Student Analysis 2-8-23: Different Types of Fractional Laplacians + Intro to non-local mean curvature

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1 Set up

Few more facts on fractional laplacian + Intro to non-local minimal surfaces

2 Sources

“Fractional Laplacians: a short survey” - Maha Daoud, El Haj Laamri

“Hitchhikers Guide to Fractional Sobolev Spaces” - Di Nezza, Palatucci, Valdinoci

“Non-local minimal surfaces” - Caffarelli, RoqueJeffre, Savin

“Non-local diffusion and applications” - Bucur, Valdinoci

- Main survey on fractional laplacian
- Caffarelli-Roque, intro to non-local minimal surfaces
- Hitchhikers guide to fractional sobolev spaces
- Maybe this?
- Guide to fractional laplacian via semigroup
- Better notes for non-local mean curvature
- Non local stuff textbook

3 3: Semigroup

- Useful for studying PDEs
- For $u \in S(\mathbb{R}^N)$, have

\[ (-\Delta)^s u(x) = \frac{s}{\Gamma(1-x)} \int_0^\infty \left( u(x) - w(x,t) \right) \frac{dt}{t^{1+s}} \]

Where $w$ solves the heat equation:

\[ \partial_t w(x,t) = \Delta w(x,t) \quad (x,t) \in \mathbb{R}^N \times [0,\infty) \]
\[ w(x,0) = u(x) \]

- Motivation:

\[ \lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+s}} \]
Proof: set \( u = t\lambda \) and remember the definition of \( \Gamma(-s) \).

Motivates

\[ w = e^{-tL}u \]

as a solution to

\[
\begin{align*}
(\partial_t + L)w &= 0, \quad \mathbb{R}^n \times \mathbb{R}^+ \\
v(x, 0) &= u(x)
\end{align*}
\]

because

\[
\begin{align*}
\partial_t w &= (\not{L})e^{-tL}u = -Lw \\
w(x, 0) &= e^{0}u = u(x)
\end{align*}
\]

- With this, we can find a kernel for the fractional laplacian
- First, we prove an equivalence

Proposition 1. The semigroup definition of the fractional laplacian coincides with the fourier definition for \( 0 < 2s < 1 \)

Proof: We have that

\[
F(\not{\Delta})^s u = |\xi|^{2s} \hat{u}(\xi)
\]

\[
= \hat{u}(\xi) \int_0^{\infty} \left( e^{-t|\xi|^2} - 1 \right) \frac{dt}{t^{1+s}}
\]

\[
= \frac{1}{\Gamma(-s)} \int_0^{\infty} \left( e^{-t|\xi|^2} \hat{u}(\xi) - \hat{u}(\xi) \right) \frac{dt}{t^{1+s}}
\]

Now fourier invert (Not thinking too deeply about what space these functions lie in), to get

\[
\begin{align*}
\not{\Delta}^s u(x) &= \frac{1}{\Gamma(-s)} \int_0^{\infty} F^{-1}(e^{-t|\xi|^2} \hat{u}(\xi)) - F^{-1}(\hat{u}(\xi)) \frac{dt}{t^{1+s}} \\
&= \frac{1}{\Gamma(-s)} \int_0^{\infty} (e^{t\Delta} u(x) - u(x)) \frac{dt}{t^{1+s}} \\
&= \frac{1}{\Gamma(-s)} \int_0^{\infty} [(G_t * u)(x) - u(x)] \frac{dt}{t^{1+s}}
\end{align*}
\]

finishing the proof. Here \( G_t(x) \) is the gaussian heat kernel.

4 4: PV Integral

- Principal value integral

\[
\not{\Delta}^s u(x) = C(N, s) \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x, \epsilon)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy
\]

\[
C(N, s) := \frac{s4^s \Gamma(s + N/2)}{\pi^{N/2} \Gamma(1 - s)}
\]

with \( C(N, s) \) (Equivalent to previous definition that Josef gave) chosen so that

Proposition 2. Let \( u \) be a compactly supported smooth function on \( \mathbb{R}^n \). Then

\[
\lim_{s \to 0^+} \not{\Delta}^s u = u, \quad \lim_{s \to 1^-} \not{\Delta}^s u = -\Delta u
\]

Proof: (Of first one) First we need the following lemma:

Lemma 4.1 (Hitchiker, Cor 4.2). For \( C(N, s) \) the normalizing constant we have

\[
\lim_{s \to 0^+} \frac{C(N, s)}{s} = \frac{2}{\omega_{N-1}}
\]
Proof: Recall that
\[ \Gamma(x) = \frac{1}{x} + O(1) \]
as \( x \to 0 \). This tells us that
\[ \lim_{s \to 0^+} \frac{C(n, s)}{s} = \frac{4\Gamma(N/2)}{\pi^{N/2}} = \frac{2}{\omega_{N-1}} \]

Now with the lemma, let \( \text{supp}(u) \subseteq B_{R_0}(0) \). Set \( R = R_0 + |x| + 1 \) (Draw picture for audience)

\[ \left| \int_{B_R(0)} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2s}} \right| \leq ||u||_{C^2} \int_{B_R} \frac{|y|^2}{|y|^{n+2s}} \int_{0}^{R} \rho^{1-2s} d\rho \\
= \frac{\omega_{n-1}}{2(1-s)} ||u||_{C^2} R^{2-2s} \]

This tells us by the lemma that
\[ \lim_{s \to 0^+} C(n, s) \int_{B_R} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2s}} dy = 0 \]

To handle other integral, Note that \( |y| \geq R \) gives \( |x \pm y| \geq |y| - |x| \geq R - |x| > R_0 \) so \( u(x \pm y) = 0 \) and
\[ \int_{\mathbb{R}^n \setminus B_R} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2s}} dy = -2u(x) \int_{\mathbb{R}^n \setminus B_R} \frac{|y|^{-n-2s}}{|y|^{n+2s}} dy \\
= \omega_{n-1}(-2u(x)) \int_{R}^{\infty} \rho^{-2s-1} d\rho \\
= -\frac{\omega_{n-1} R^{-2s}}{s} u(x) \]

And so
\[ \lim_{s \to 0^+} C(n, s) \frac{1}{2} \int_{\mathbb{R}^n \setminus B_R} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2s}} dy = \frac{C(n, s) \omega_{n-1} R^{-2s}}{2s} u(x) \\
= u(x) \]

Recalling the definition of the regional laplacian for \( C^2 \) functions gives us the result. \( \square \)

5 Nonlocal mean curvature

- How to get minimal surfaces? One way: minimize
\[ E_\epsilon(u) = \int |\nabla u|^2 + \frac{1}{\epsilon} F(u) \]
for each \( u \). Consider \( Y_\epsilon = u_{\text{min}, \epsilon}^{-1} \), and take
\[ Y_\epsilon \xrightarrow{\Gamma_{\epsilon \to 0}} Y \]

- Other way: Discretized heat flow:
\[ u = \chi_\Omega - \chi_{\Omega^c} \]
as an initial condition. Define
\[ t_{k+1} = t_k + \delta \]
And let \( S_{k+1} \) be generated as follows: given \( u_k(x) \), solve
\[ u_t - \Delta u = 0, \quad u(\cdot, 0) = u_k(x), \quad t \in [0, \epsilon] \]
Then
\[ u_{k+1}(x) = u(x, \epsilon) = (G_{\epsilon} * u_k)(x) \]
and define
\[ \Omega_{k+1} = \{ u_{k+1} > 0 \}, \quad S_{k+1} = \partial \Omega_{k+1} \]

- If \( \delta \sim \epsilon^2 \), this is a discretized approximation to mean curvature flow for time \( t \sim k\delta \)

![Figure 1: State of geometry is sad! No one likes to draw pictures like this (mathcha.io)](image)

- Question: What about fractional heat equation? (Motivated by Levy Processes which are like brownian motion but different)

- Now solve
\[ \partial_t - \Delta^s u = 0 \]

- (On Allen–Cahn side, now makes sense to )
define
\[ \tilde{E}(u) = (1 - s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, dx \, dy + \int_{\mathbb{R}^n} F(u) \, dx \]

having replaced gradient term by fractional derivative!

- Q: Let’s say \( F(u) = (1 - u^2)^2 \) - Are there any minimizers of \( \tilde{E} \) such that \( F(u) \equiv 0 \) a.e.?

- Forces \( u = \pm 1 \), so we define
\[ \bar{E}(u) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \]
but only search on sets, i.e.
\[ \Omega \subseteq \mathbb{R}^n, \quad \bar{E}(\Omega) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\chi_E(x) - \chi_E(y))^2}{|x - y|^{n+2s}} \]

- How to minimize \( E \) with some fixed “boundary data”?

- In classic minimal surfaces, this is like finding area minimizers with fixed boundary → Plateau problem (Draw pic of hyperboloid bubble wand set up)

- Boundary data: in non-local setting, idea is to fix \( E \) outside of some set \( \Omega \)

- Let \( \Omega \subseteq \mathbb{R}^n \), and consider sets \( E \subseteq \mathbb{R}^n \) such that \( E \cap \Omega^c \) is fixed

- New stuff:
Definition 5.1. The overlap of two sets $A$ and $B$ is given by

$$L_s(A,B) = \int_{A \times B} \frac{1}{|x-y|^{n+s}} dx dy$$

Some properties:

$$L(A,B) = L(B,A)$$
$$L(A_1 \sqcup A_2, B) = L(A_1, B) + L(A_2, B)$$

Definition 5.2. The $s$ perimeter of some set $E$ in $\Omega$ is

$$P_{s,\Omega}(E) := [\chi_E]^2_{H^{n/2}(\Omega)} = \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega \times \Omega^c)} \frac{|\chi_E(x) - \chi_E(y)|^2}{|x-y|^{n+s}} dx dy$$

• Note that

$$P_{s,\Omega}(E) = L(E \cap \Omega, E^c) + L(E \cap \Omega^c, E^c \cap \Omega)$$

Definition 5.3. We say that $E$ is $s$-nonlocal minimal (or $\partial E$ is a nonlocal minimal surface) in a bounded lipschitz domain $\Omega$ if

$$P_{s,\Omega}(E) \leq P_{s,\Omega}(F) \quad \text{if} \quad E \cap \Omega^c = F \cap \Omega^c$$

Proposition 3. $E$ is $s$-nonlocal minimal if and only if it satisfies the

1. Subsolution property
   $$\forall A \subseteq E \cap \Omega, \quad L(A, E \setminus A) - L(A, E^c) \geq 0$$

2. Supersolution property
   $$\forall A \subseteq E^c \cap \Omega, \quad L(A, E) - L(A, E^c \setminus A) \leq 0$$

Proof: (Draw picture) The idea for the first is that if $A \subseteq E \cap \Omega$ then consider $F = E \setminus A$ and break down

The RHS is

$$P_{s,\Omega}(E \setminus A) = P_{s,\Omega}(E \cap \Omega)$$

Figure 2
So that
\[ P_{s,\Omega}(E\setminus A) - P_{s,\Omega}(E) = L(E, A) - L(E^c, A) - L(A, A) \]
\[ = L(E\setminus A, A) - L(E^c, A) \geq 0 \]
ending proof.

- supersolution condition is similar, if \( A \subseteq E^c \cap \Omega \), then \( F = E \cup A \) is comparable

- Intuition, let \( A = \{x_0\} \) for \( x_0 \in \partial E \) (Assume \( \partial E \) smooth):
\[ L(E\setminus A, A) - L(E^c, A) = L(E, \{x_0\}) - L(E^c, \{x_0\}) \geq 0 \]
\[ L(A, E) - L(A, E^c\setminus A) = L(\{x_0\}, E) - L(\{x_0\}, E^c) \leq 0 \]
Nonlocal Minimal \( \implies L(E, \{x_0\}) - L(E^c, \{x_0\}) = 0 \)

We define
\[ H_s(x_0) = L(E, \{x_0\}) - L(E^c, \{x_0\}) = \int_{\mathbb{R}^n} \chi_E(x) - \chi_{E^c}(x) \frac{1}{|x - x_0|^{n+s}} dx \]
Find \( E \) such that the above is true at every boundary point

- Nonlocal minimal surfaces weird because interaction between \( E \cap \Omega \) and \( E^c \), as well as \( E \cap \Omega^c \) and \( E^c \cap \Omega \)

\[ \]
Figure 4