

Math Camp

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Multivariable Calculus

Functions of many variables:

- 1) Policies may be multidimensional (policy provision and pork buy off)
- 2) Countries may invest in offensive and defensive resources for fighting wars
- 3) Ethnicity and resources could affect investment

Today:

- 0) Determinant
- 1) Multivariate functions
- 2) Partial Derivatives, Gradients, Jacobians, and Hessians
- 3) Total Derivative, Implicit Differentiation, Implicit Function Theorem
- 4) Multivariate Integration

Determinant

Suppose we have a **square** ($n \times n$) matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

A determinant is a function that assigns a number to square matrices

Determinant

Facts needed to define determinant :

Definition

A *permutation* of the set of integers $\{1, 2, \dots, J\}$ is an arrangement of these integers in some order without omissions or repetition.

For example, consider $\{1, 2, 3, 4\}$

$\{3, 2, 1, 4\}$

$\{4, 3, 2, 1\}$

If we have J integers then there are $J!$ permutations

Determinant

Definition

An *inversion* occurs when a larger integer occurs before a smaller integer in a permutation

Even permutation: total inversions are even

Odd permutation: total inversions are odd

Count the inversions

{3, 2, 1}

{1, 2, 3}

{3, 1, 2}

{2, 1, 3}

{1, 3, 2}

{2, 3, 1}

Determinant

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For a square $n \times n$ matrix A , we will call an *elementary product* an n element long product, with no two components coming from the same row or column. We will call a *signed elementary product* one that multiplies odd permutations of the column numbers by -1 .

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There are $n!$ elementary products

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*Suppose A is an $n \times n$ matrix. Define the determinant function $\det(A)$ to be the sum of signed elementary products from A . Call $\det(A)$ the **determinant** of A*

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$$\begin{aligned}\det(A) &= \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= a_{11}a_{22} - a_{12}a_{21}\end{aligned}$$

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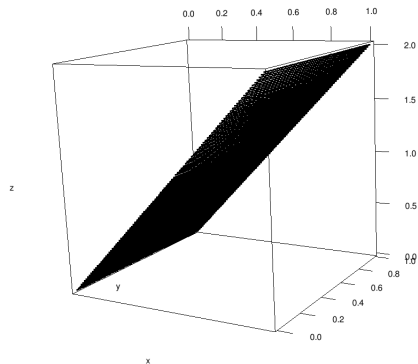
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R Code!

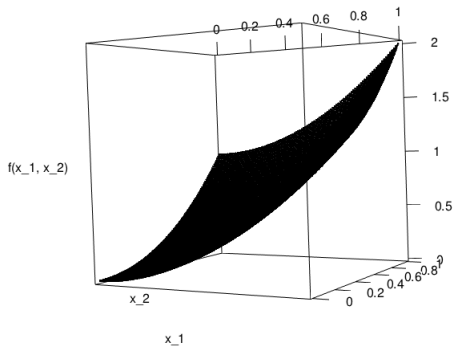
Multivariate Functions

$$f(x_1, x_2) = x_1 + x_2$$



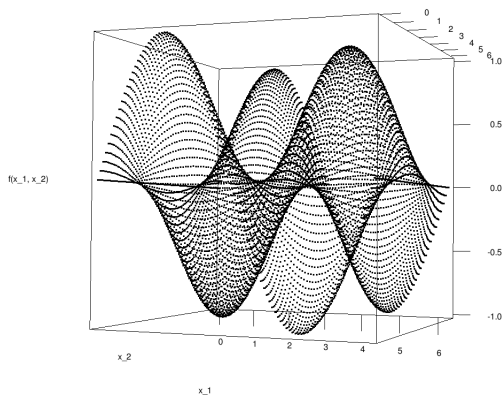
Multivariate Functions

$$f(x_1, x_2) = x_1^2 + x_2^2$$



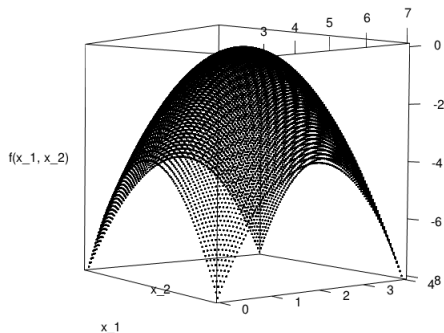
Multivariate Functions

$$f(x_1, x_2) = \sin(x_1) \cos(x_2)$$



Multivariate Functions

$$f(x_1, x_2) = -(x - 5)^2 - (y - 2)^2$$



Multivariate Functions

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

Multivariate Functions

$$\begin{aligned}f(\mathbf{x}) &= f(x_1, x_2, \dots, x_N) \\ &= x_1 + x_2 + \dots + x_N \\ &= \sum_{i=1}^N x_i\end{aligned}$$

Multivariate Functions

Definition

Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$. We will call f a *multivariate* function. We will commonly write,

$$f(\mathbf{x}) = f(x_1, x_2, x_3, \dots, x_n)$$

- $\mathbb{R}^n = \mathbb{R} \underbrace{\times}_{\text{cartesian}} \mathbb{R} \times \mathbb{R} \times \dots \mathbb{R}$

- The function we consider will take n inputs and output a single number (that lives in \mathbb{R}^1 , or the real line)

Example 1

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

Evaluate at $\mathbf{x} = (x_1, x_2, x_3) = (2, 3, 2)$

$$\begin{aligned} f(2, 3, 2) &= 2 + 3 + 2 \\ &= 7 \end{aligned}$$

Example 1

$$f(x_1, x_2) = x_1 + x_2 + x_1 x_2$$

Evaluate at $\mathbf{w} = (w_1, w_2) = (1, 2)$

$$\begin{aligned} f(w_1, w_2) &= w_1 + w_2 + w_1 w_2 \\ &= 1 + 2 + 1 \times 2 \\ &= 5 \end{aligned}$$

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Regression Models and Randomized Treatments

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Multivariate Derivative

Definition

Suppose $f : X \rightarrow \mathbb{R}^1$, where $X \subset \mathbb{R}^n$. $f(\mathbf{x}) = f(x_1, x_2, \dots, x_N)$. If the limit,

$$\begin{aligned}\frac{\partial}{\partial x_i} f(\mathbf{x}_0) &= \frac{\partial}{\partial x_i} f(x_{01}, x_{02}, \dots, x_{0i}, x_{0i+1}, \dots, x_{0N}) \\ &= \lim_{h \rightarrow 0} \frac{f(x_{01}, x_{02}, \dots, x_{0i} + h, \dots, x_{0N}) - f(x_{01}, x_{02}, \dots, x_{0i}, \dots, x_{0N})}{h}\end{aligned}$$

exists then we call this the partial derivative of f with respect to x_i at the value $\mathbf{x}_0 = (x_{01}, x_{02}, \dots, x_{0N})$.

Rules for Taking Partial Derivatives

Partial Derivative: $\frac{\partial f(\mathbf{x})}{\partial x_i}$

- Treat each instance of x_i as a **variable** that we would differentiate before
- Treat each instance of $\mathbf{x}_{-i} = (x_1, x_2, x_3, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ as a **constant**

Example Partial Derivatives

$$\begin{aligned}f(\mathbf{x}) &= f(x_1, x_2) \\ &= x_1 + x_2\end{aligned}$$

Partial derivative, with respect to x_1 at (x_{01}, x_{02})

$$\begin{aligned}\frac{\partial f(x_1, x_2)}{\partial x_1} \Big|_{(x_{01}, x_{02})} &= 1 + 0 \Big|_{x_{01}, x_{02}} \\ &= 1\end{aligned}$$

Example Partial Derivatives

$$\begin{aligned}f(\mathbf{x}) &= f(x_1, x_2, x_3) \\ &= x_1^2 \log(x_1) + x_2 x_1 x_3 + x_3^2\end{aligned}$$

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What is the partial derivative with respect to x_1 ?
 $\mathbf{x}_0 = (x_{01}, x_{02}, x_{03})$.

Evaluated at

$$\begin{aligned}\frac{\partial f(\mathbf{x})}{\partial x_1} \Big|_{\mathbf{x}_0} &= 2x_1 \log(x_1) + x_1^2 \frac{1}{x_1} + x_2 x_3 \Big|_{\mathbf{x}_0} \\ &= 2x_{01} \log(x_{01}) + x_{01} + x_{02} x_{03}\end{aligned}$$

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What is the partial derivative with respect to x_1 ? x_2 ? Evaluated at $\mathbf{x}_0 = (x_{01}, x_{02}, x_{03})$.

$$\begin{aligned}\frac{\partial f(\mathbf{x})}{\partial x_2} \Big|_{\mathbf{x}_0} &= x_1 x_3 \Big|_{\mathbf{x}_0} \\ &= x_{01} x_{03}\end{aligned}$$

Example Partial Derivatives

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What is the partial derivative with respect to x_1 ? x_2 ? x_3 ? Evaluated at $\mathbf{x}_0 = (x_{01}, x_{02}, x_{03})$.

$$\begin{aligned}\frac{\partial f(\mathbf{x})}{\partial x_3} \Big|_{\mathbf{x}_0} &= x_1 x_2 + 2x_3 \Big|_{\mathbf{x}_0} \\ &= x_{01} x_{02} + 2x_{03}\end{aligned}$$

Rate of Change, Linear Regression

Suppose we regress **Approval**_{*i*} rate for Obama in month *i* on **Employ**_{*i*} and **Gas**_{*i*}. We obtain the following model:

$$\text{Approval}_i = 0.8 - 0.5\text{Employ}_i - 0.25\text{Gas}_i$$

We are modeling $\text{Approval}_i = f(\text{Employ}_i, \text{Gas}_i)$. What is partial derivative with respect to employment?

$$\frac{\partial f(\text{Employ}_i, \text{Gas}_i)}{\partial \text{Employ}_i} = -0.5$$

Gradient

Definition

Suppose $f : X \rightarrow \mathbb{R}^1$ with $X \subset \mathbb{R}^n$ is a differentiable function. Define the gradient vector of f at \mathbf{x}_0 , $\nabla f(\mathbf{x}_0)$ as,

$$\nabla f(\mathbf{x}_0) = \left(\frac{\partial f(\mathbf{x}_0)}{\partial x_1}, \frac{\partial f(\mathbf{x}_0)}{\partial x_2}, \frac{\partial f(\mathbf{x}_0)}{\partial x_3}, \dots, \frac{\partial f(\mathbf{x}_0)}{\partial x_n} \right)$$

- The gradient points in the direction that the function is **increasing** in the fastest direction
- We'll use this to do optimization (both analytic and computational)

Example Gradient Calculation

Suppose

$$\begin{aligned}f(\mathbf{x}) &= f(x_1, x_2, \dots, x_n) \\ &= x_1^2 + x_2^2 + \dots + x_n^2 \\ &= \sum_{i=1}^n x_i^2\end{aligned}$$

Then $\nabla f(\mathbf{x}^*)$ is

$$\nabla f(\mathbf{x}^*) = (2x_1^*, 2x_2^*, \dots, 2x_n^*)$$

So if $\mathbf{x}^* = (3, 3, \dots, 3)$ then

$$\begin{aligned}\nabla f(\mathbf{x}^*) &= (2 * 3, 2 * 3, \dots, 2 * 3) \\ &= (6, 6, \dots, 6)\end{aligned}$$

Second Partial Derivative

Definition

Suppose $f : X \rightarrow \mathfrak{R}$ where $X \subset \mathfrak{R}^n$ and suppose that $\frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_i}$ exists. Then we define,

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_j \partial x_i} \equiv \frac{\partial}{\partial x_j} \left(\frac{\partial f(\mathbf{x})}{\partial x_i} \right)$$

- Second derivative could be with respect to x_j or with some other variable x_i
- Nagging question: does order matter?

Second Partial Derivative: Order Doesn't Matter

Theorem

Young's Theorem Let $f : X \rightarrow \mathfrak{R}$ with $X \subset \mathfrak{R}^n$ be a twice differentiable function on all of X . Then for any i, j , at all $\mathbf{x}^* \in X$,

$$\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}^*) = \frac{\partial^2}{\partial x_j \partial x_i} f(\mathbf{x}^*)$$

Second Order Partial Derivates

$$f(\mathbf{x}) = x_1^2 x_2^2$$

Then,

$$\frac{\partial^2}{\partial x_1 \partial x_1} f(\mathbf{x}) = 2x_2^2$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} f(\mathbf{x}) = 4x_1 x_2$$

$$\frac{\partial^2}{\partial x_2 \partial x_2} f(\mathbf{x}) = 2x_1^2$$

Hessians

Definition

Suppose $f : X \rightarrow \mathbb{R}^1$, $X \subset \mathbb{R}^n$, with f a twice differentiable function. We will define the **Hessian** matrix as the matrix of second derivatives at $\mathbf{x}^* \in X$,

$$\mathbf{H}(f)(\mathbf{x}^*) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{x}^*) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}^*) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{x}^*) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}^*) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(\mathbf{x}^*) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{x}^*) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{x}^*) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{x}^*) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

- Hessians are **symmetric**
- They describe **curvature** of a function (think, how bended)
- Will be the basis for second derivative test for multivariate optimization

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Functions with Multidimensional Codomains

Definition

Suppose $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$. We will call f a *multivariate* function. We will commonly write,

$$f(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix}$$

Example Functions

Suppose $f : \mathbb{R} \rightarrow \mathbb{R}^2$,

$$f(t) = (t^2, \sqrt{t})$$

Example Functions

Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined as

$$f(r, \theta) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$$

Example Functions

Suppose we have some policy $\mathbf{x} \in \mathbb{R}^M$. Suppose we have N legislators where legislator i has utility

$$U_i(\mathbf{x}) = \sum_{j=1}^M -(x_j - \mu_{ij})^2$$

We can describe the utility of all legislators to the proposal as

$$f(\mathbf{x}) = \begin{pmatrix} \sum_{j=1}^M -(x_j - \mu_{1j})^2 \\ \sum_{j=1}^M -(x_j - \mu_{2j})^2 \\ \vdots \\ \sum_{j=1}^M -(x_j - \mu_{Nj})^2 \end{pmatrix}$$

Jacobian

Definition

Suppose $f : X \rightarrow \mathbb{R}^n$, where $X \subset \mathbb{R}^m$, with f a differentiable function. Define the **Jacobian** of f at \mathbf{x} as

$$\mathbf{J}(f)(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

Example of Jacobian

$$f(r, \theta) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$$

$$\mathbf{J}(f)(r, \theta) = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

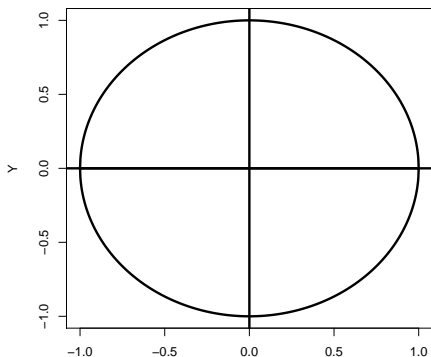
Implicit Functions and Differentiation

We have defined functions **explicitly**

$$Y = f(x)$$

We might also have an **implicit** function:

$$1 = x^2 + y^2$$



Implicit Function Theorem (From Avi Acharya's Notes)

Definition

Suppose $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}$. Let $f : X \cup Y \rightarrow \mathbb{R}$ be a differentiable function (with continuous partial derivatives). Let $(\mathbf{x}^*, y^*) \in X \cup Y$ such that

$$\begin{aligned}\frac{\partial f(\mathbf{x}^*, y^*)}{\partial y} &\neq 0 \\ f(\mathbf{x}^*, y^*) &= 0\end{aligned}$$

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Then there exists $B \subset \mathbb{R}^n$ such that there is a differentiable function $g : B \rightarrow \mathbb{R}$ such that $x^* \in B$ then $g(x^*) = y^*$ and $f(x, g(x)) = 0$. The derivative of g for $x \in B$ is given by

$$\frac{\partial g}{\partial x_j} = -\frac{\frac{\partial f}{\partial x_j}}{\frac{\partial f}{\partial y}}$$

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The intuition from the Implicit Function Theorem is that any function $g(x) = y$ there would need an “infinite” slope.

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In this example, the negative sign is “moving things to the other side”. In general, the negative sign will capture that we want to measure the **compensatory** behavior of the function: how y moves in response to some x_i **along a level curve**

Example 2: Implicit Function Theorem (From Jim Fearon)

Suppose there n individuals, each individual i earns pre-tax income $y_i > 0$.

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Suppose:

$$U_i(t, y_i) = y_i(1 - t^2) + t\bar{y}$$

Example 2: Implicit Function Theorem (From Jim Fearon)

An individual's optimal tax rate is:

$$\begin{aligned}\frac{\partial U_i(t, y_i)}{\partial t} &= -2y_i t + \bar{y} \\ 0 &= -2y_i t^* + \bar{y} \\ \frac{\bar{y}}{2y_i} &= t_i^*\end{aligned}$$

Checking the second derivative:

$$\frac{\partial^2 U_i(t, y_i)}{\partial^2 t} = -2y_i$$

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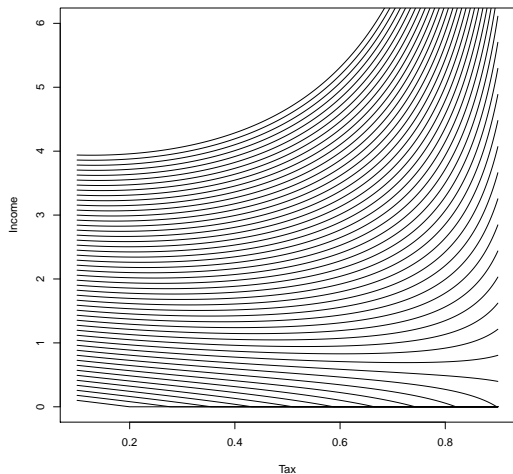
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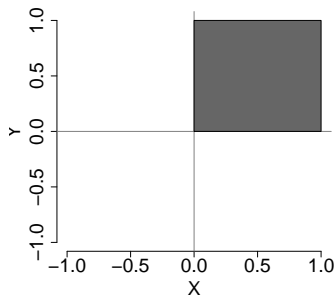
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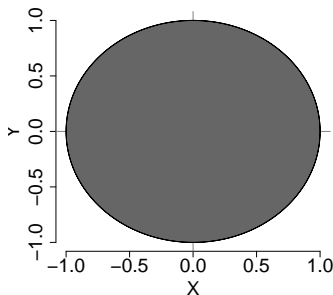
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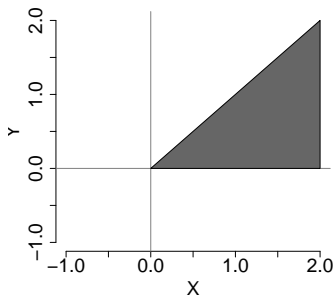
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- $A = \{x, y : x^2 + y^2 \leq 1\}$
- $A = \{x, y : x < y, x, y \in (0, 2)\}$



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How do calculate the area under the function over these regions?

Multivariate Integration

Definition

Suppose $f : X \rightarrow \mathfrak{R}$ where $X \subset \mathfrak{R}^n$. We will say that f is integrable over $A \subset X$ if we are able to calculate its area with refined partitions of A and we will write the integral $I = \int_A f(\mathbf{x})d\mathbf{A}$

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That's horribly abstract. There is an extremely helpful theorem that makes this manageable.

Theorem

Fubini's Theorem Suppose $A = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$ and that $f : A \rightarrow \mathfrak{R}$ is *integrable*. Then

$$\int_A f(\mathbf{x})d\mathbf{A} = \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \dots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(\mathbf{x})dx_1 dx_2 \dots dx_{n-1} dx_n$$

Multivariate Integration Recipe

$$\int_A f(\mathbf{x}) d\mathbf{A} = \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \dots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(\mathbf{x}) dx_1 dx_2 \dots dx_{n-1} dx_n$$

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- 1) Start with the inside integral x_1 is the variable, everything else a constant
- 2) Work inside to out, **iterating**
- 3) At the last step, we should arrive at a number

Intuition: Three Dimensional Jello Molds, a discussion

Multivariate Uniform Distribution

Suppose $f : [0, 1] \times [0, 1] \rightarrow \Re$ and $f(x_1, x_2) = 1$ for all $x_1, x_2 \in [0, 1] \times [0, 1]$. What is $\int_0^1 \int_0^1 f(x) dx_1 dx_2$?

$$\begin{aligned} \int_0^1 \int_0^1 f(x) dx_1 dx_2 &= \int_0^1 \int_0^1 1 dx_1 dx_2 \\ &= \int_0^1 x_1 \Big|_0^1 dx_2 \\ &= \int_0^1 (1 - 0) dx_2 \\ &= \int_0^1 1 dx_2 \\ &= x_2 \Big|_0^1 \\ &= 1 \end{aligned}$$

Example 2

Suppose $f : [a_1, b_1] \times [a_2, b_2] \rightarrow \mathfrak{R}$ is given by

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Example 3: Exponential Distributions

Suppose $f : \mathfrak{R}_+^2 \rightarrow \mathfrak{R}$ and that

$$f(x_1, x_2) = 2 \exp(-x_1) \exp(-2x_2)$$

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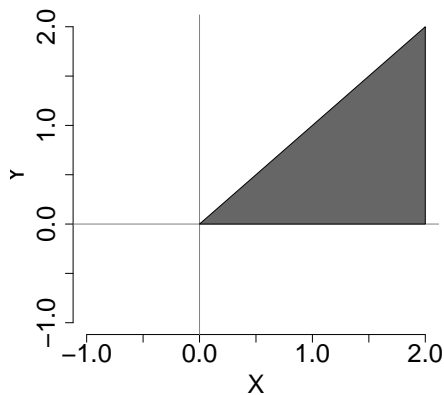
Challenge Problems

- 1) Find $\int_0^1 \int_0^1 x_1 + x_2 dx_1 dx_2$
- 2) Demonstrate that

$$\int_0^b \int_0^a x_1 - 3x_2 dx_1 dx_2 = \int_0^a \int_0^b x_1 - 3x_2 dx_2 dx_1$$

More Complicated Bounds of Integration

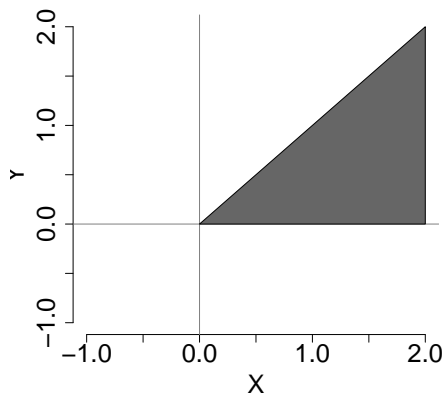
So far, we have integrated over **rectangles**. But often, we are interested in more complicated regions



How do we do this?

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Example 4: More Complicated Regions

Suppose $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$, $f(x_1, x_2) = x_1 + x_2$. Find area of function where $x_1 < x_2$.

Trick: we need to determine bound. If $x_1 < x_2$, x_1 can take on any value from 0 to x_2

$$\begin{aligned}\iint_{x_1 < x_2} f(\mathbf{x}) &= \int_0^1 \int_0^{x_2} x_1 + x_2 dx_1 dx_2 \\ &= \int_0^1 x_2 x_1 \Big|_0^{x_2} dx_2 + \int_0^1 \frac{x_1^2}{2} \Big|_0^{x_2} \\ &= \int_0^1 x_2^2 dx_2 + \int_0^1 \frac{x_2^2}{2} \\ &= \frac{x_2^3}{3} \Big|_0^1 + \frac{x_2^3}{6} \Big|_0^1 \\ &= \frac{1}{3} + \frac{1}{6} \\ &= \frac{3}{6} = \frac{1}{2}\end{aligned}$$

Consider the same function and let's switch the bounds.

$$\begin{aligned}\iint_{x_1 < x_2} f(\mathbf{x}) &= \int_0^1 \int_{x_1}^1 x_1 + x_2 dx_2 dx_1 \\ &= \int_0^1 x_1 x_2 \Big|_{x_1}^1 + \int_0^1 \frac{x_2^2}{2} \Big|_{x_1}^1 dx_1 \\ &= \int_0^1 x_1 - x_1^2 + \int_0^1 \frac{1}{2} - \frac{x_1^2}{2} dx_1 \\ &= \frac{x_1^2}{2} \Big|_0^1 - \frac{x_1^3}{3} \Big|_0^1 + \frac{x_1}{2} \Big|_0^1 - \frac{x_1^3}{6} \Big|_0^1 \\ &= \frac{1}{2} - \frac{1}{3} + \frac{1}{2} - \frac{1}{6} \\ &= 1 - \frac{3}{6} \\ &= \frac{1}{2}\end{aligned}$$

Example 5: More Complicated Regions

Suppose $f[0, 1] \times [0, 1] \rightarrow \mathfrak{R}$, $f(x_1, x_2) = 1$. What is the area of $x_1 + x_2 < 1$? Where is $x_1 + x_2 < 1$? Where, $x_1 < 1 - x_2$

$$\begin{aligned}\iint_{x_1+x_2 < 1} f(\mathbf{x}) d\mathbf{x} &= \int_0^1 \int_0^{1-x_2} 1 dx_1 dx_2 \\ &= \int_0^1 x_1 \Big|_0^{1-x_2} dx_2 \\ &= \int_0^1 (1 - x_2) dx_2 \\ &= x_2 \Big|_0^1 - \frac{x_2^2}{2} \Big|_0^1 \\ &= 1 - \left(\frac{1}{2}\right) \\ &= \frac{1}{2}\end{aligned}$$

Where we're going

Tomorrow: Multivariate Optimization + Review of Mathematical Concepts
Thursday: Probability Theory, Day 1