Math Camp

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September 15th, 2016

Conditional Probability

Political scientists almost always examine conditional relationships

- Given highway and partisanship, what is the probability of moving? (Clayton Nall)
- Given racial background, what is the probability of holding liberal political views? (Lauren Davenport)
- Given small donor base, what is the probability of extreme positions? (Adam Bonica)

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Let's formalize this idea.

Conditional Probability: Definition

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$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

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- $P(F)$ normalize: we know $P(F)$ already occurred

Some Examples

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- In words?

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Everything we proved yesterday holds for $P(\cdot|B)$.

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- $P(S|B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$
- Suppose $E_1, E_2, \ldots, E_N$ are mutually exclusive.
  Recall: $(\cup_{i=1}^{N} E_i) \cap B = \cup_{i=1}^{N} E_i \cap B$

$$
\begin{aligned}
P(\cup_{i=1}^{N} E_i | B) &= \frac{P(\cup_{i=1}^{N} E_i \cap B)}{P(B)} \\
&= \frac{\sum_{i=1}^{N} P(E_i \cap B)}{P(B)} \\
&= \sum_{i=1}^{N} P(E_i|B)
\end{aligned}
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We are calculating probabilities in the new "universe" B

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\begin{aligned}
P(\text{Cutoff Shirt}|\text{Southwest Airlines}) &= 0.2 \\
P(\text{Southwest Airlines}|\text{Cutoff Shirt}) &\approx 1
\end{aligned}
$$

Proposition

Multiplication Rule: Suppose $E_1, E_2, \ldots, E_N$ *is a sequence of events.*

$$P(E_1 \cap E_2 \cap \cdots \cap E_N) = P(E_1)P(E_2|E_1)P(E_3|E_2, E_1) \times \cdots \times P(E_N|E_{N-1}, E_{N-2}, \ldots, E_1)$$

Proof.

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Repeating for all probabilities proves the proposition

□

Law of Total Probability

Proposition

Suppose that we have a set of events $F_1, F_2, \ldots, F_N$ such that the events are mutually exclusive and together comprise the entire sample space $\cup_{i=1}^{N} F_i =$ Sample Space. Then, for any event E

$$P(E) = \sum_{i=1}^{N} P(E|F_i) \times P(F_i)$$

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Suppose $F_1, F_2, \ldots, F_N$ are mutually exclusive and $\cup_{i=1}^{N} F_i = S$. Then we can write E as:

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Example: Law of Total Probability

Infer $P(\text{vote})$ after mobilization campaign

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Sample space (one person) =
{ (mobilized, vote), (mobilized, not vote), (not mobilized, vote) , (not mobilized, not vote) }
Mobilization partitions the space (mutually exclusive and exhaustive)

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\begin{aligned}
P(\text{vote}) &= P(\text{mob.}) \times P(\text{vote}|\text{mob.}) + P(\text{not mob} \times P(\text{vote}|\text{not mob}) \\
&= 0.6 \times 0.75 + 0.4 \times 0.25 \\
&= 0.55
\end{aligned}
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Mixture of two coins

Bayes' Rule

- $P(B|A)$ may be easy to obtain
- $P(A|B)$ may be harder to determine
- Bayes' rule provides a method to move from $P(B|A)$ to $P(A|B)$.

Definition

Bayes' Rule: For two events A and B,

$$P(A|B) = \frac{P(A) \times P(B|A)}{P(B)}$$

Proof.

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Bayes' Rule: Example

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For example, Washington is the "blackest" name in America.

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For example, Washington is the "blackest" name in America.

- P(black)= 0.126.
- P(not black) = 1 - P(black) = 0.874.
- P(Washington| black) = 0.00378.
- P(Washington|nb) = 0.000060615.

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$$
\begin{aligned}
P(\text{black}|\text{Wash}) &= \frac{P(\text{black})P(\text{Wash}|\text{black})}{P(\text{Wash})} \\
&= \frac{P(\text{black})P(\text{Wash}|\text{black})}{P(\text{black})P(\text{Wash}|\text{black}) + P(\text{nb})P(\text{Wash}|\text{nb})} \\
&= \frac{0.126 \times 0.00378}{0.126 \times 0.00378 + 0.874 \times 0.000060616} \\
&\approx 0.9
\end{aligned}
$$


MAKE
A
DEAL


LET'S MAK
A DEAL


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"You blew it, and you blew it big! Since you seem to have difficulty grasping the basic principle at work here, I'll explain. After the host reveals a goat, you now have a one-in-two chance of being correct. Whether you change your selection or not, the odds are the same. There is enough mathematical illiteracy in this country, and we don't need the world's highest IQ propagating more. Shame!" Scott Smith, Ph.D. University of Florida (From Wikipedia)

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`R Code!`

Testing for a Rare Disease

Suppose there is a medical test

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P(\text{positive}|\text{disease}) &= 0.99 \\
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Independence and Information

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- Independence is symetric: if F is independent of E, then E is indepenent of F

Example Independence Relationship

Flip a fair coin twice.

$E =$ first flip heads

$F =$ second flip heads

$$
\begin{aligned}
P(E \cap F) &= P(\{(H,H),(H,T)\} \cap \{(H,H),(T,H)\}) \\
&= P(\{(H,H)\}) \\
&= \frac{1}{4} \\
P(E) &= \frac{1}{2} \\
P(F) &= \frac{1}{2} \\
P(E)P(F) &= \frac{1}{2}\frac{1}{2} = \frac{1}{4} = P(E \cap F)
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Independence: No Information

Suppose E and F are independent. Then,

$$\begin{aligned} P(E|F) &= \frac{P(E \cap F)}{P(F)} \\ &= \frac{P(E)P(F)}{P(F)} \\ &= P(E) \end{aligned}$$

Conditioning on the event F does not modify the probability of E.
No information about E in F

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Mutually exclusive $\neq$ Independent

Suppose E and F are mutually exclusive events:

$E = \{(H, H), (H, T)\}$; $F = \{(T, H), (T, T)\}$

$E \cap F = \emptyset$

$P(E|F) = 0$; $P(E) = \frac{1}{2}$.

Independence and Complements

Proposition

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Example: Independence and Causal Inference

Selection and Observational Studies

- We often want to infer the effect of some treatment
  - Incumbency on vote return
  - Democracy on war
- Observational studies: observe what we see to make inference
- Problem: units select into treatment
  - Simple example: enroll in job training if I think it will help
  - P(job|training in study) $\neq$ P(job|forced training)
- Background characteristic: difference between treatment and control groups
- Experiments (second greatest discovery of 20th century): make background characteristics and treatment status independent

Conditional Probability

Definition

Let E_1 and E_2 be two events. We will say that the events are conditionally independent given E_3 if

$$P(E_1 \cap E_2 | E_3) = P(E_1|E_3)P(E_2|E_3)$$

Proposition

Suppose E_1 and E_2 and E_3 are events such that $P(E_1 \cap E_2) > 0$ and $P(E_2 \cap E_3) > 0$. Then E_1 and E_2 are conditionally independent given E_3 if and only if $P(E_1|E_2 \cap E_3) = P(E_1|E_3)$.

Proof.

Suppose E_1 and E_2 are conditionally independent given E_3. Then

$$\begin{aligned}
P(E_1 \cap E_2|E_3) &= \frac{P(E_1 \cap E_2 \cap E_3)}{P(E_3)} \\
&= \frac{P(E_3)P(E_2|E_3)P(E_1|E_2 \cap E_3)}{P(E_3)} \\
P(E_1|E_3)P(E_2|E_3) &= P(E_2|E_3)P(E_1|E_2 \cap E_3) \\
P(E_1|E_3) &= P(E_1|E_2 \cap E_3)
\end{aligned}$$

□

Proof.

Suppose $P(E_1|E_2 \cap E_3) = P(E_1|E_3)$

$$\begin{aligned} P(E_1 \cap E_2|E_3) &= P(E_2|E_3)P(E_1|E_2 \cap E_3) \\ &= P(E_2|E_3)P(E_1|E_3) \end{aligned}$$

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But

$$\begin{aligned}
P(H_1) &= P(E_1)P(H_1|E_1) + P(E_1^c)P(H_1|E_1^c) = 1/2(0.99) + 1/2(0.01 \\
P(H_2) &= 1/2 \\
P(H_1 \cap H_2) &= P(E_1)P(H_1 \cap H_2|E_1) + P(E_1^c)P(H_1 \cap H_2|E_1^c) \\
&= 0.5(0.99 \times 0.99) + 0.5(0.01 \times 0.01) \approx 0.5
\end{aligned}$$

Definition

Suppose we have a sequence of events $E_1, E_2, \ldots, E_n$. *We say the sequence of events is* *mutually indepenent* *if for each subset of the sequence,* $E_{i_1}, E_{i_2}, \ldots, E_{i_j}$

$$P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_j}) = \prod_{m=1}^{j} P(E_{i_m})$$

For a sequence to be independent, every subset is independent

Definition

Define the odds of some event E as

$$odds_E = \frac{P(E)}{1 - P(E)}$$

Suppose F is another event. Define the odds ratio of E to F as

$$\begin{aligned} odds\ ratio_{E:F} &= \frac{odds_E}{odds_F} \\ &= \frac{\frac{P(E)}{1-P(E)}}{\frac{P(F)}{1-P(F)}} \end{aligned}$$

- Big: implies E is very likely
- Small: implies E is unlikely
- Problem: big changes in odd ratio may correspond to very small changes in chance something will happen $\rightsquigarrow$ baseline problem

Where we're going

Today

- Conditional probability
- Bayes' Rule
- Independence

Next lecture: Random variables (discrete and continuous)