Student Analysis 2-8-23: Different Types of Fractional Laplacians + Intro to non-local mean curvature

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1 Set up

Few more facts on fractional laplacian + Intro to non-local minimal surfaces

2 Sources

"Fractional Laplacians: a short survey" - Maha Daoud, El Haj Laamri

"Hitchhikers Guide to Fractional Sobolev Spaces" - Di Nezza, Palatucci, Valdinoci

"Non-local minimal surfaces" - Caffarelli, RoqueJeffre, Savin

"Non-local diffusion and applications" - Bucur, Valdinoci

- Main survey on fractional laplacian
- Caffarelli-Roque, intro to non-local minimal surfaces
- Hitchhikers guide to fractional sobolev spaces
- Maybe this?
- Guide to fractional laplacian via semigroup
- Better notes for non-local mean curvature
- Non local stuff textbook

3 3: Semigroup

- Useful for studying PDEs
- For $u \in S(\mathbb{R}^N)$, have

  $$(-\Delta)^s u(x) = \frac{s}{\Gamma(1-x)} \int_0^\infty (u(x) - w(x,t)) \frac{dt}{t^{1+s}}$$

  Where w solves the heat equation:

  $$\partial_t w(x,t) = \Delta w(x,t) \quad (x,t) \in \mathbb{R}^N \times [0,\infty)$$
  $$w(x,0) = u(x)$$

- Motivation:

  $$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+s}}$$

Proof: set $u = t\lambda$ and remember the definition of $\Gamma(-s)$. □

Motivates

$$\text{“}w = e^{-tL}u\text{”}$$

as a solution to

$$\begin{aligned} (\partial_t + L)w = 0, \qquad & \mathbb{R}^n \times \mathbb{R}^+ \\ v(x,0) = u(x) & \end{aligned}$$

because

$$\begin{aligned} \partial_t w &= (-L)e^{-tL}u = -Lw \\ w(x,0) &= e^{-0}u = u(x) \end{aligned}$$

- With this, we can find a kernel for the fractional laplacian
- First, we prove an equivalence

Proposition 1. *The semigroup definition of the fractional laplacian coincides with the fourier definition for* $0 < 2s < 1$

Proof: We have that

$$\begin{aligned} F((-\Delta)^s u) &= |\xi|^{2s}\hat{u}(\xi) \\ &= \hat{u}(\xi)\frac{1}{\Gamma(-s)}\int_0^\infty (e^{-t|\xi|^2} - 1)\frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)}\int_0^\infty (e^{-t|\xi|^2}\hat{u}(\xi) - \hat{u}(\xi))\frac{dt}{t^{1+s}} \end{aligned}$$

Now fourier invert (Not thinking too deeply about what space these functions lie in), to get

$$\begin{aligned} (-\Delta)^s u(x) &= \frac{1}{\Gamma(-s)}\int_0^\infty F^{-1}(e^{-t|\xi|^2}\hat{u}(\xi)) - F^{-1}(\hat{u}(\xi)))\frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)}\int_0^\infty (e^{t\Delta}u(x) - u(x))\frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)}\int_0^\infty [(G_t * u)(x) - u(x)]\frac{dt}{t^{1+s}} \end{aligned}$$

finishing the proof. Here $G_t(x)$ is the gaussian heat kernel.

4 4: PV Integral

- Principal value integral

$$(-\Delta)^s u(x) = C(N,s)\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N\setminus B(x,\epsilon)}\frac{u(x)-u(y)}{||x-y||^{N+2s}}dy$$

$$C(N,s) := \frac{s4^s\Gamma(s+N/2)}{\pi^{N/2}\Gamma(1-s)}$$

with $C(N,s)$ (Equivalent to previous definition that Josef gave) chosen so that

Proposition 2. *Let u be a compactly supported smooth function on $\mathbb{R}^n$. Then*

$$\lim_{s\to 0^+}(-\Delta)^s u = u, \qquad \lim_{s\to 1^-}(-\Delta)^s u = -\Delta u$$

Proof: (Of first one) First we need the following lemma:

Lemma 4.1 (Hitchiker, Cor 4.2)**.** *For $C(N,s)$ the normalizing constant we have*

$$\lim_{s\to 0^+}\frac{C(N,s)}{s} = \frac{2}{\omega_{N-1}}$$

Proof: : Recall that

$$\Gamma(x) = \frac{1}{x} + O(1)$$

as $x \to 0$. This tells us that

$$\lim_{s\to 0^+} \frac{C(n,s)}{s} = \frac{4\Gamma(N/2)}{\pi^{N/2}} = \frac{2}{\omega_{N-1}}$$

Now with the lemma, let $\mathrm{supp}(u) \subseteq B_{R_0}(0)$. Set $R = R_0 + |x| + 1$ (Draw picture for audience)

$$\Big| \int_{B_R(0)} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}} \Big| \leq ||u||_{C^2} \int_{B_R} \frac{|y|^2}{|y|^{n+2s}}$$
$$\omega_{n-1} ||u||_{C^2} \int_0^R \rho^{1-2s} d\rho$$
$$= \frac{\omega_{n-1}||u||_{C^2} R^{2-2s}}{2(1-s)}$$

This tells us by the lemma that

$$\lim_{s\to 0+} C(n,s) \int_{B_R} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}} dy = 0$$

To handle other integral, Note that $|y| \geq R$ gives $|x \pm y| \geq |y| - |x| \geq R - |x| > R_0$ so $u(x \pm y) = 0$ and

$$\int_{\mathbb{R}^n \setminus B_R} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}} dy = -2u(x) \int_{\mathbb{R}^n \setminus B_R} |y|^{-n-2s} dy$$
$$= \omega_{n-1}(-2u(x)) \int_R^\infty \rho^{-2s-1} d\rho$$
$$= -\frac{\omega_{n-1} R^{-2s}}{s} u(x)$$

And so

$$\lim_{s\to 0^+} \frac{C(n,s)}{2} \int_{\mathbb{R}^n \setminus B_R} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}} dy = \frac{C(n,s)\omega_{n-1} R^{-2s}}{2s} u(x)$$
$$= u(x)$$

Recalling the definition of the regional laplacian for C^2 functions gives us the result. □

5 Nonlocal mean curvature

- How to get minimal surfaces? One way: minimize

$$E_\epsilon(u) = \int \epsilon |\nabla u|^2 + \frac{1}{\epsilon} F(u)$$

for each u. Consider $Y_\epsilon = u^{-1}_{min,\epsilon}(0)$, and take

$$Y_\epsilon \overset{\Gamma, \epsilon\to 0}{Y}$$

- Other way: Discretized heat flow:

$$u = \chi_\Omega - \chi_{\Omega^c}$$

as an initial condition. Define

$$t_{k+1} = t_k + \delta$$

And let S_{k+1} be generated as follows: given $u_k(x)$, solve

$$u_t - \Delta u = 0, \qquad u(\cdot, 0) = u_k(x), \qquad t \in [0, \epsilon]$$

Then

$$u_{k+1}(x) = u(x, \epsilon) = (G_\epsilon * u_k)(x)$$

And define

$$\Omega_{k+1} = \{u_{k+1} > 0\}, \qquad S_{k+1} = \partial\Omega_{k+1}$$

- If $\delta \sim \epsilon^2$, **this is a discretized approximiation to mean curvature flow for time** $t \sim k\delta$

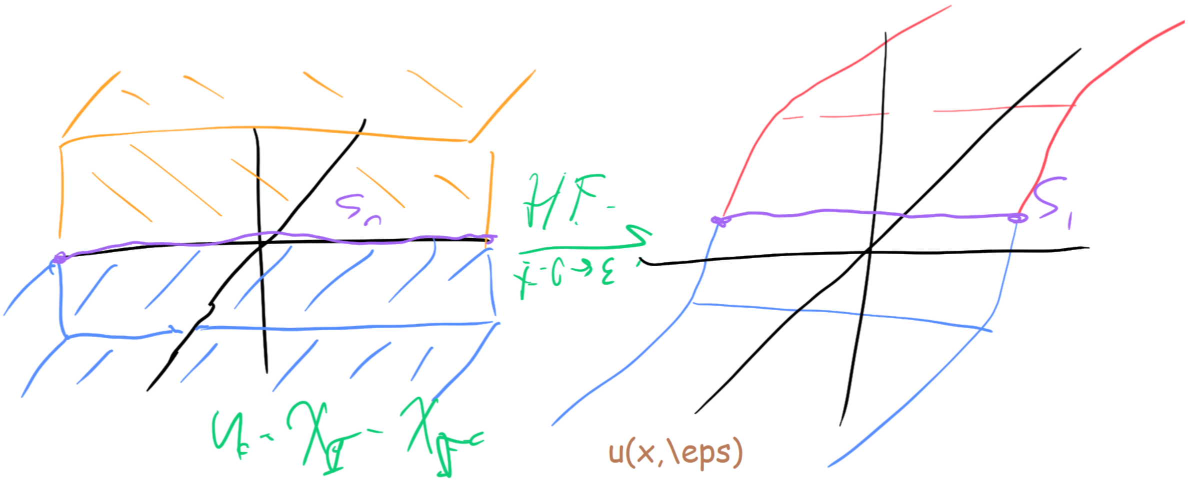

Figure 1: State of geometry is sad! No one likes to draw pictures like this (mathcha.io)

- Question: What about fractional heat equation? (Motivated by Levy Processes which are like brownian motion but different)
- Now solve

$$\partial_t - \Delta^s u = 0$$

- (On Allen–Cahn side, now makes sense to)
define

$$\tilde{E}(u) = (1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dxdy + \int_{\mathbb{R}^n} F(u)dx$$

having replaced gradient term by *fractional derivative*!

- Q: Let's say $F(u) = (1-u^2)^2$ - Are there any minimizers of $\tilde{E}$ such that $F(u) \equiv 0$ a.e.?
- Forces $u = \pm 1$, so we define

$$\overline{E}(u) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}$$

but only search on sets, i.e.

$$\Omega \subseteq \mathbb{R}^n, \qquad \overline{E}(\Omega) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\chi_E(x)-\chi_E(y))^2}{|x-y|^{n+2s}}$$

- How to minimize E with some fixed "boundary data"?
- In classic minimal surfaces, this is like finding area minimizers with fixed boundary $\to$ Plateau problem (Draw pic of hyperboloid bubble wand set up)
- Boundary data: in non-local setting, idea is to fix E outside of some set Ω
- Let $\Omega \subseteq \mathbb{R}^n$, and consider sets $E \subseteq \mathbb{R}^n$ such that $E \cap \Omega^c$ is fixed
- New stuff:

Definition 5.1. *The* *s overlap* *of two sets* A *and* B *is given by*

$$L_s(A,B) = \int_{A\times B} \frac{1}{|x-y|^{n+s}} dxdy$$

Some properties:

$$L(A,B) = L(B,A)$$
$$L(A_1 \sqcup A_2, B) = L(A_1, B) + L(A_2, B)$$

Definition 5.2. *The* s *perimeter of some set* E *in* Ω *is*

$$P_{s,\Omega}(E) := [\chi_E]^2_{H^{s/2}(\Omega)} = \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\Omega^c\times\Omega^c)} \frac{|\chi_E(x)-\chi_E(y)|^2}{|x-y|^{n+s}} dxdy$$

- Note that

$$P_{s,\Omega}(E) = L(E\cap\Omega, E^c) + L(E\cap\Omega^c, E^c\cap\Omega)$$

Definition 5.3. *We say that* E *is s-nonlocal minimal (or* ∂E *is a nonlocal minimal surface) in a bounded lipschitz domain* Ω *if*

$$P_{s,\Omega}(E) \le P_{s,\Omega}(F) \qquad \text{if} \quad E\cap\Omega^c = F\cap\Omega^c$$

Proposition 3. *E is s-nonlocal minimal if and only if it satisfies the*

1. *Subsolution property*
$$\forall A\subseteq E\cap\Omega, \qquad L(A, E\setminus A) - L(A, E^c) \ge 0$$
2. *Supersolution property*
$$\forall A\subseteq E^c\cap\Omega, \qquad L(A, E) - L(A, E^c\setminus A) \le 0$$

Proof: (Draw picture) The idea for the first is that if $A\subseteq E\cap\Omega$ then consider $F = E\setminus A$ and break down

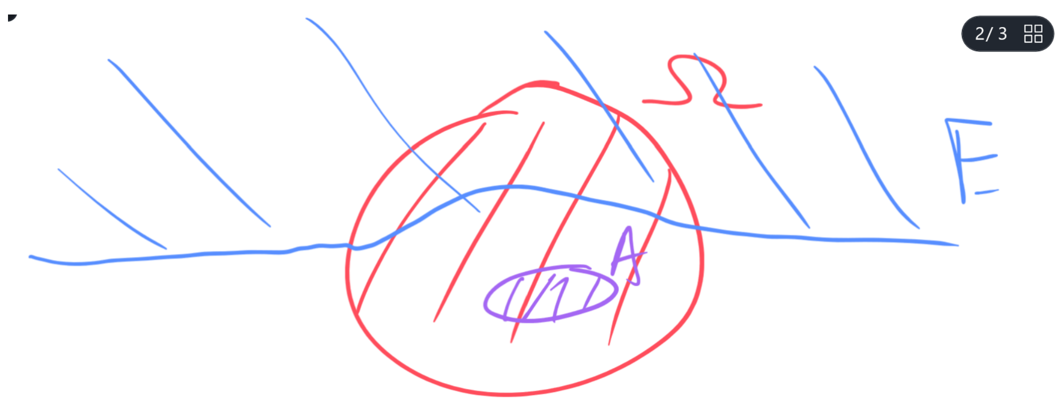

Figure 2

$$P_{s,\Omega}(E) \le P_{s,\Omega}(E\setminus A)$$

The RHS is

$$\begin{aligned}
P_{s,\Omega}(E\setminus A) &= L((E\setminus A)\cap\Omega, (E\setminus A)^c) + L((E\setminus A)\cap\Omega^c, (E\setminus A)^c, \Omega) \\
&= I_1 + I_2 \\
I_1 &= L(E\cap\Omega, E^c\cup A) - L(A, E^c\cup A) \\
&= L(E\cap\Omega, E^c) + L(E\cap\Omega, A) - L(E^c, A) - L(A, A) \\
I_2 &= L(E\cap\Omega^c, (E^c\cup A)\cap\Omega) \\
&= L(E\cap\omega^c, E^c\cap\Omega) + L(E\cap\Omega^c, A)
\end{aligned}$$

So that

$$\begin{aligned} P_{s,\Omega}(E\backslash A) - P_{s,\Omega}(E) &= L(E,A) - L(E^c,A) - L(A,A) \\ &= L(E\backslash A, A) - L(E^c, A) \\ &\geq 0 \end{aligned}$$

ending proof. □

- supersolution condition is similar, if $A \subseteq E^c \cap \Omega$, then $F = E \cup A$ is comparable
- Intuition, let $A = \{x_0\}$ for $x_0 \in \partial E$ (Assume ∂E smooth):

$$\begin{aligned} L(E\backslash A, A) - L(E^c, A) &= L(E, \{x_0\}) - L(E^c, \{x_0\}) \geq 0 \\ L(A, E) - L(A, E^c\backslash A) &= L(\{x_0\}, E) - L(\{x_0\}, E^c) \leq 0 \end{aligned}$$

$$\text{Nonlocal Minimal} \implies L(E, \{x_0\}) - L(E^c, \{x_0\}) = 0$$

We define

$$H_s(x_0) = L(E, \{x_0\}) - L(E^c, \{x_0\}) = \int_{\mathbb{R}^n} \frac{\chi_E(x) - \chi_{E^c}(x)}{|x - x_0|^{n+s}} dx$$

Find E such that the above is true at every boundary point

- Nonlocal minimal surfaces weird because interaction between $E \cap \Omega$ and E^c, as well as $E \cap \Omega^c$ and $E^c \cap \Omega$

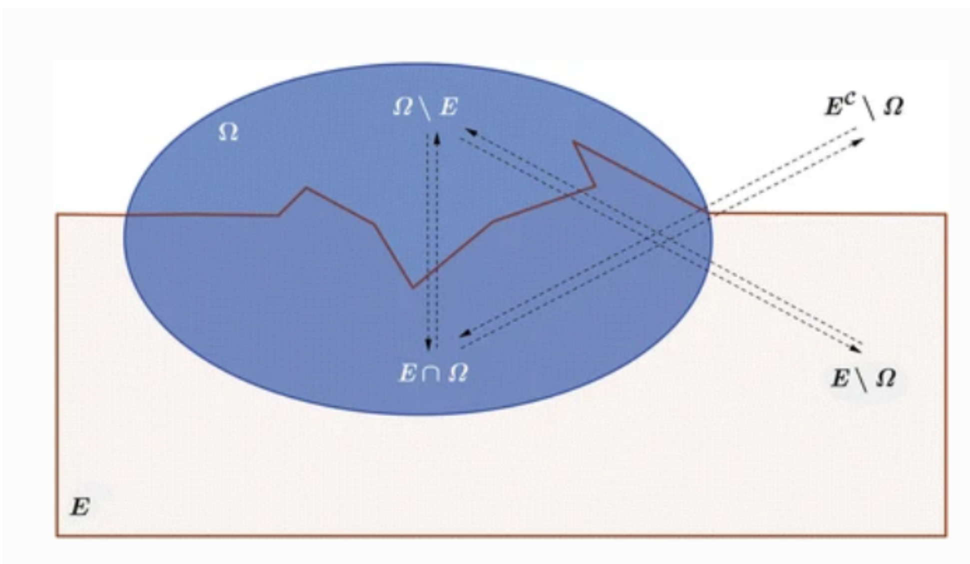

Figure 3

- Connecting to fractional laplacian: Define

$$\tilde{\chi}_E = \begin{cases} \chi_E - \chi_{E^c} & x \notin \partial E \\ 0 & x \in \partial E \end{cases}$$

intuitively, this function averages out to 0 near points on ∂E. Then for

$$\begin{aligned} H_s(x_0) &= \frac{1}{2} \int_{\mathbb{R}^n} \frac{\tilde{\chi}_E(x_0 + y) + \tilde{\chi}_E(x_0 - y)}{|y|^{n+s}} dy \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \frac{\tilde{\chi}_E(x_0 + y) + \tilde{\chi}_E(x_0 - y) - 2\tilde{\chi}_E(x_0)}{|y|^{n+s}} dy \\ &= \frac{(-\Delta)^s \tilde{\chi}_E(x_0)}{C(n,s)} \end{aligned}$$

- Maybe mention something how nonlocal minimal surfaces are weird - sticking? Regularity?
- Pictures

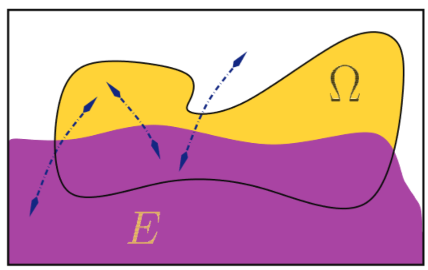

Figure 1. Long-range interactions leading to the fractional perimeter of the set E in Ω.

Figure 4