Math Camp

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Optimization

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How to Optimize

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- More difficult problems $\leadsto$ computational solutions

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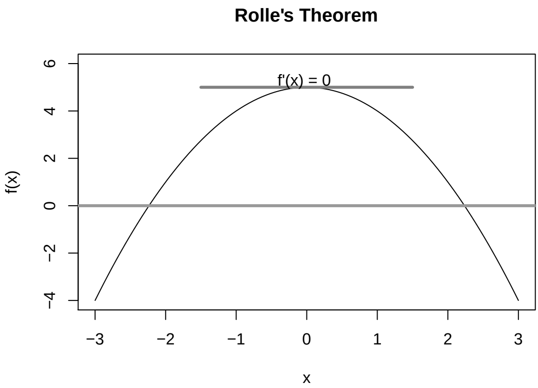
Intuition: Optimization with Derivatives, Known well behaved functions



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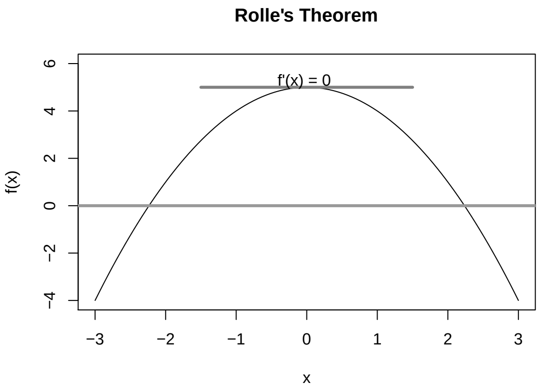

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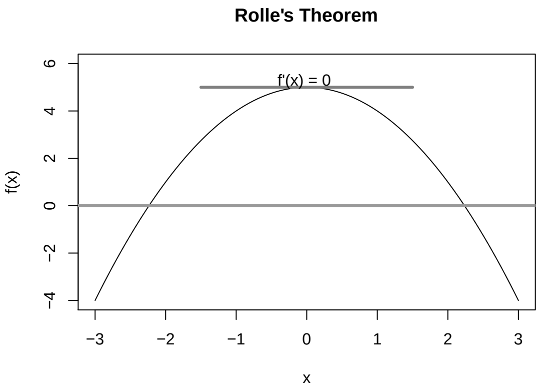

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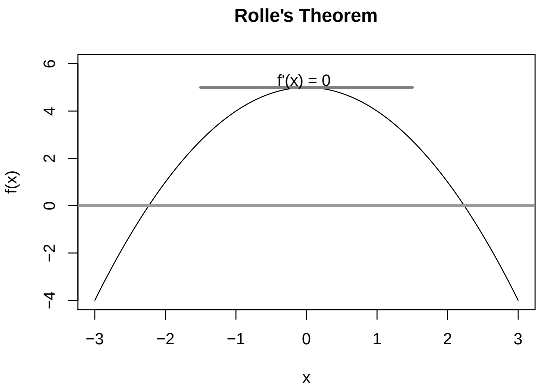

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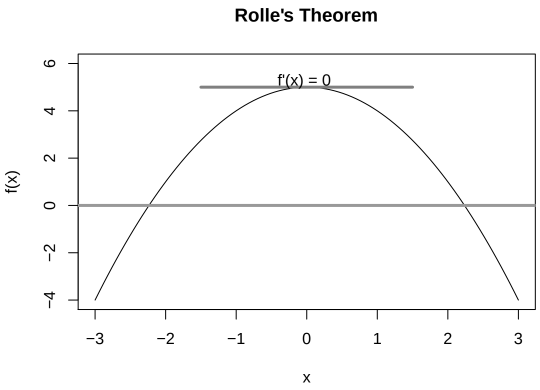

- Rolle's theorem guarantee's that, at some point, $f^{'}(x) = 0$
- Intuition from proof—what happens as we approach from the left?
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- critical intuition first, second derivatives

Second Derivatives

Definition

Suppose $f : \Re \rightarrow \Re$ *is differentiable. Recall we write this as* $f^{'}$ *and suppose that* $f^{'} : \Re \rightarrow \Re$*. Then if the limit,*

$$\lim_{x \rightarrow x_0} R(x) \quad = \quad \frac{f^{'}(x) - f^{'}(x_0)}{x - x_0}$$

exists, we call this the second derivative at $x_0, f^{''}(x_0)$*.*

Example of Second Derivatives

$$
\begin{aligned}
f(x) &= x \\
f^{'}(x) &= 1 \\
f^{''}(x) &= 0
\end{aligned}
$$

Example of Second Derivatives

$$
\begin{aligned}
f(x) &= e^x \\
f^{'}(x) &= e^x \\
f^{''}(x) &= e^x
\end{aligned}
$$

Example of Second Derivatives

$$
\begin{aligned}
f(x) &= \log(x) \\
f^{'}(x) &= \frac{1}{x} \\
f^{''}(x) &= \frac{-1}{x^2}
\end{aligned}
$$

Example of Second Derivatives

$$
\begin{aligned}
f(x) &= \frac{1}{x} \\
f^{'}(x) &= \frac{-1}{x^2} \\
f^{''}(x) &= \frac{2}{x^3}
\end{aligned}
$$

Example of Second Derivatives

$$
\begin{aligned}
f(x) &= -x^2 + 20 \\
f^{'}(x) &= -2x \\
f^{''}(x) &= -2
\end{aligned}
$$

Approximating functions and second order conditions

Theorem

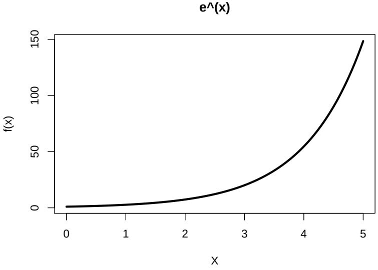
Taylor's Theorem *Suppose* $f : \Re \rightarrow \Re$, $f(x)$ *is infinitely differentiable function. Then, the taylor expansion of* $f(x)$ *around* a *is given by*

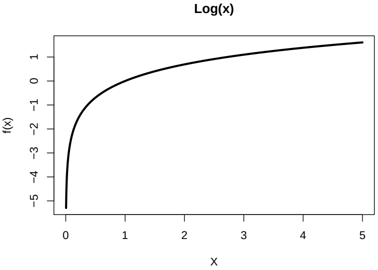
$$
\begin{aligned}
f(x) &= f(a) + \frac{f^{'}(a)}{1!}(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \frac{f^{'''}(a)}{3!}(x-a)^3 + \ldots \\
f(x) &= \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^n
\end{aligned}
$$

R Code!

Concavity, Convexity, Inflections

Second derivatives provide further information about functions





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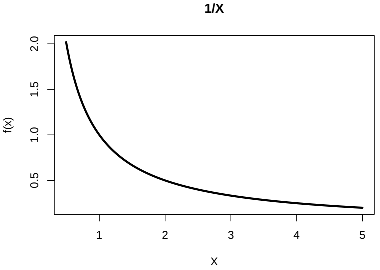
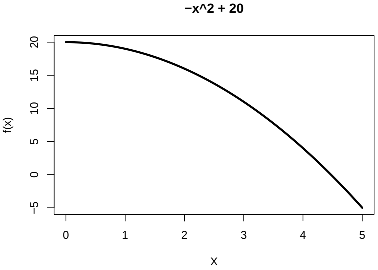




Concave Up/ Convex

Definition

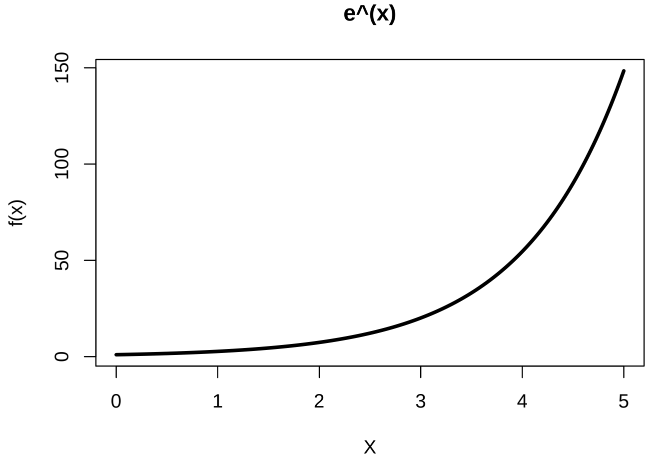
Suppose $f : [a, b] \rightarrow \Re$ *is a twice differentiable function. If, for all* $x \in [a, b]$ *and* $y \in [a, b]$ *and* $t \in (0, 1)$

$$f((1-t)x + ty) \quad < \quad (1-t)f(x) + tf(y)$$

We say that f *is strictly concave up or convex. Equivalently if* $f^{''}(x) > 0$ *for all* $x \in [a, b]$*, we say that* f *is strictly concave up.*

Concave Up, Graphical Test

$f(x) = e^x$, $[1, 4]$



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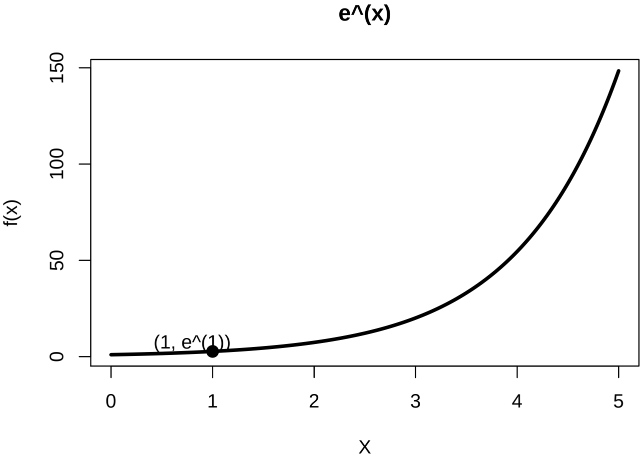

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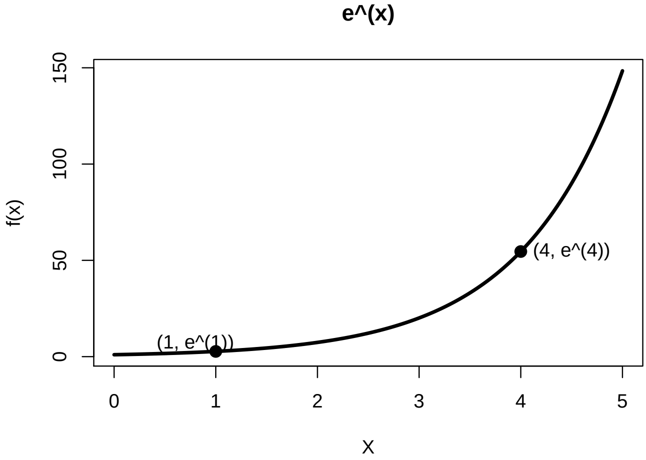

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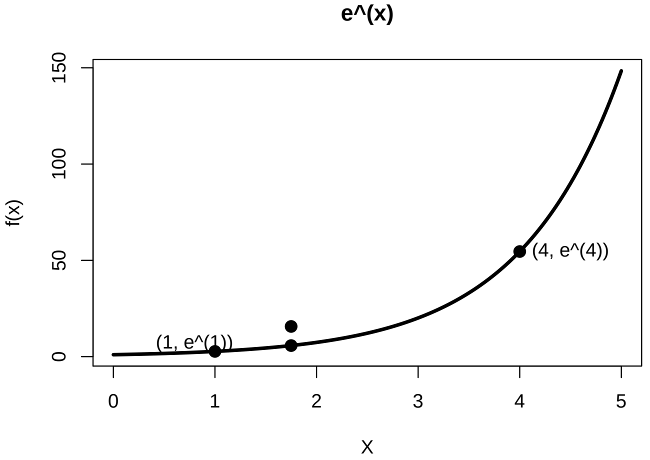

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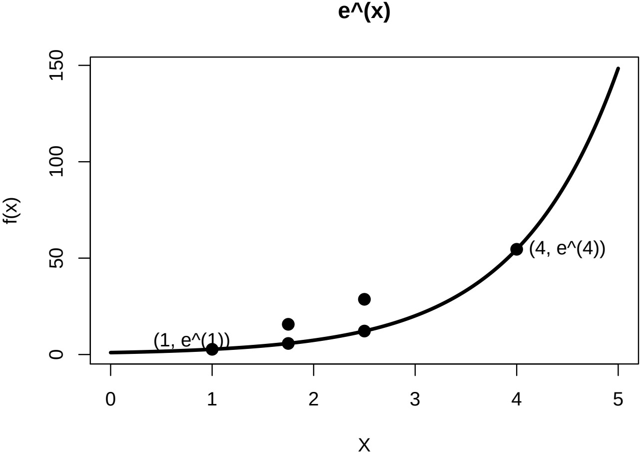

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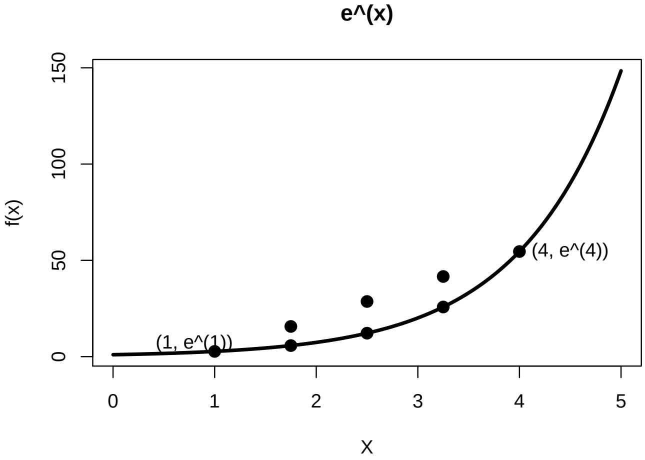

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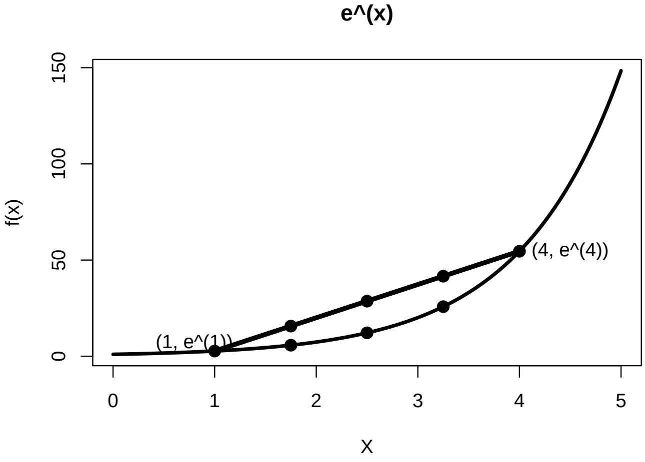
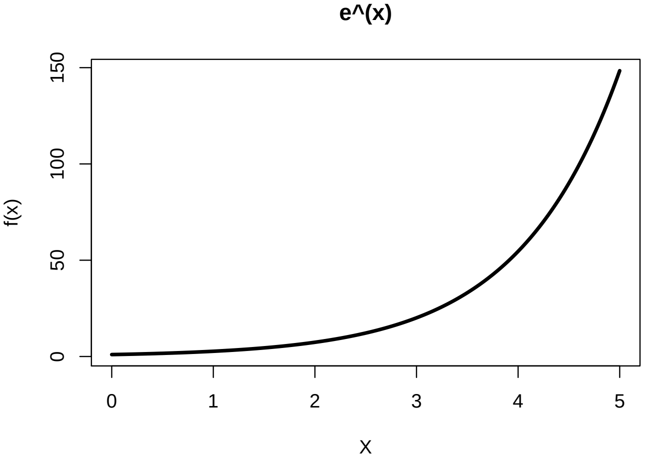


Concave Up, Second Derivative



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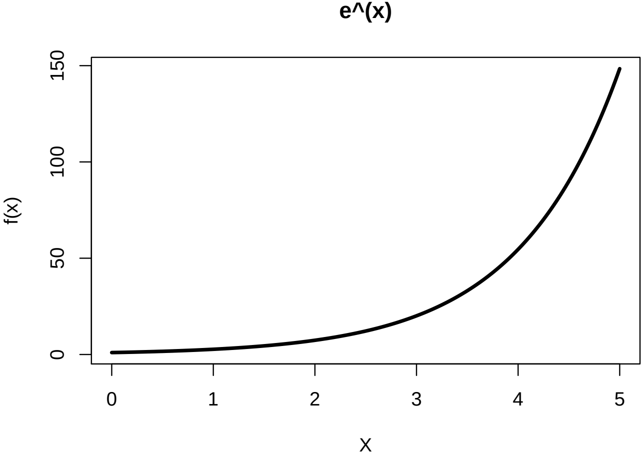
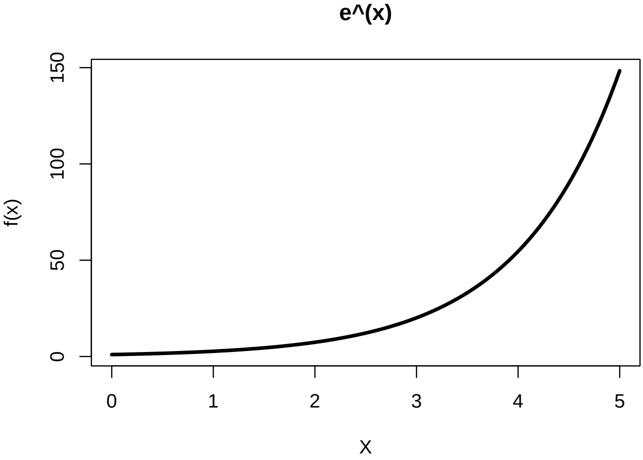


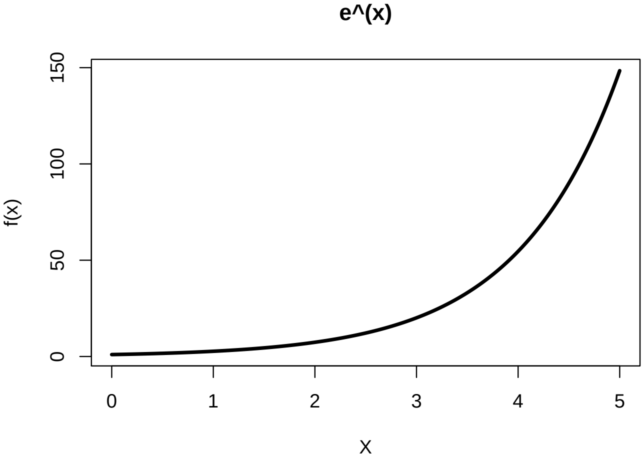
$$f(x) = e^x$$

Concave Up, Second Derivative



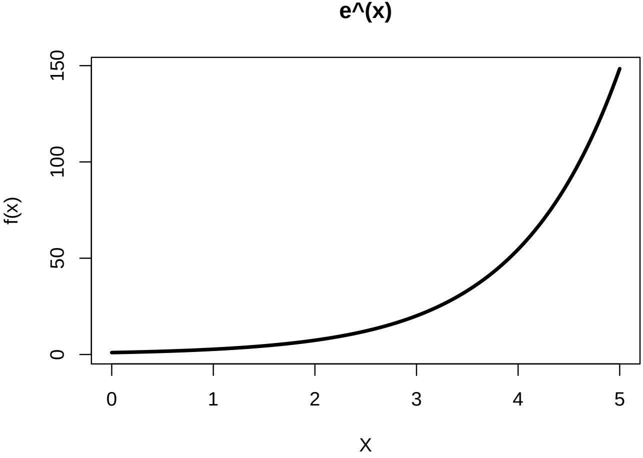
$$
\begin{aligned}
f(x) &= e^x \\
f^{'}(x) &= e^x
\end{aligned}
$$

Concave Up, Second Derivative



$$f(x) = e^x$$
$$f^{'}(x) = e^x$$
$$f^{''}(x) = e^x$$

Concave Up, Second Derivative



$$f(x) = e^x$$
$$f^{'}(x) = e^x$$
$$f^{''}(x) = e^x$$

$e^x > 0$ for all $x \in [1, 4]$

Concave Down

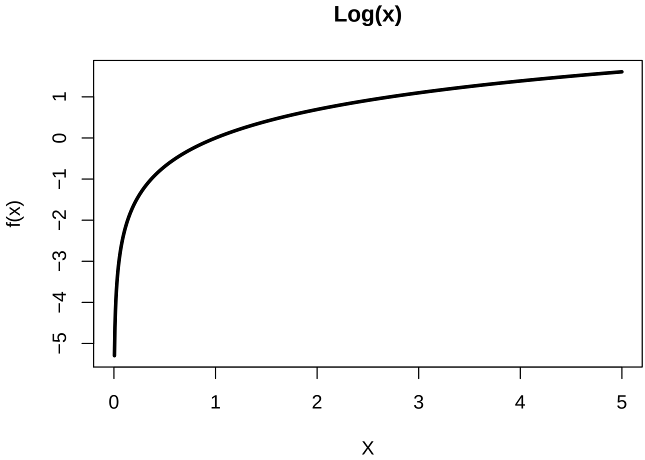
Definition

Suppose $f : [a, b] \rightarrow \Re$ *is a twice differentiable function. If, for all* $x \in [a, b]$ *and* $y \in [a, b]$ *and* $t \in (0, 1)$

$$f((1-t)x + ty) \quad > \quad (1-t)f(x) + tf(y)$$

We say that f *is strictly concave down. Equivalently if* $f^{''}(x) < 0$ *for all* $x \in [a, b]$*, we say that* f *is strictly concave down.*

Concave Down



- Show Concave down with graph test for $x \in [1, 4]$
- Show concave down with second derivative test for $x \in [1, 4]$

Optimization

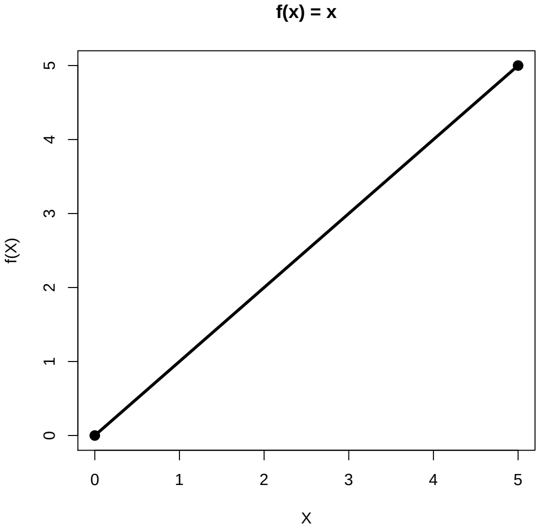
Theorem

Extreme Value Theorem *Suppose* $f : [a, b] \rightarrow \Re$ *and that* f *is continuous. Then* f *obtains its extreme value on* $[a, b]$.

Corollary

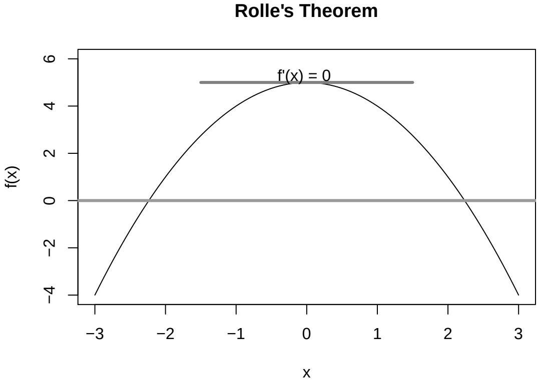
Suppose $f : [a, b] \rightarrow \Re$, *that* f *is continuous and differentiable, and that* $f(a)$ *nor* $f(b)$ *is the extreme value. Then* f *obtains its maximum on* (a, b) *and if* $f(x_0)$ *is the extreme value of* f $x_0 \in (a, b)$ *then,* $f^{'}(x_0) = 0$.

Extrema on End Points



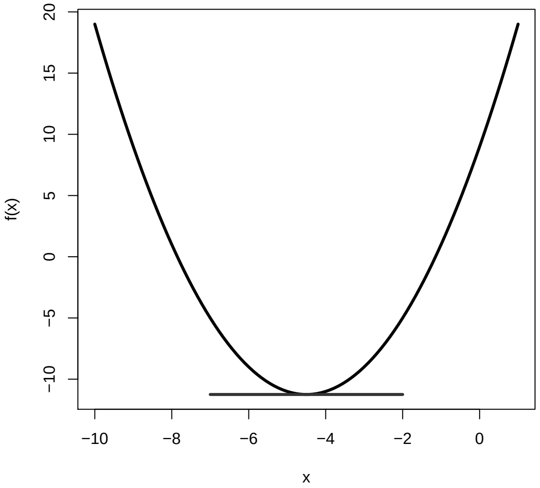
Maximum in Middle, Concave Down

$f(x) = -x^2 + 5.$



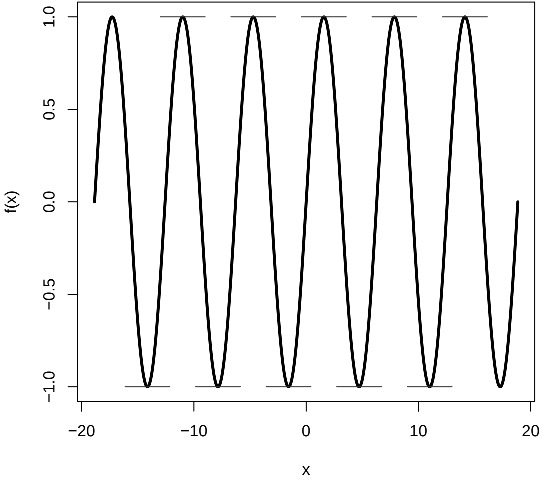
Minimum in Interior, Concave Up

$f(x) = x^2 + 9x + 9$



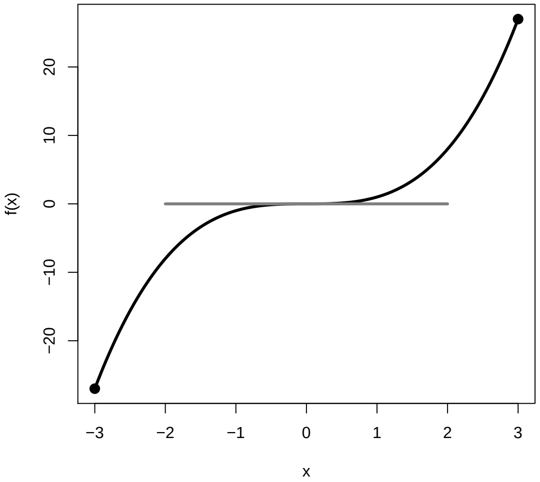
Local Optima

$f(x) = \sin(x)$



Inflection points

$f(x) = x^3$



Framework for Optimization

Recipe for optimization

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- Find $f^{'}(x)$.

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- Find $f^{'}(x)$.
- Set $f^{'}(x) = 0$ and solve for x. Call all x_0 such that $f^{'}(x_0) = 0$ critical values.

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Framework for Optimization

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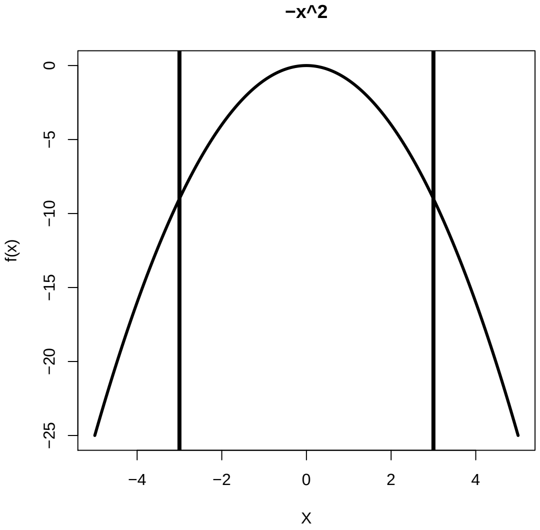
- Check End Points!

Example 1: $f(x) = -x^2$, $x \in [-3, 3]$



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1) Critical Value:

$$\begin{aligned} f^{'}(x) &= -2x \\ 0 &= -2x^{*} \\ x^{*} &= 0 \end{aligned}$$

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2) Second Derivative:

$$\begin{aligned} f^{'}(x) &= -2x \\ f^{''}(x) &= -2 \end{aligned}$$

$f^{''}(x) < 0$, local maximum

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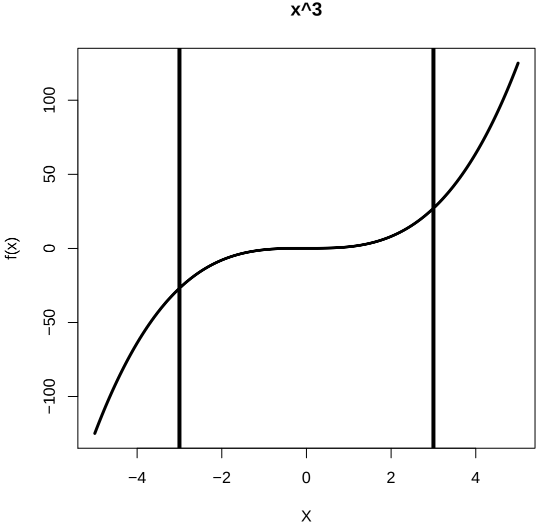
Example 1: $f(x) = -x^2$, $x \in [-3, 3]$

3) Check end points

$$\begin{aligned} f(0) &= -0^2 = 0 \\ f(-3) &= -(-3)^2 = -9 \\ f(3) &= -(3)^2 = -9 \end{aligned}$$

Example 2: $f(x) = x^3$, $x \in [-3, 3]$



Example 2: $f(x) = x^3$, $x \in [-3, 3]$

1) Critical Values:

$$\begin{aligned} f^{'}(x) &= 3x^2 \\ 0 &= 3(x^*)^2 \\ x^* &= 0 \end{aligned}$$

Example 2: $f(x) = x^3$, $x \in [-3, 3]$

1) Critical Values:

$$\begin{aligned} f^{'}(x) &= 3x^2 \\ 0 &= 3(x^*)^2 \\ x^* &= 0 \end{aligned}$$

2) Second Derivative:

$$\begin{aligned} f^{''}(x) &= 6x \\ f^{''}(0) &= 0 \end{aligned}$$

No information

Example 2: $f(x) = x^3$, $x \in [-3, 3]$

3) Check End Points:

$$\begin{aligned} f(0) &= 0^3 = 0 \\ f(-3) &= -3^3 = -27 \\ f(3) &= 3^3 = 27 \end{aligned}$$

Neither maximum nor minimum, saddle point

Example 3: Spatial Model

A large literature in Congress supposes legislators and policies can be situated in policy space

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$$U_i(x) = -(x - \mu)^2$$

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Define legislator i's utility as, $U : \Re \rightarrow \Re$,

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U_i(x) &= -(x-\mu)^2 \\
U_i(x) &= -x^2 + 2x\mu - \mu^2
\end{aligned}
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What is i's optimal policy over the range $x \in [\mu - 2, \mu + 2]$?

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$$U_i^{'}(x) = -2(x-\mu)$$

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What is i's optimal policy over the range $x \in [\mu - 2, \mu + 2]$?

$$\begin{aligned} U_i^{'}(x) &= -2(x-\mu) \\ 0 &= -2x^{*} + 2\mu \end{aligned}$$

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U_i^{'}(x) &= -2(x-\mu) \\
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Example 3: Spatial Model

A large literature in Congress supposes legislators and policies can be situated in policy space

Suppose legislator i and policies $x, i \in \Re$.

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We call μ legislator i's ideal point

Example 3: Spatial Model

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\begin{aligned}
U_i(\mu) &= -(\mu - \mu)^2 = 0 \\
U_i(\mu - 2) &= -(\mu - 2 - \mu)^2 = -4 \\
U_i(\mu + 2) &= -(\mu + 2 - \mu)^2 = -4
\end{aligned}
$$

Maximize utility at μ

Example 4: Maximum Likelihood Estimation

In 350a, we'll learn about parameters from data.

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Theorem

Suppose $f : \Re \rightarrow (0, \infty)$. *If* x_0 *maximizes* f, *then* x_0 *maximizes* $\log(f(x))$.

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&= -\frac{1}{2}\left(\sum_{i=1}^{N} Y_i^2 - 2\mu\sum_{i=1}^{N} Y_i + N\times\mu^2\right)
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$$\begin{aligned} f^{'}(\mu) &= -\frac{1}{2}\left(-2\sum_{i=1}^{N} Y_i + 2N\mu\right) \\ f^{''}(\mu) &= -N \end{aligned}$$

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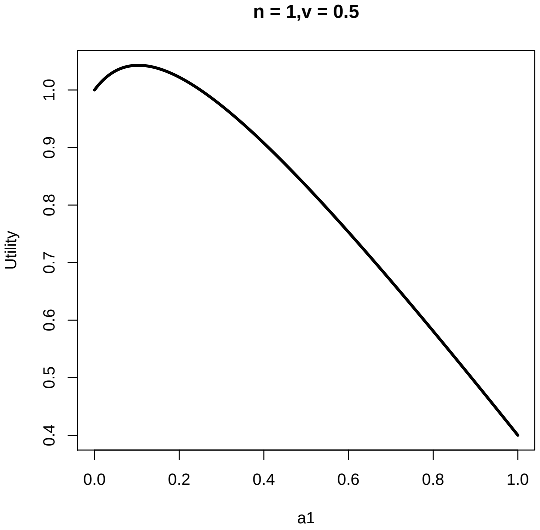
- Suppose country 2 selected value x. What should country 1 invest to maximize utility?

Example 5: IR Bargaining (from Jim Fearon, Part 1)



Example 5: IR War (from Jim Fearon, Part 1)

$$\begin{aligned}\frac{\partial U_1(a_1)}{\partial a_1} &= -1 + \frac{na_1^{n-1}(a_1^n + x^n) - (na_1^{n-1}a_1^n)}{(a_1^n + x^n)^2}v \\ &= -1 + \frac{na_1^{n-1}x^n}{(a_1^n + x^n)^2}v\end{aligned}$$

Set $n = 1$ (for simplicity)

$$\begin{aligned}0 &= -1 + \frac{x}{(a_1 + x)^2}v \\ a_1^* &= \sqrt{v}\sqrt{x} - x\end{aligned} \quad (0.1)$$

Second derivative!

$$U_1^{''}(a_1) = \frac{-2vx}{(a_1 + x)^3}$$

Example 5: IR Bargaining (from Jim Fearon, Part 1)

One more—check endpoints

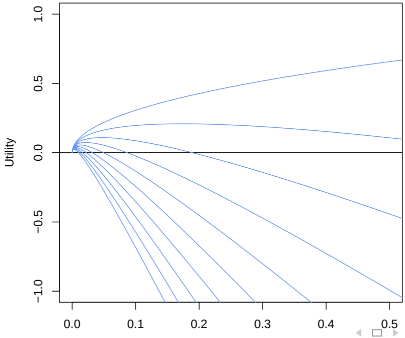
$$
\begin{aligned}
a_1^* &= 0, \text{ if } \sqrt{v}\sqrt{x} - x < 0 \\
a_1^* &= 0, \text{ if } \sqrt{v} < \sqrt{x} \\
a_1^* &= \sqrt{v}\sqrt{x} - x \text{ otherwise}
\end{aligned}
$$

Optimization Challenge Problem

- Suppose a candidate is attempting to mobilize voters. Suppose that for each investment of $x \in [0, \infty)$ the candidate receives return of $x^{1/2}$, but incurs cost of ax. So, candidate utility is,

$$U_i = x^{1/2} - ax$$

What is the optimal investment x^*?



Computational Optimization Approaches

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  - Branch and Bound ...

Newton-Raphson Method

Iterative procedure to find a root

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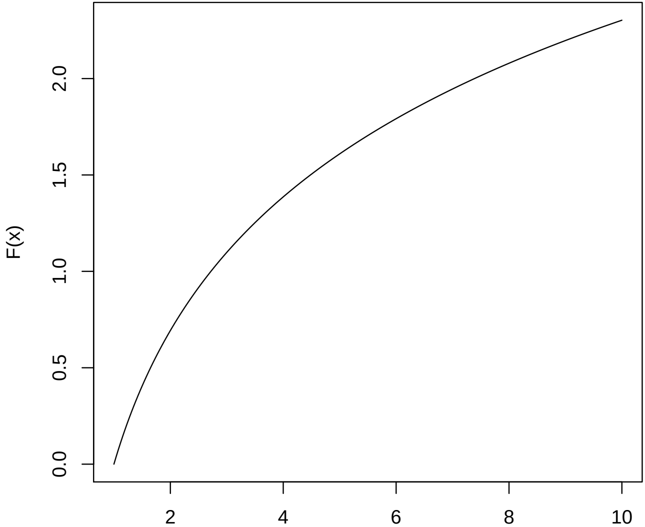
Approximate with tangent line, iteratively update

Tangent Line



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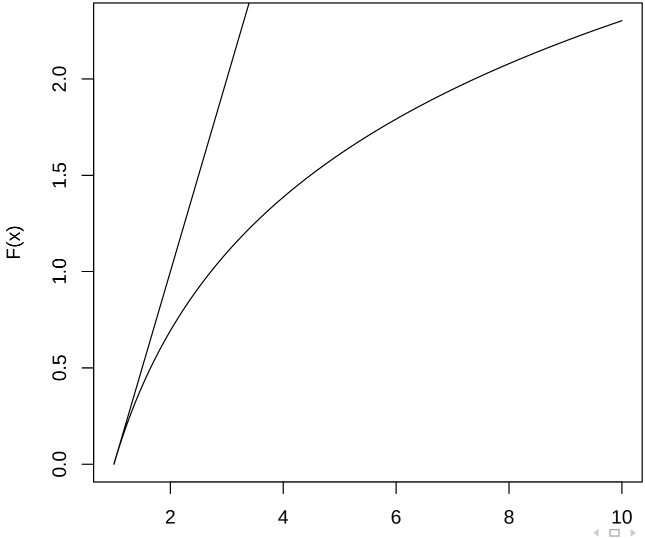

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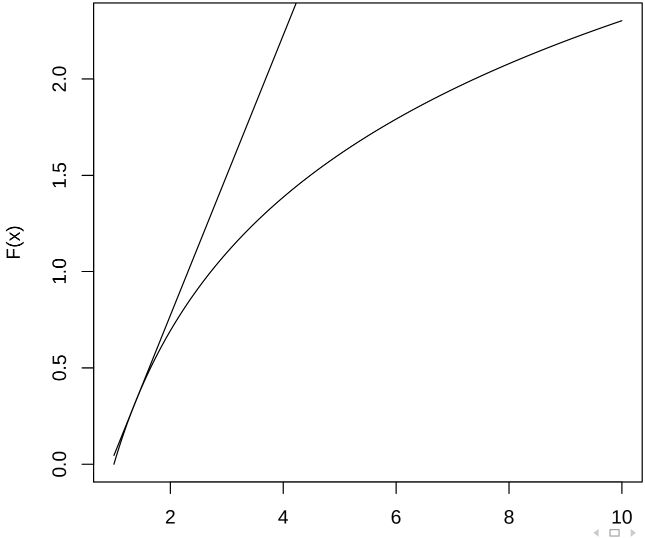

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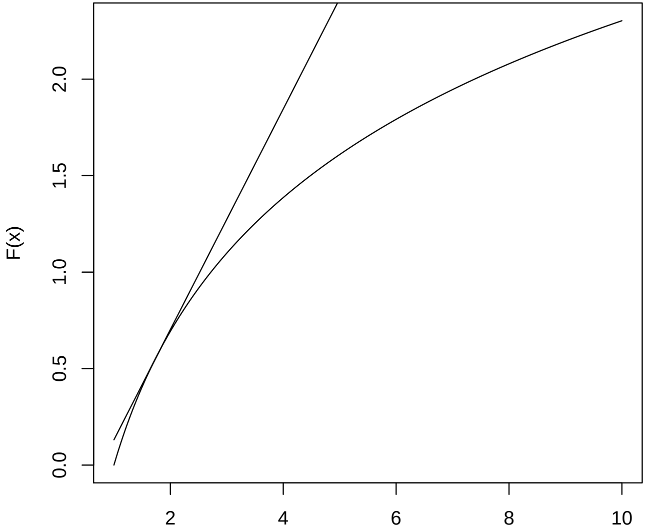

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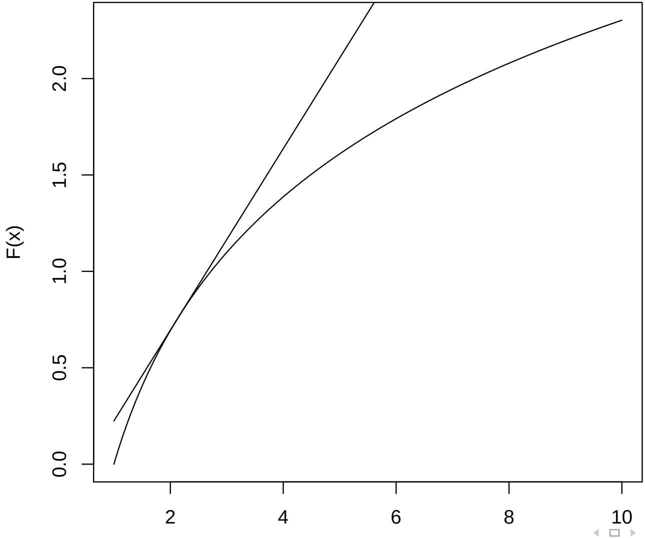

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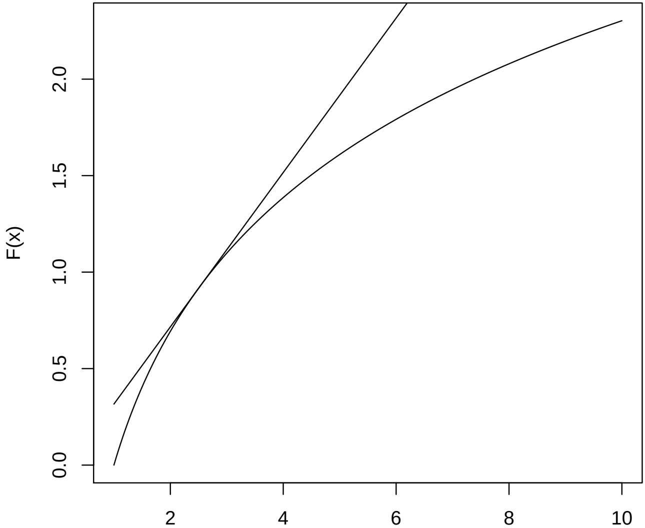

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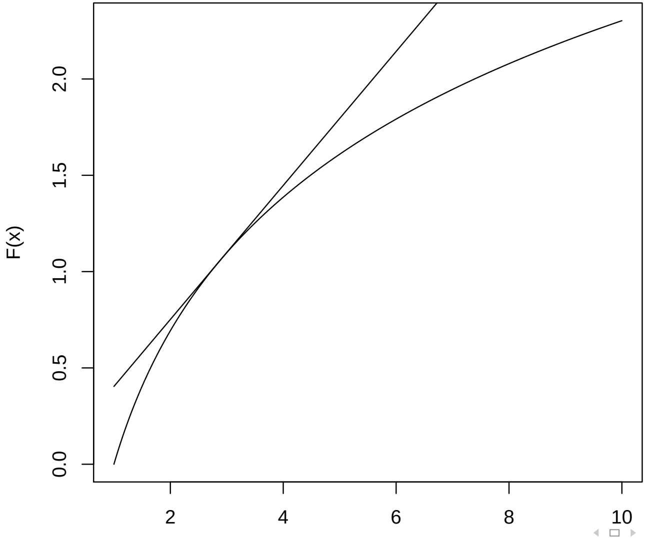

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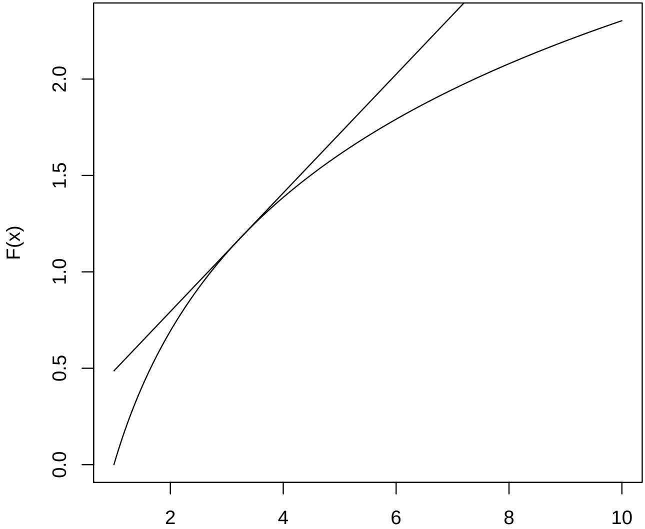

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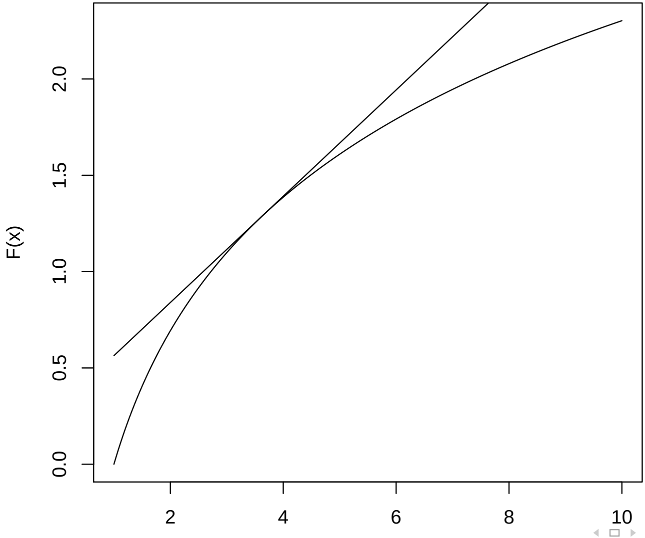

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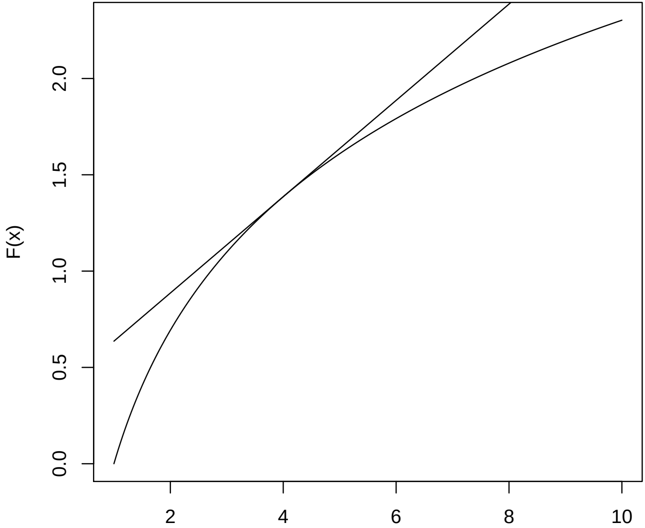

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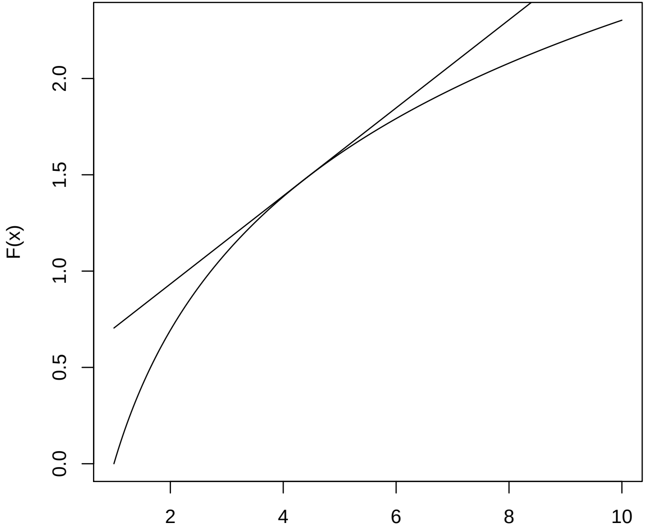

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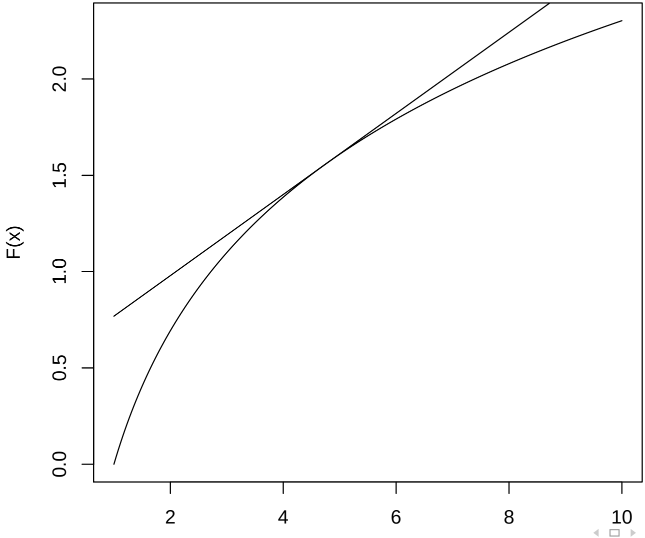

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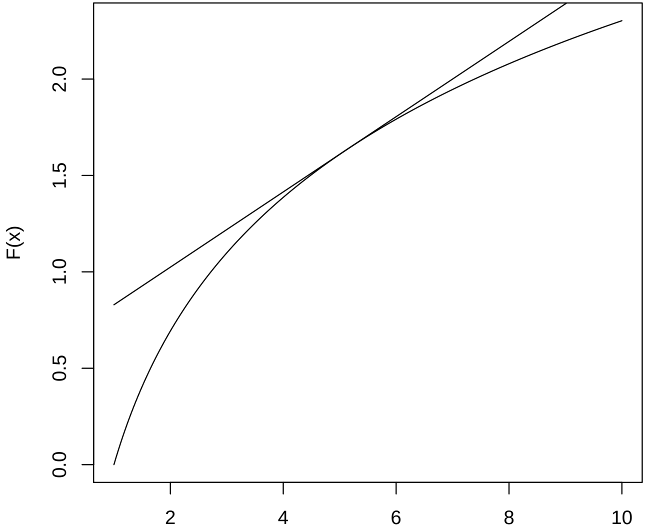

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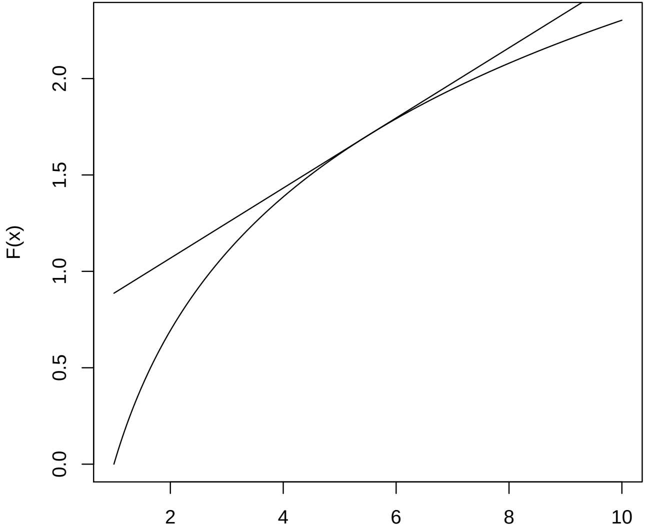

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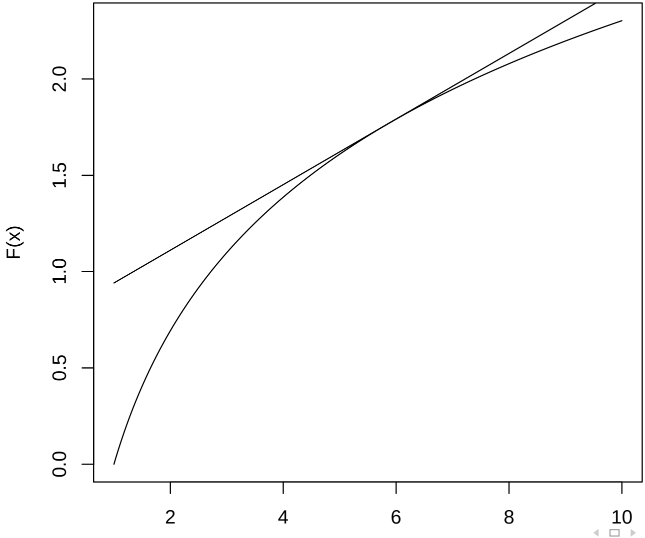

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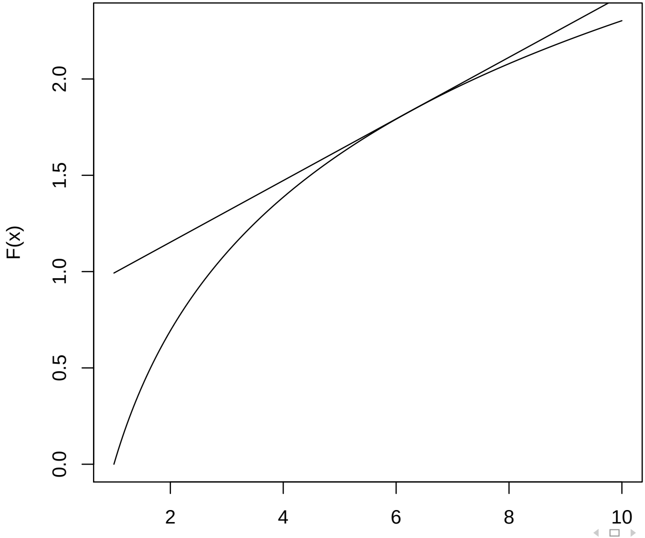

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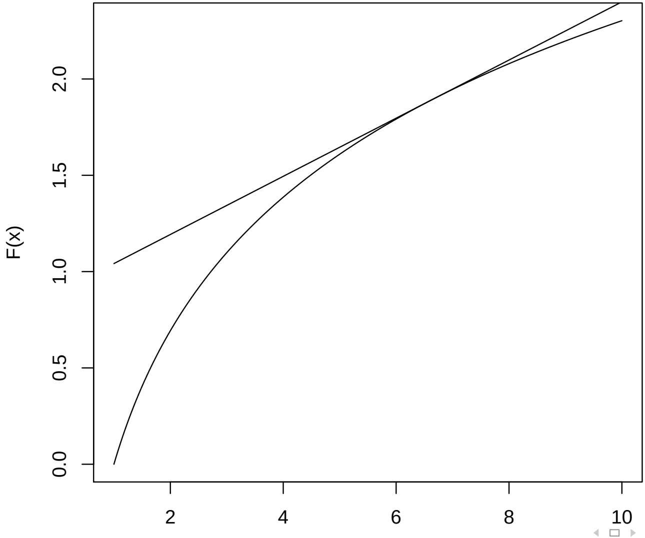

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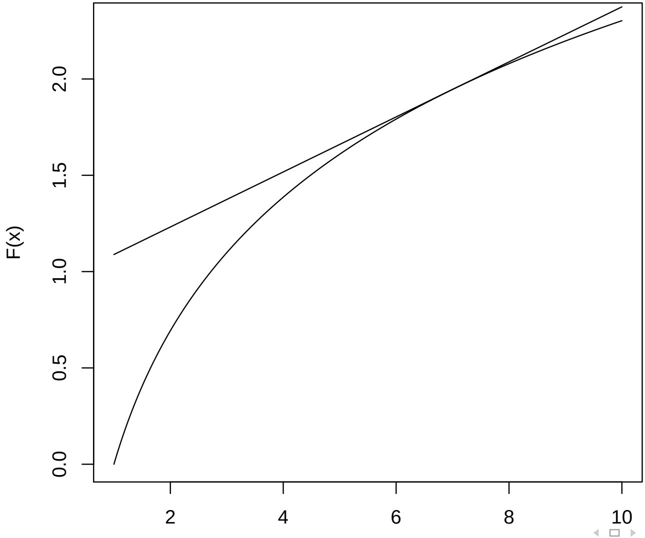

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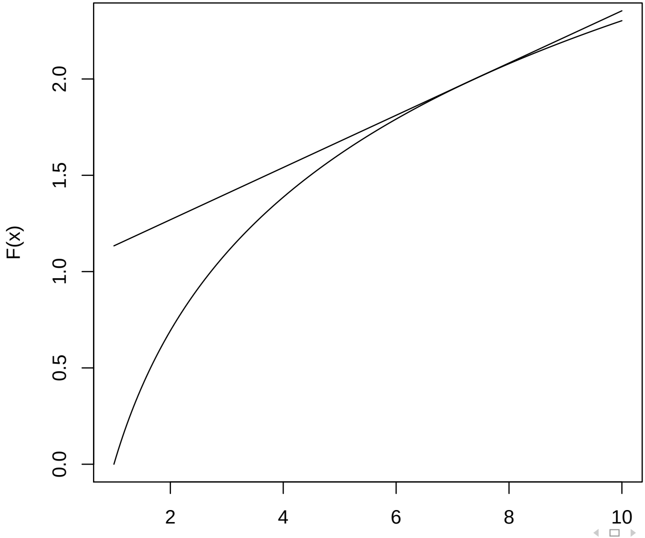

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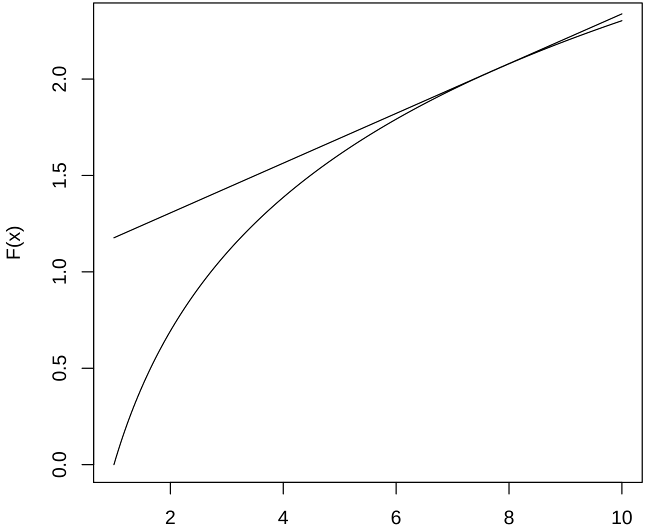

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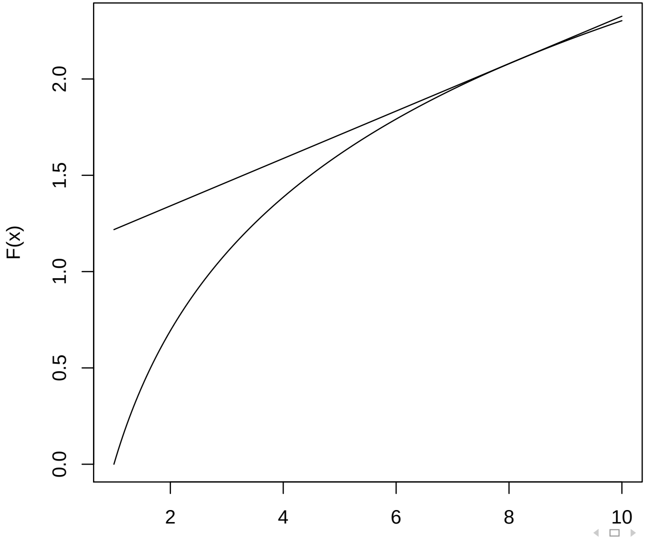

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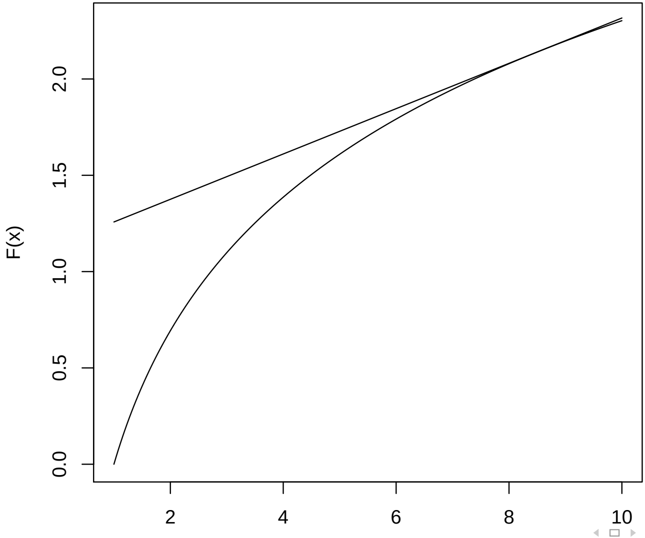

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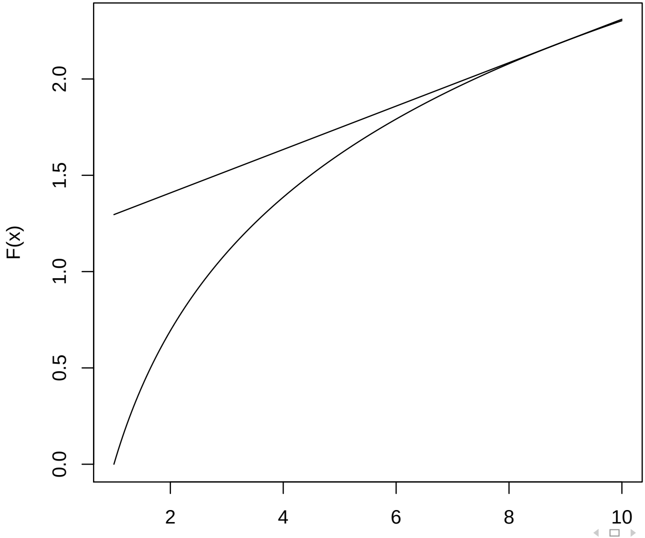

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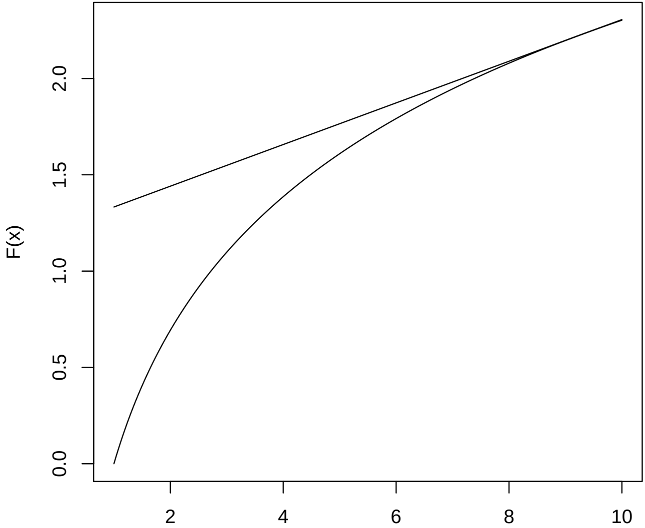

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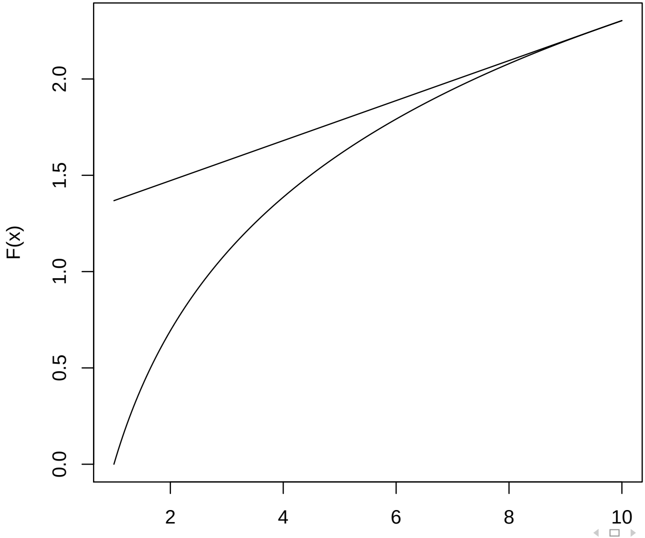

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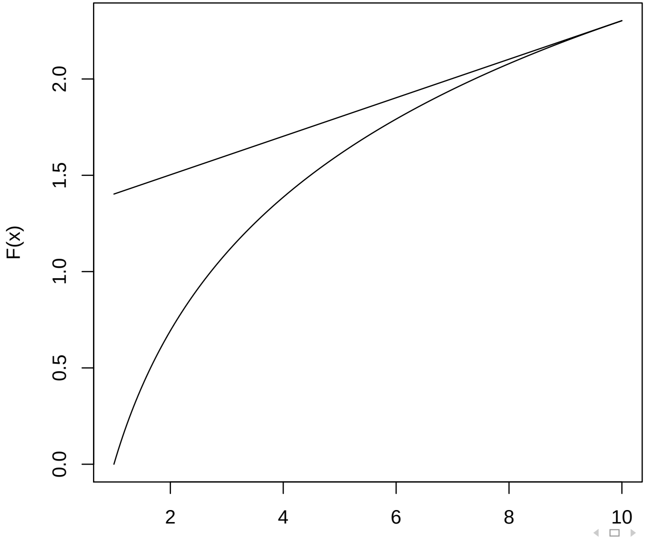

Tangent Line

Formula for Tangent line at x_0:

Tangent Line

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Suppose we have some initial guess x_0. We're going to approximate $f^{'}(x)$ with the tangent line to generate a new guess

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$$
\begin{aligned}
g(x) &= f^{''}(x_0)(x - x_0) + f^{'}(x_0) \\
0 &= f^{''}(x_0)(x_1 - x_0) + f^{'}(x_0)
\end{aligned}
$$

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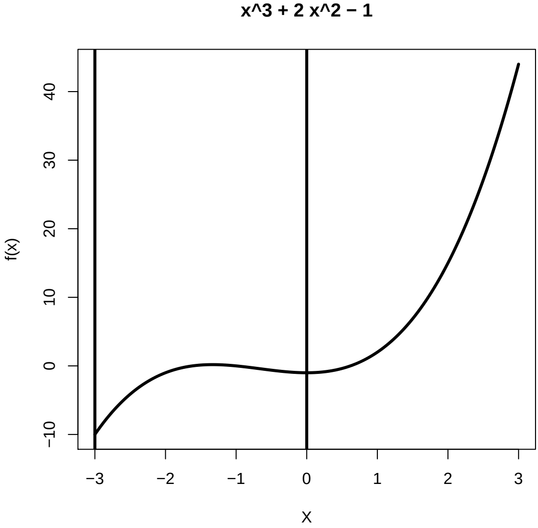
$$\begin{aligned} g(x) &= f^{''}(x_0)(x - x_0) + f^{'}(x_0) \\ 0 &= f^{''}(x_0)(x_1 - x_0) + f^{'}(x_0) \\ x_1 &= x_0 - \frac{f^{'}(x_0)}{f^{''}(x_0)} \end{aligned}$$

Example Function

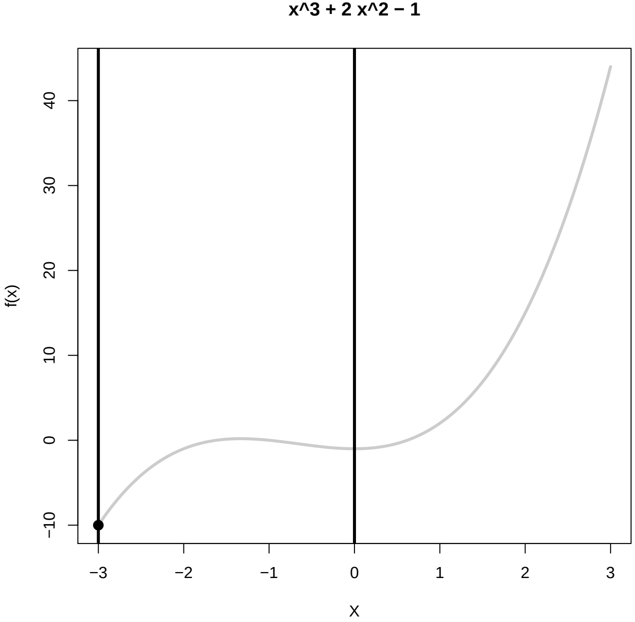
$f(x) = x^3 + 2x^2 - 1$ find x that maximizes $f(x)$ with $x \in [-3, 0]$



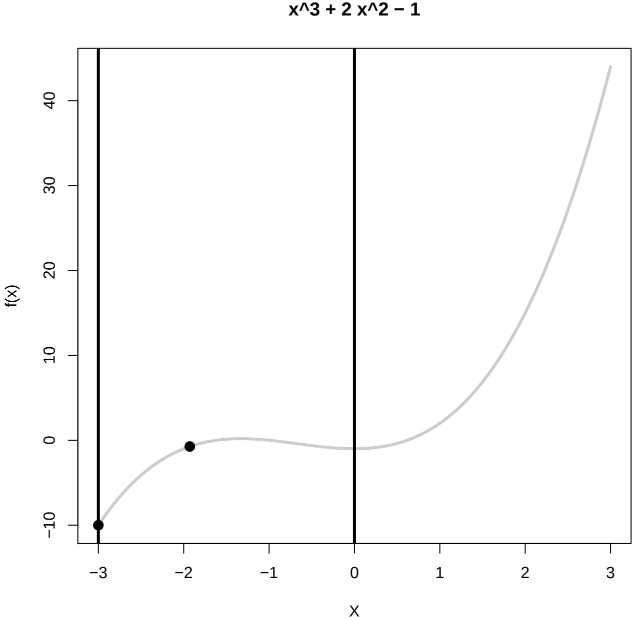
$$
\begin{aligned}
f^{'}(x) &= 3x^2 + 4x \\
f^{''}(x) &= 6x + 4
\end{aligned}
$$

Suppose we have guess x_t then the next step is:

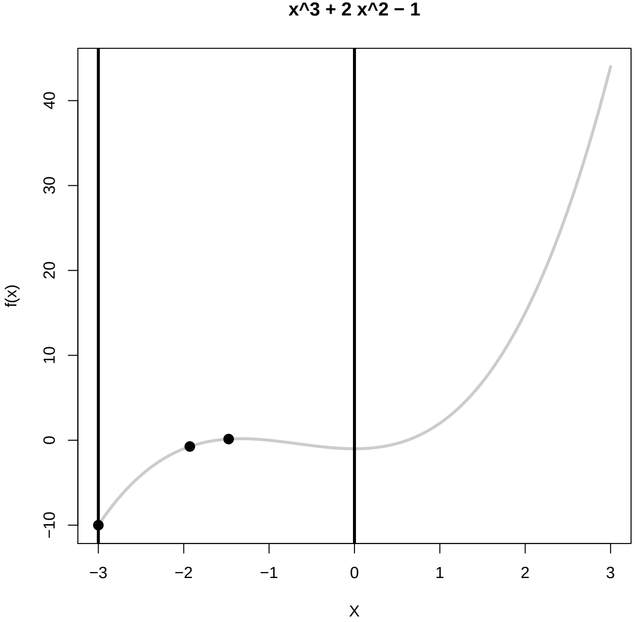
$$
x_{t+1} = x_t - \frac{3x_t^2 + 4x_t}{6x_t + 4}
$$



x^3 + 2 x^2 − 1
f(x)
40
30
20
10
0
−10
−3
−2
−1
0
1
2
3
X



x^3 + 2 x^2 − 1
f(x)
−10
0
10
20
30
40
−3
−2
−1
0
1
2
3
X


x^3 + 2 x^2 − 1
f(x)
40
30
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10
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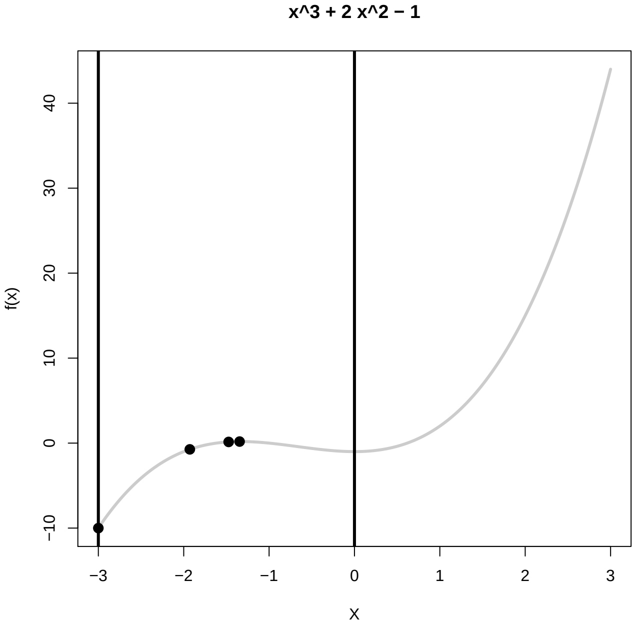
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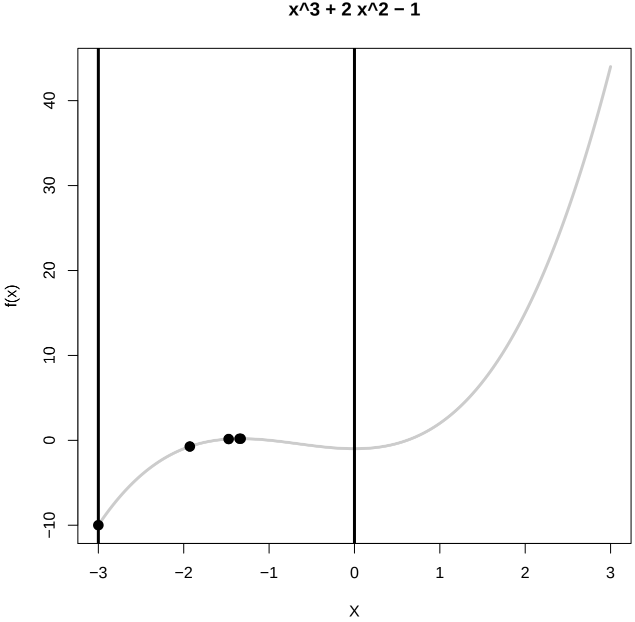
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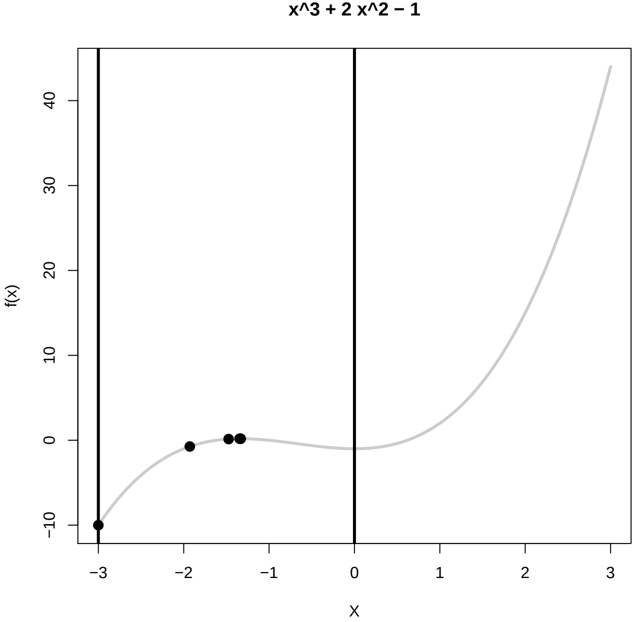
x^3 + 2 x^2 − 1
f(x)
−10
0
10
20
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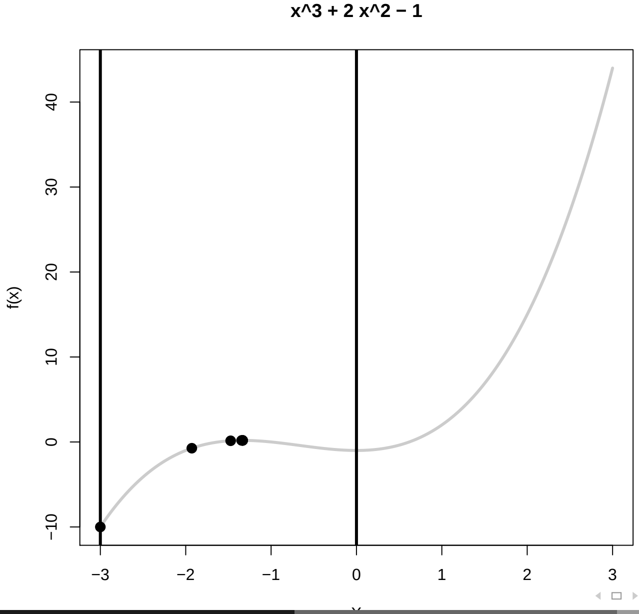
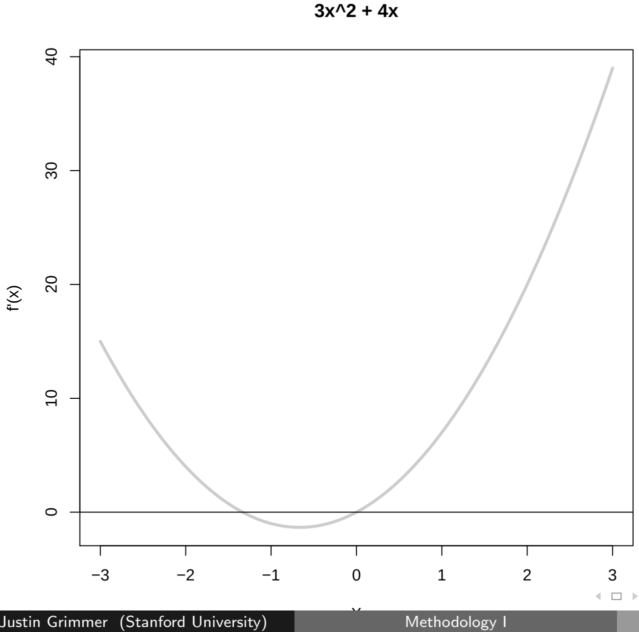
X

$x^* = -1.3333$



What is Happening with the Roots



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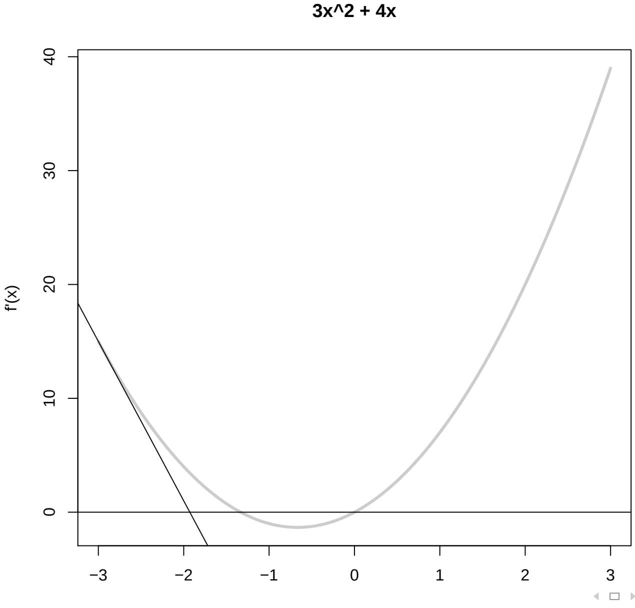

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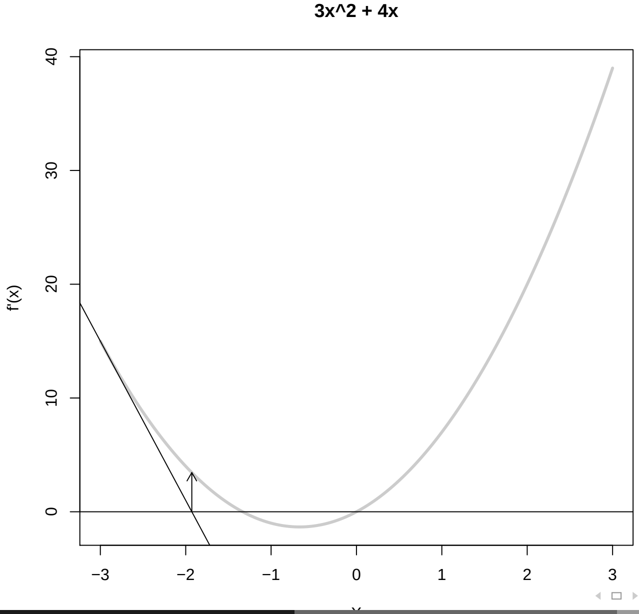

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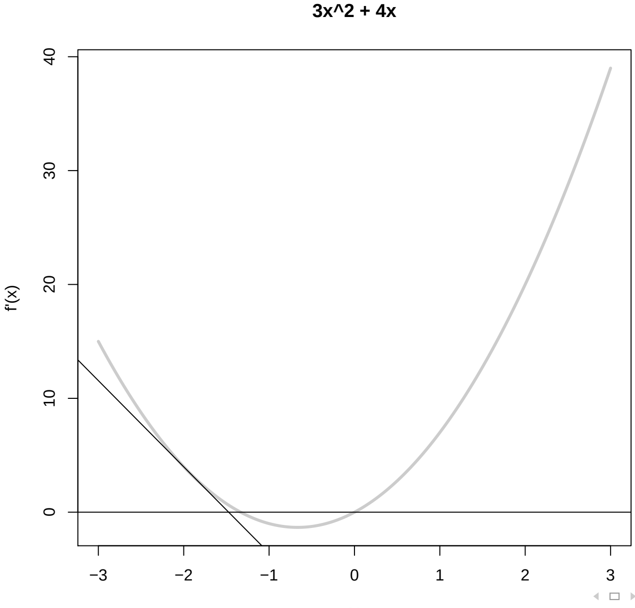

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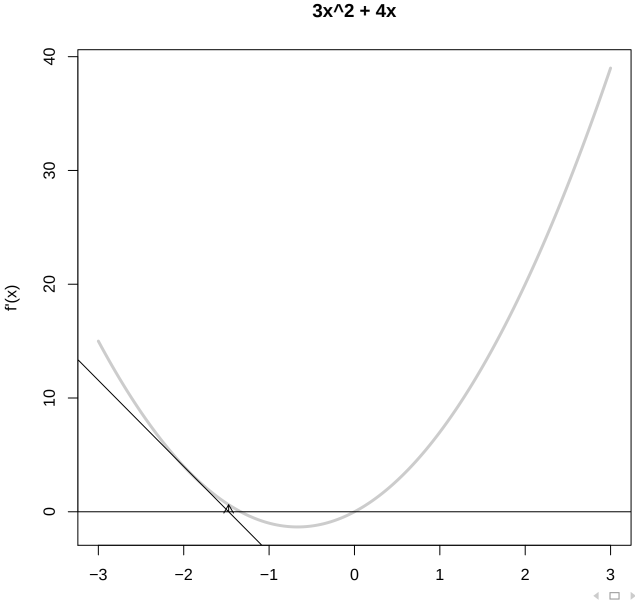

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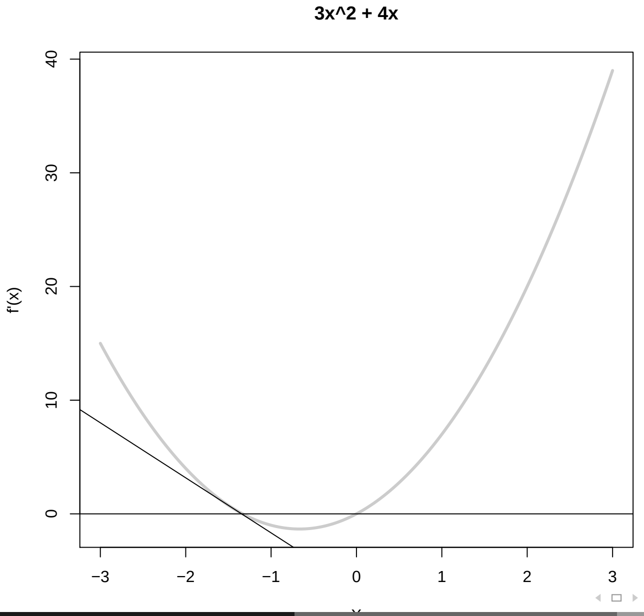

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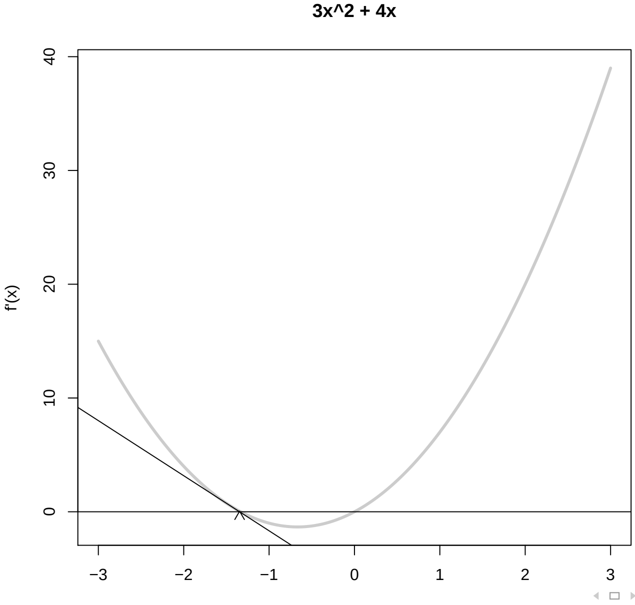

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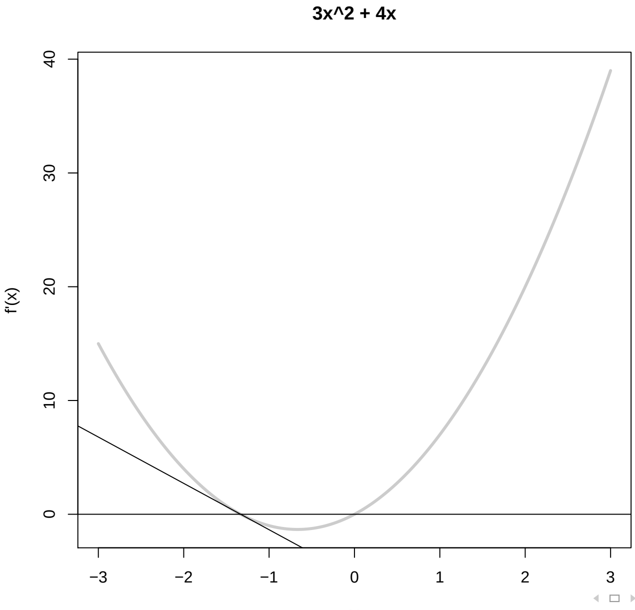

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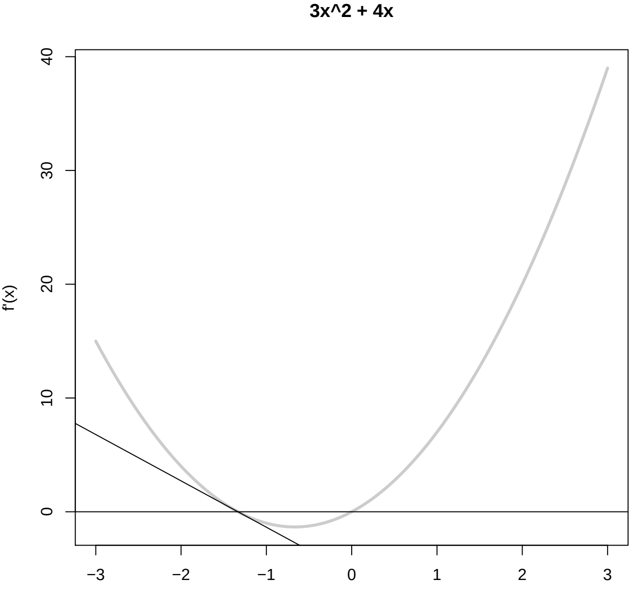

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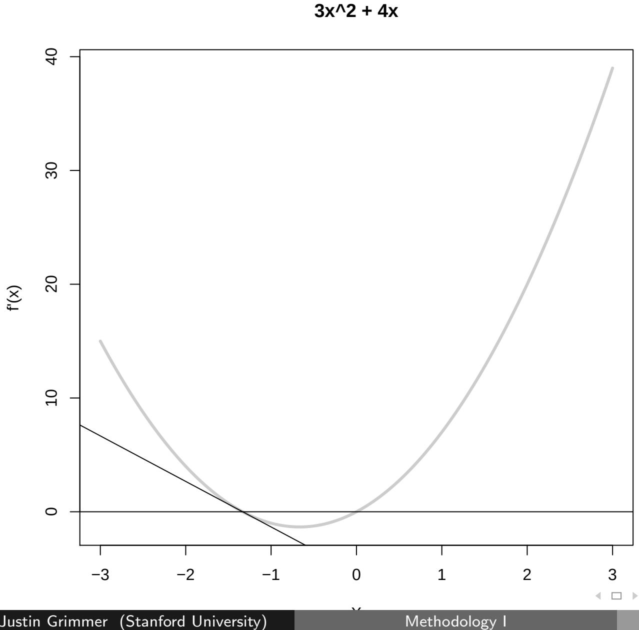

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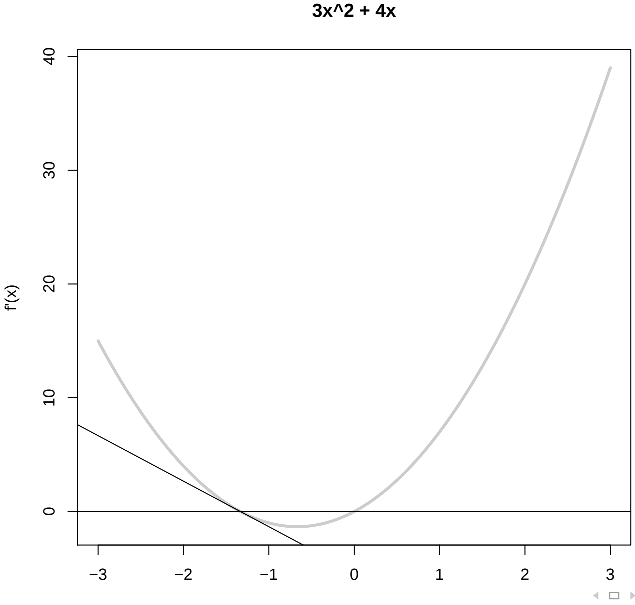

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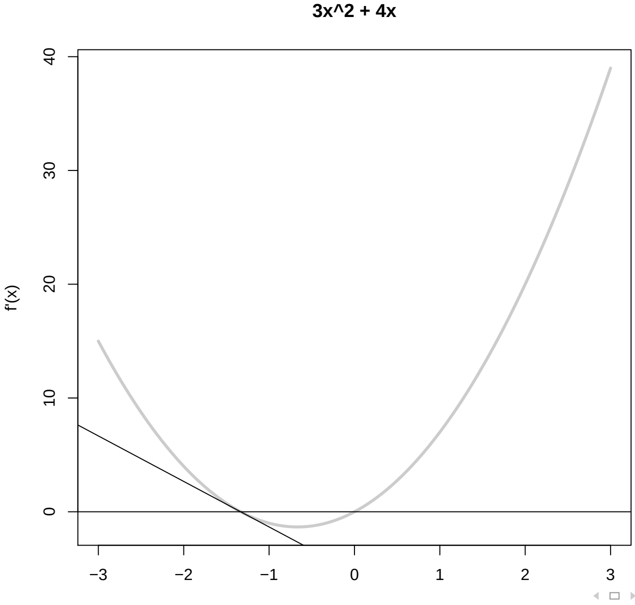

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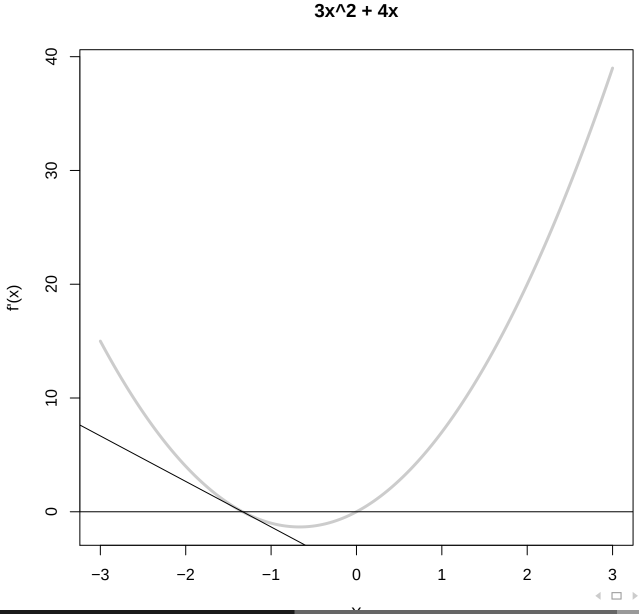

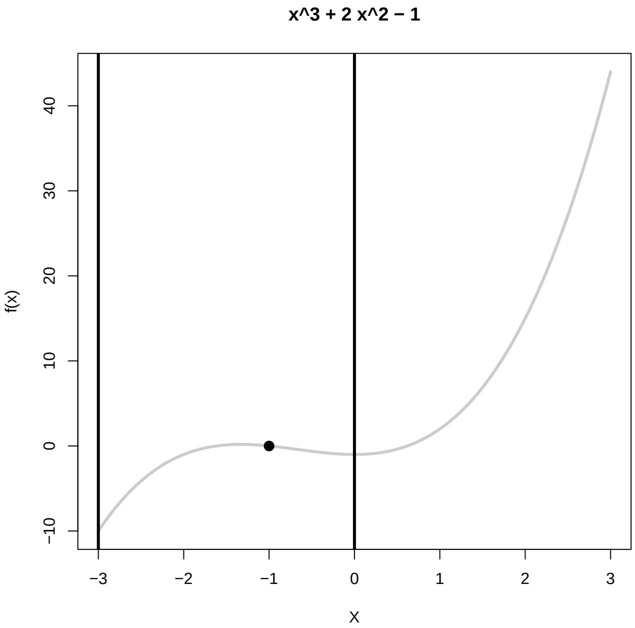
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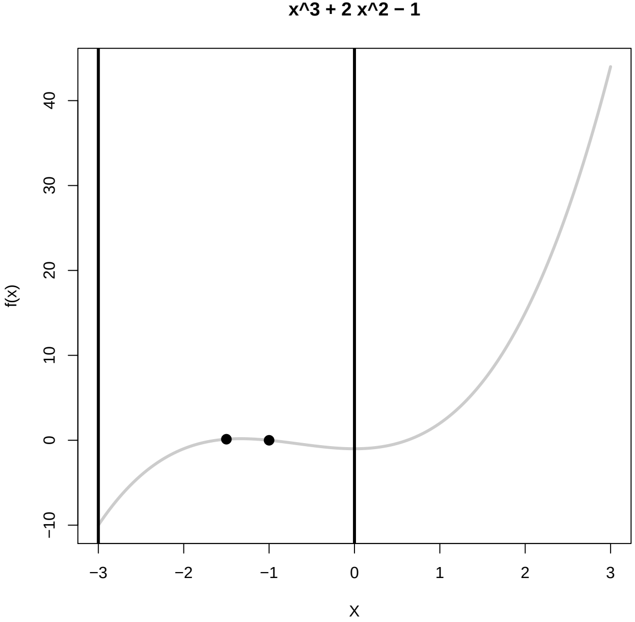
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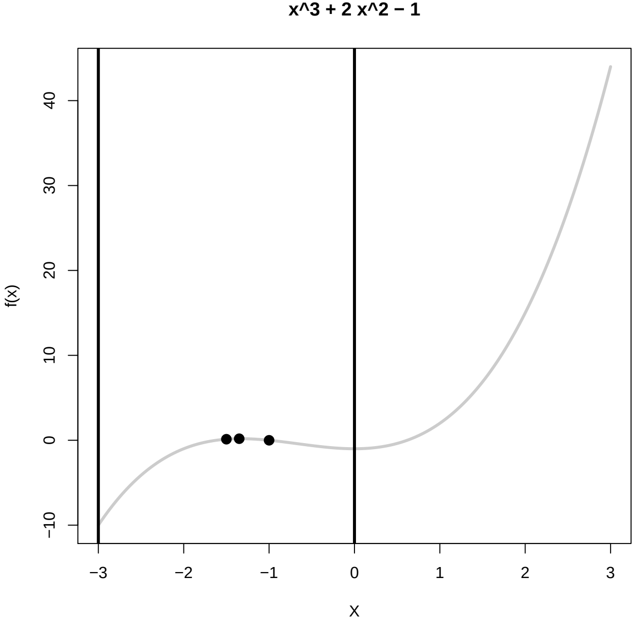

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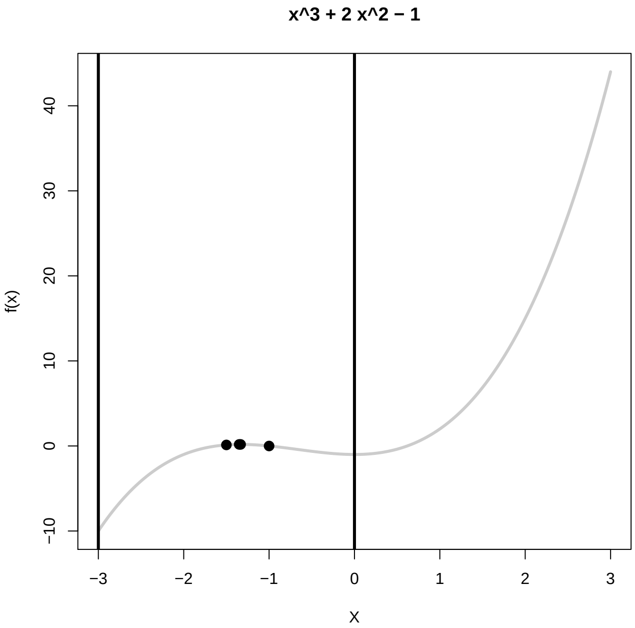
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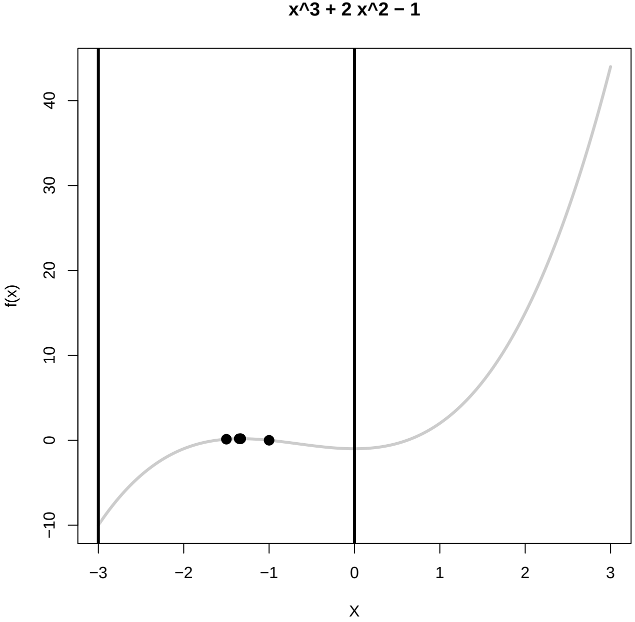
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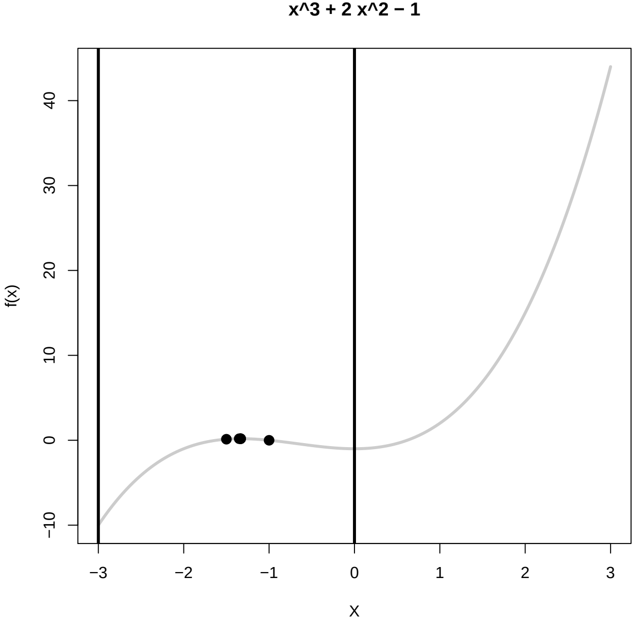
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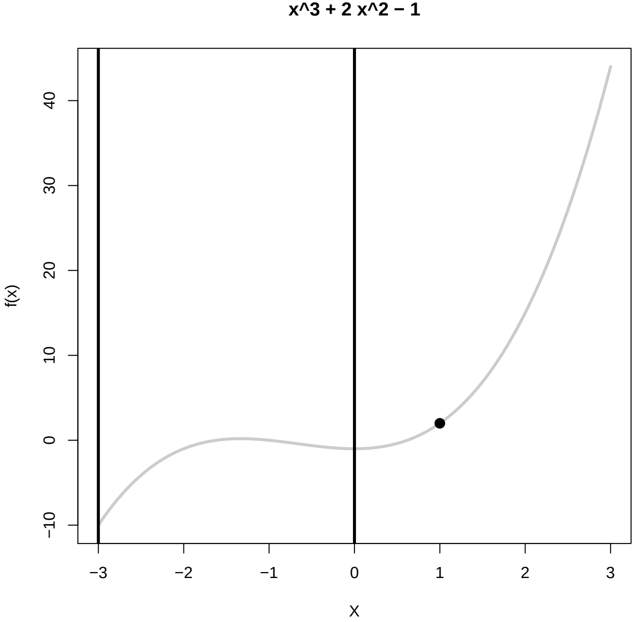
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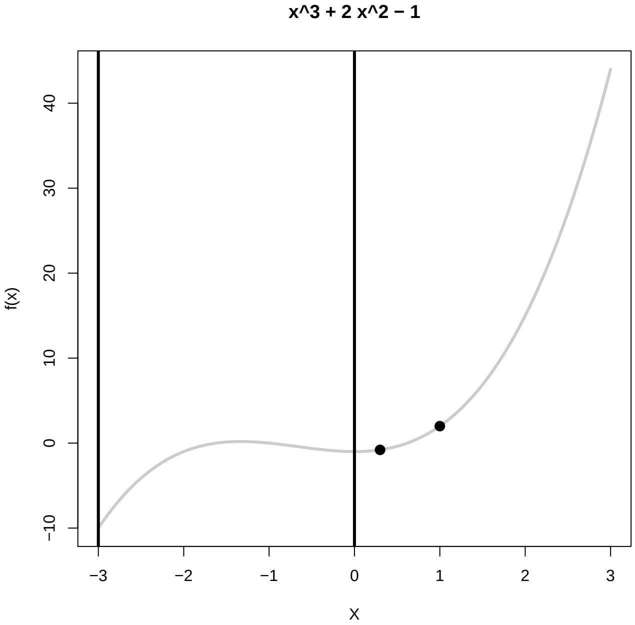
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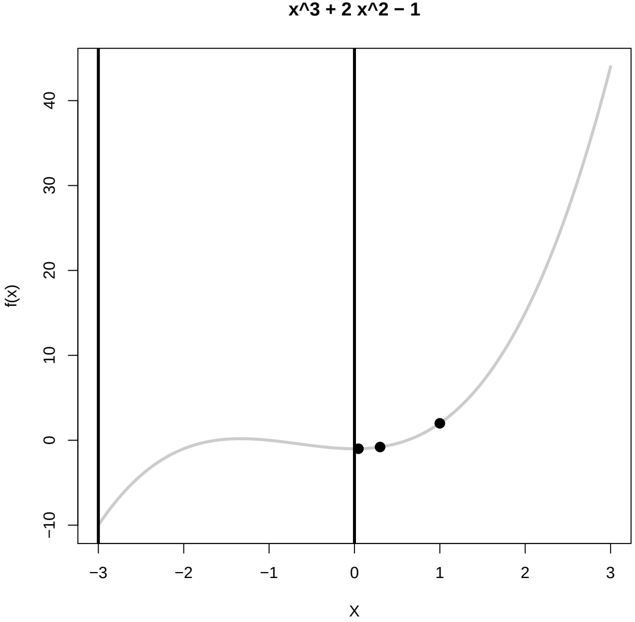
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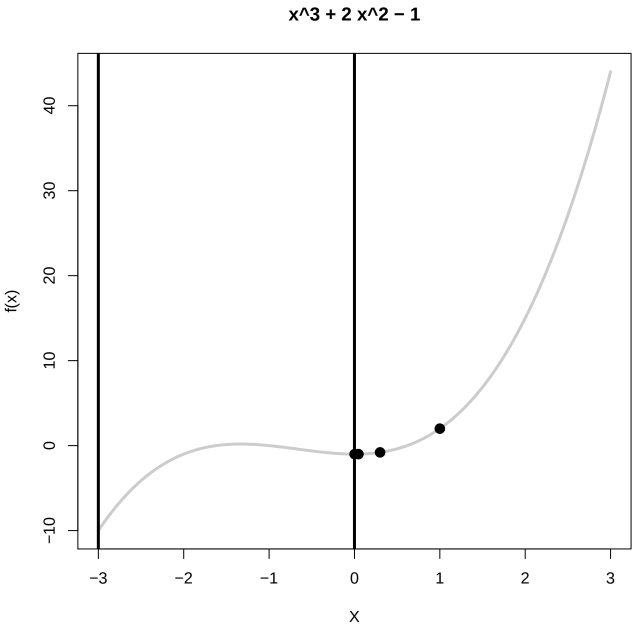
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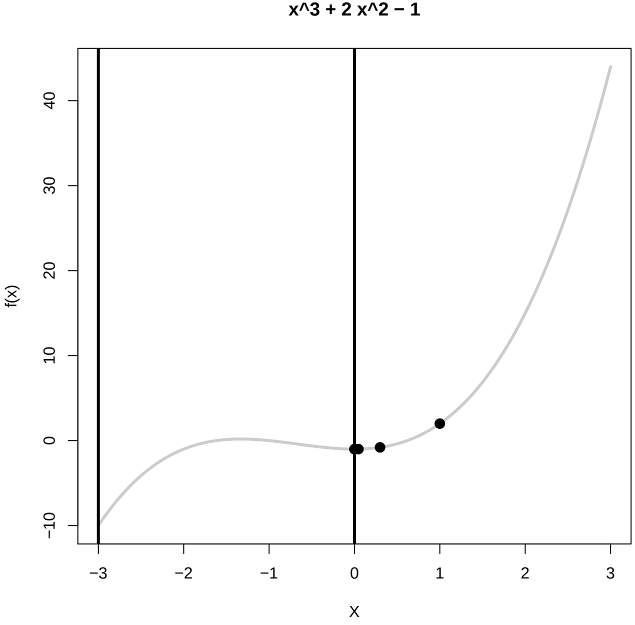
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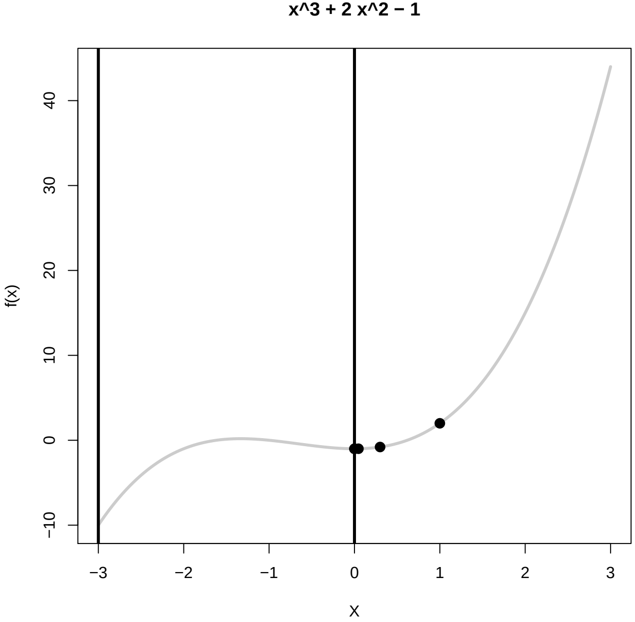
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To the R Code!

Today/Tomorrow

- A Framework for optimization
  - Analytic: pencil and paper math
  - Computational: iterative algorithm that aids in solution
- Integration: antidifferentation/area finding