and the images to 0). Alternatively, $\varepsilon(z) \neq \varepsilon'(z)$, and then Lemma 5 applies. Then φ is an isomorphism, i.e. $N = 1$ and $\ker(\varphi) = 0$. □

5.2. **Proof of the main result - Theorem 4.** The proof of this result is in various steps, since there are plenty of things to verify.

Step 1. We first introduce the following terminology: given $M \in \mathcal{D}(\varepsilon)$, define $n(M)$ to be the minimal n so that $(z - \varepsilon(z))^n M = 0 \; \forall z \in R$. Similarly define $n'(m) = n'_M(m)$ for $m \in M \in \mathcal{C}(\varepsilon)$; this is well-defined because if $m \in M \cap N$ in $\mathcal{C}(\varepsilon)$, then $n'_M(m) = n'_{Am}(m) = n'_N(m)$.

Then the first observation is that a priori, if $M \in \mathcal{D}$, then $M(\varepsilon) \in \mathcal{C}(\varepsilon)$, but since M is killed by $\prod_i (\ker \varepsilon_i)^{n_i}$, hence (using Lemma 5) $M(\varepsilon_i) \in \mathcal{D}(\varepsilon_i)$, since $n(M(\varepsilon_i)) \leq n_i$ for all i.

Finally, we can see via Lemma 5 that for any $\varepsilon \neq \varepsilon' \in \widehat{R}$ and $M \in \mathcal{C}$, $(M(\varepsilon))(\varepsilon) = M(\varepsilon)$ and $(M(\varepsilon))(\varepsilon') = 0$.

Step 2. Next, we note that each $\mathcal{D}(\varepsilon)$ (and also each $\mathcal{C}(\varepsilon)$) is closed under taking submodules, quotients, and extensions, since if $0 \to M' \to M \to M'' \to 0$, then $M \in \mathcal{D}(\varepsilon)$ if and only if $M', M'' \in \mathcal{D}(\varepsilon)$, and

$$n(M') + n(M'') \geq n(M) \geq n(M'), n(M'')$$

Also, there are no nontrivial module maps between (objects in) different blocks. This is because given $\varphi : M(\in \mathcal{C}(\varepsilon)) \to M'(\in \mathcal{C}(\varepsilon'))$, choose $z \notin \ker(\varepsilon - \varepsilon')$; then $z - \varepsilon(z)$ is central in A, hence an endomorphism of the identity functor. Now given $m \in M \in \mathcal{C}(\varepsilon)$, note that $Am \in \mathcal{D}(\varepsilon)$, with $n(Am) = n'(m)$. Hence

$$\varphi(Am) \cong (z - \varepsilon(z))^{n'(m)} \varphi(Am) = \varphi\left((z - \varepsilon(z))^{n'(m)} \cdot Am\right) = \varphi(0) = 0$$

where the (first) isomorphism holds since by Lemma 5, $z - \varepsilon(z)$ is invertible on M'. Hence $\varphi(m) = 0 \ \forall m \in M$.

Step 3. It remains to write each $M \in \mathcal{C}'$ as a direct sum of components in various blocks (note the "first observation" in Step 1 above). We first show that we can write M merely as a *sum* of components. This uses some commutative algebra; for the next two lemmas, let R be any unital commutative ring (e.g. "our" k-algebra R).

Lemma 6. *If $R = \sum_{j=1}^s I_j$ is a finite sum of ideals, then $R = \sum_{j=1}^s I_j^n$ for all $n \in \mathbb{N}$.*

Proof. We look at the equality $R = R^{s(n-1)+1} = (\sum_j I_j)^{s(n-1)+1}$. Clearly, any monomial on the right has, by the pigeon-hole principle, at least n terms of a given kind, and hence is contained in I_j^n for some j. Hence we get

$$R = R^{s(n-1)+1} = \left(\sum_j I_j\right)^{s(n-1)+1} \subset \sum_{j=1}^s I_j^n \subset R$$

because we already have the summands $I_j^{s(n-1)+1} \subset I_j^n$ in the summation, for all j. Hence equality is attained everywhere. □

Lemma 7. *If $\{\mathfrak{m}_i : 1 \leq i \leq s\}$ are distinct maximal ideals in R for some $s > 1$, then $\sum_i (\mathfrak{m}_1 \dots \widehat{\mathfrak{m}_i} \dots \mathfrak{m}_s)^n = R$ for all $n \in \mathbb{N}$.*

Proof. Since each summand is an ideal of R (even for $n = 1$), hence by the previous lemma it suffices to show the claim for the case $n = 1$. We now show the claim by induction on s. For $s = 2$, the claim is again trivial.

Suppose we have shown the claim for $s-1$. We then have

$$\begin{aligned} \mathfrak{m}_s &= R\mathfrak{m}_s \subset \text{ (by induction) } \left[\sum_{i=1}^{s-1} (\mathfrak{m}_1 \dots \widehat{\mathfrak{m}_i} \dots \mathfrak{m}_{s-1})\right] \cdot \mathfrak{m}_s \\ &\subset \sum_{i=1}^{s} (\mathfrak{m}_1 \dots \widehat{\mathfrak{m}_i} \dots \mathfrak{m}_s) \end{aligned}$$

so that $\mathfrak{m}_s$ is contained in the sum. By symmetry, so is $\mathfrak{m}_1$, and hence their sum, which is obviously the whole of R, is also contained in this sum, whence we have equality as claimed. □

We are now in a position to show this step. Given $m \in M \in \mathcal{C}'$, we know that $\prod_i (\ker \varepsilon_i)^{n_i}$ kills m (for some ε_i, n_i); hence it also kills the *submodule* $M' := A \cdot m$. Hence so does $\prod_i \mathfrak{m}_i^n$, where $n = \max_i(n_i)$ and $\mathfrak{m}_i := \ker \varepsilon_i$.

This implies: $(\prod_{j \neq i} \mathfrak{m}_j)^n \cdot M' \subset M'(\varepsilon_i) \subset M(\varepsilon_i)$ for all i. Now apply Lemma 7 above. Then $1 = \sum_i r_i$, with $r_i \in (\prod_{j \neq i} \mathfrak{m}_j)^n$, so $m = \sum_i r_i m$, with each $r_i m \in M(\varepsilon_i)$.

Step 4. It remains to show that any sum of components is direct. We do this in two different ways.

First Method: Suppose $\sum_{i=1}^l m_i = 0$, with $m_i \in M(\varepsilon_i)$ for all i (with the ε_i's distinct). We show that each $m_i = 0$, by induction on l. If $l = 1$, the result is trivial; if $l > 1$, then for each $i < l$, choose $z_i \notin \ker(\varepsilon_i - \varepsilon_l)$. Then $\prod_{i=1}^{l-1} (z_i - \varepsilon_i(z_i))^{n_i}$ kills $m_1, m_2, \dots, m_{l-1}$, if the ith factor kills m_i for each i. Hence we get:

$$\prod_{i=1}^{l-1} (z_i - \varepsilon_i(z_i))^{n_i} \cdot m_l = 0$$

But each factor is invertible on $M(\varepsilon_l)$ by Lemma 5, whence $m_l = 0$, and by induction, every other m_i also vanishes.

Second Method: (This only works in $\mathcal{D}$.) We first show that extensions between two different blocks are trivial. Suppose we have an extension: $0 \to M(\varepsilon) \to M \to M(\varepsilon') \to 0$ in $\mathcal{D}$. Choose $z \notin \ker(\varepsilon - \varepsilon')$, and define $\varphi := (z - \varepsilon(z))^{n(M(\varepsilon))} : M \to M$ (so $\ker \varphi \supset M(\varepsilon)$).

Then φ factors through a map $\overline{\varphi} : M(\varepsilon') \to M$, and composing this with $M \twoheadrightarrow M(\varepsilon')$ yields the map $\varphi = (z - \varepsilon(z))^{n(M(\varepsilon))}$ on $M(\varepsilon')$. By Lemma 5, this is an isomorphism; thus we produce a splitting to the above short exact sequence, namely the composite:

$$M(\varepsilon') \xrightarrow{\varphi^{-1}} M(\varepsilon') \xrightarrow{\overline{\varphi}} M$$

To finish the proof, use induction on the number of ε's, and the following general lemma:

Lemma 8. *Given a ring S and a short exact sequence of S-modules $0 \to A \oplus B' \to C \to B'' \to 0$, suppose $\mathrm{Ext}^1_S(B'', A) = 0$. Then $C = A \oplus B$, where $0 \to B' \to B \to B'' \to 0$.*

To prove the lemma, apply $\mathrm{Hom}_S(B'', -)$ to: $0 \to B' \to B' \oplus A \to A \to 0$. The lemma follows by considering the long exact sequence of Ext_S's (in the abelian category $S - \mathrm{Mod}$). □

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