Math Camp

Justin Grimmer

Associate Professor
Department of Political Science
Stanford University

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Inequalities and Limit Theorems

Limit Theorems

- What happens when we consider a long sequence of random variables ?
- What can we reasonably infer from data?
  - Laws of large numbers: averages of random variables converge on expected value?
  - Central Limit Theorems: sum of random variables have normal distribution?
- We'll focus on intuition for both, but we'll prove some stuff too.

Review Session

Weak Law of Large Numbers

Proof plan:

- Markov's Inequality
- Chebyshev's Inequality
- Weak Law of Large Numbers

Markov's Inequality

Proposition

Suppose X is a random variable that takes on non-negative values. Then, for all $a > 0$,

$$P(X \geq a) \quad \leq \quad \frac{E[X]}{a}$$

Markov's Inequality

Proof.

For $a > 0$,

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For $a > 0$,

$$E[X] = \int_0^{\infty} x f(x) dx$$

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\begin{aligned}
E[X] &= \int_0^{\infty} xf(x)dx \\
&= \int_0^{a} xf(x)dx + \int_a^{\infty} xf(x)dx
\end{aligned}
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Because $X \geq 0$,

$$E[X] \geq \int_a^\infty x f(x) dx \geq \int_a^\infty a f(x) dx = aP(X \geq a)$$

□

Markov's Inequality

Proof.

For $a > 0$,

$$E[X] = \int_0^\infty xf(x)dx$$
$$= \int_0^a xf(x)dx + \int_a^\infty xf(x)dx$$

Because $X \geq 0$,

$$E[X] \geq \int_a^\infty xf(x)dx \geq \int_a^\infty af(x)dx = aP(X \geq a)$$
$$\frac{E[X]}{a} \geq P(X \geq a)$$

□

Chebyshev's Inequality

Proposition

If X is a random variable with mean μ and variance σ^2, then, for any value $k > 0$,

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

Chebyshev's Inequality

Proof.

Define the random variable

$$Y = (X - \mu)^2$$

Where $\mu = E[X]$.

Chebyshev's Inequality

Proof.

Define the random variable

$$Y \quad = \quad (X - \mu)^2$$

Where $\mu = E[X]$.
Then we know Y is a non-negative random variable. Set $a = k^2$.

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$$P(Y \geq k^2) \leq \frac{E[Y]}{k^2}$$

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Applying the inequality:

$$P(Y \geq k^2) \leq \frac{E[Y]}{k^2}$$
$$P((X-\mu)^2 \geq k^2) \leq \frac{E[(X-\mu)^2]}{k^2}$$

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Proof.

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$$Y = (X - \mu)^2$$

Where $\mu = E[X]$.
Then we know Y is a non-negative random variable. Set $a = k^2$.
Applying the inequality:

$$P(Y \geq k^2) \leq \frac{E[Y]}{k^2}$$
$$P((X-\mu)^2 \geq k^2) \leq \frac{E[(X-\mu)^2]}{k^2}$$
$$P((X-\mu)^2 \geq k^2) \leq \frac{\sigma^2}{k^2}$$

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Sequence of Random Variables

Sequence of Independent and Identically, Distributed Random variables.

- Sequence: $X_1, X_2, \ldots, X_n, \ldots$
- Think of a sequence as sampled data:
  - Suppose we are drawing a sample of N observations
  - Each observation will be a random variable, say X_i
  - With realization x_i

Mean/Variance of Sample Mean

Proposition

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Let $\bar{X}_n$ be the sample mean. Then $E[\bar{X}_n] = \mu$ and $var(\bar{X}_n) = \frac{\sigma^2}{n}$

Proof.

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Proposition

Let $X_1, X_2, \ldots, X_n$ *be a random sample from a distribution with mean* μ *and variance* σ^2*. Let* $\bar{X}_n$ *be the sample mean. Then* $E[\bar{X}_n] = \mu$ *and* $var(\bar{X}_n) = \frac{\sigma^2}{n}$

Proof.

$$E[\bar{X}_n] = \frac{1}{n}\sum_{i=1}^{n} E[X_i]$$

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Proof.

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\begin{aligned}
E[\bar{X}_n] &= \frac{1}{n}\sum_{i=1}^{n} E[X_i] \\
&= \frac{1}{n} n\mu = \mu
\end{aligned}
$$

□

Mean/Variance of Sample Mean

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$$\text{var}(\bar{X}_n) = \frac{1}{n^2}\text{var}(\sum_{i=1}^{n} X_i)$$

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$$\begin{aligned} \text{var}(\bar{X}_n) &= \frac{1}{n^2}\text{var}(\sum_{i=1}^{n} X_i) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\text{var}(X_i) \end{aligned}$$

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&= \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}
\end{aligned}
$$

Weak Law of Large Numbers

Proposition

Suppose $X_1, X_2, \ldots, X_n$ *is a random sample from a distribution with mean* μ *and* $Var(X_i) = \sigma^2$*. Then, for all* $\epsilon > 0$,

$$P\left\{\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| \geq \epsilon\right\} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Weak Law of Large Numbers

Proof.

From our previous proposition

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$$\frac{\mathrm{E}[X_1 + X_2 + \cdots + X_n]}{n} \quad = \quad \frac{\sum_{i=1}^{n} E[X_i]}{n} = \mu$$

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Proof.

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$$\frac{\mathrm{E}[X_1 + X_2 + \cdots + X_n]}{n} = \frac{\sum_{i=1}^{n} E[X_i]}{n} = \mu$$

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Weak Law of Large Numbers

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From our previous proposition

$$\frac{\text{E}[X_1 + X_2 + \cdots + X_n]}{n} \quad = \quad \frac{\sum_{i=1}^{n} E[X_i]}{n} = \mu$$

Further,

$$\text{E}[(\frac{\sum_{i=1}^{n} X_i - \mu}{n})^2] = \frac{\text{Var}(X_1 + X_2 + \cdots + X_n)}{n^2} \quad = \quad \frac{\sum_{i=1}^{n} \text{Var}(X_i)}{n^2} = \frac{\sigma^2}{n}$$

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Apply Chebyshev's Inequality:

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Apply Chebyshev's Inequality:

$$P\left\{\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| \geq \epsilon\right\} \leq \frac{\sigma^2}{n\epsilon^2}$$

□

Suppose $X_1, X_2, \ldots$ are iid normal distributions,

$$X_i \sim \text{Normal}(0, 10)$$

$$P\left\{\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| \geq 0.1\right\} \text{ as } n \rightarrow \infty$$

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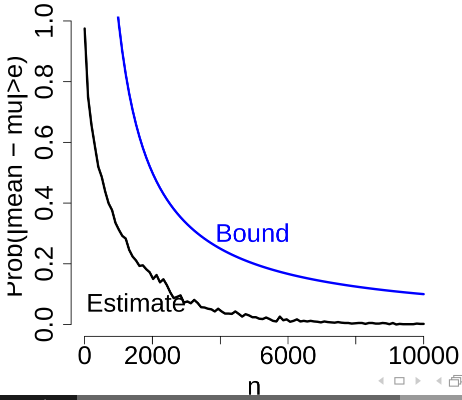

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Suppose we want to guarantee that we have at most a 0.01 probability of being more than 0.1 away from the true μ. How big do we need n?

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$$n = \frac{1000}{0.01}$$

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Suppose we want to guarantee that we have at most a 0.01 probability of being more than 0.1 away from the true μ. How big do we need n?

$$\begin{aligned} 0.01 &= \frac{10}{n(0.1^2)} \\ n &= \frac{1000}{0.01} \\ n &= 100,000 \end{aligned}$$

Sequences and Convergence

Sequence (refresher):

$$\{a_i\}_{i=1}^{\infty} \quad = \quad \{a_1, a_2, a_3, \ldots, a_n, \ldots, \}$$

Definition

We say that the sequence $\{a_i\}_{i=1}^{\infty}$ *converges to real number A if for each* $\epsilon > 0$ *there is a positive integer N such that for* $n \geq N$, $|a_n - A| < \epsilon$

Sequences and Convergence

Sequence of functions:

$$\{f_i\}_{i=1}^{\infty} \quad = \quad \{f_1, f_2, f_3, \ldots, f_n, \ldots, \}$$

Definition

Suppose $f_i : X \rightarrow \Re$ for all i. Then $\{f_i\}_{i=1}^{\infty}$ converges pointwise to f if, for all $x \in X$ and $\epsilon > 0$, there is an N such that for all $n \geq N$,

$$|f_n(x) - f(x)| < \epsilon$$

This is as strong of a statement as we're likely to make in statistics

Convergence Definitions

Define $\widehat{\theta}_n$ to be estimator for θ based on n observations.

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Sequence of estimators: increasing sample size

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Question: What can we say about $\left\{\widehat{\theta}_i\right\}_{i=1}^{n}$ as $n \rightarrow \infty$?

- What is the probability $\widehat{\theta}_n$ differs from θ?
- What is the probability $\left\{\widehat{\theta}_i\right\}_{i=1}^{n}$ converges to θ?
- What is sampling distribution of $\widehat{\theta}_n$ as $n \rightarrow \infty$?

Convergence in Probability

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Definition

We will say the sequence $\widehat{\theta}_n$ converges in probability to θ (perhaps a non-degenerate RV) if,

$$\lim_{n \to \infty} Prob(|\widehat{\theta}_n - \theta| > \epsilon) = 0$$

For any $\epsilon > 0$

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- ϵ is a tolerance parameter: how much error around θ?
- In the limit, convergence in probability implies sampling distribution collapses on a spike at θ
- $\left\{\widehat{\theta}_i\right\}$ need not actually converge to θ, only P($|\theta_n$ - $\theta| > \epsilon) = 0$

Example (Cassella and Burger)

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$$X_1(s) = s + I(s \in [0, 1]) \quad , \quad X_2(s) = s + I(s \in [0, 1/2])$$

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Does $X_n(s)$ pointwise converge to $X(s)$?
Does $X_n(s)$ converge in probability to $X(s)$?

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Does $X_n(s)$ pointwise converge to $X(s)$?
Does $X_n(s)$ converge in probability to $X(s)$?

$$P(|X_n - X| > \epsilon) \quad = \quad P(s \in [l_n, u_n])$$

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Does $X_n(s)$ converge in probability to $X(s)$?

$$P(|X_n - X| > \epsilon) \quad = \quad P(s \in [l_n, u_n])$$

Length of $[l_n, u_n] \rightarrow 0 \Rightarrow P(s \in [L_n, U_n]) = 0$

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- Almost sure says that, for all outcomes (s) in sample space (S) $s \in S$,

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Except for a subset $\mathcal{N} \subset S$ such that $P(\mathcal{N}) = 0$.

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No!: the sequence doesn't converge for each s
For each value of s the sequence varies between s and $s+1$ infinitely often

Convergence in Distribution

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- Weakest form of convergence almost sure $\rightarrow$ probability $\rightarrow$ distribution
- Says that cdfs are equal, says nothing about convergence of underlying RV
- Useful for justifying use of some sampling distributions

Convergence in Distribution $\not\Rightarrow$ Convergence in Probability

Define $X \sim N(0,1)$ and each $X_n = -X$. Then:
$X_n \sim N(0,1)$ for all n so X_n trivially converges to X. But,

$$\begin{aligned} P(|X_n - X| > \epsilon) &= P(|X + X| > \epsilon) \\ &= P(|2X| > \epsilon) \\ &= P(|X| > \epsilon/2) \not\rightsquigarrow 0 \end{aligned}$$

Central Limit Theorem

Proposition

Let X_1, $X_2, \ldots$ be a sequence of independent random variables with mean μ and variance σ^2. Let X_i have a cdf $P(X_i \leq x) = F(x)$ and moment generating function $M(t) = E[e^{tX_i}]$. Let $S_n = \sum_{i=1}^{n} X_i$. Then

$$\lim_{n\to\infty} P\left(\frac{S_n - \mu n}{\sigma\sqrt{n}} \leq x\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\frac{z^2}{2}\right) dz$$

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Proof plan:

1) Rely on Fact that convergence of MGFs $\rightsquigarrow$ convergence in CDFs
2) Show that MGFs, in limit, converge on normal MGF

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Let F_n be a sequence of cumulative distribution functions with the corresponding moment generating functions M_n. F be a cdf with the moment generating functions M. If $\lim_{n\rightarrow\infty} M_n(t) \rightarrow M(t)$ for all t in some interval, then $F_n(x) \rightsquigarrow F(x)$ for all x (when F is continuous).

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Proposition

Suppose $M(t)$ is a moment generating function some random variable X. Then $M(0) = 1$.

Proof of Central Limit Theorem (Courtsey of Swarthmore Notes)

Proof. Suppose $X_1, \ldots, X_n$ are iid variables with $E[X] = 0$, variance σ_x^2, Moment Generating Function (MGF) $M_x(t)$.

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$M_{S_n} = (M_x(t))^n$ and $M_{Z_n}(t) = \left(M_x\left(\frac{t}{\sigma_x\sqrt{n}}\right)\right)^n$

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Using Taylor's Theorem we can write

$$M_x(s) \;\; = \;\; M_x(0) + sM_x^{'}(0) + \frac{1}{2}s^2 M_x^{''}(0) + e_s$$

$e_s/s^2 \rightarrow 0$ as $s \rightarrow 0$.

$$M_x(s) = M_x(0) + sM_x^{'}(0) + \frac{1}{2}s^2 M_x^{''}(0) + e_s$$

Filling in the values we have

$$M_x(s) = 1 + 0 + \frac{\sigma_x^2}{2}s^2 + \underbrace{e_s}_{\text{Goes to zero}}$$

Set $s = \frac{t}{\sigma_x \sqrt{n}}$ $\lim_{n\rightarrow\infty} s \rightarrow 0$. Then

$$\begin{aligned} M_{Z_n}(t) &= \left(1 + \frac{\sigma_x^2}{2}\left(\frac{t}{\sigma_x\sqrt{n}}\right)^2\right)^n \\ &= \left(1 + \frac{t^2/2}{n}\right)^n \\ \lim_{n\rightarrow\infty} M_{Z_n}(t) &= e^{\frac{t^2}{2}} \end{aligned}$$

Review Time