Math Camp

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September 21st, 2016

Questions?
(Dose response curve and conditional density functions)

Define following terms:

- Suppose $f : \Re \rightarrow \Re$. Provide the definition of a continuous function f
- Suppose $f : \Re \rightarrow \Re$. Define the derivative of function f at x_0.
- Convergence of a sequence $\{a_n\}_{n=1}^{\infty}$
- Suppose $f : \Re^2 \rightarrow \Re$, $f(x_1, x_2)$. Define $\nabla f(\boldsymbol{x}_0)$ where $\boldsymbol{x}_0 = (x_{01}, x_{02})$.

Where We've Been, Where We're Going

Finishing Up Yesterday:

5) The Multivariate Normal Distribution and You

Today:

1) Properties of Expectations
2) Changing Coordinates
3) Moment Generating Functions

Definition

Suppose $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$ *is a vector of random variables. If* $\boldsymbol{X}$ *has pdf*

$$f(\boldsymbol{x}) \quad = \quad (2\pi)^{-N/2} det(\boldsymbol{\Sigma})^{-1/2} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\right)$$

Then we will say $\boldsymbol{X}$ *is a Multivariate Normal Distribution,*

$$\boldsymbol{X} \quad \sim \quad \textit{Multivariate Normal}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

- Regularly used for likelihood, Bayesian, and other parametric inferences

Properties of the Multivariate Normal Distribution

Suppose $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$
$X \sim$ Multivariate Normal$(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$\begin{aligned} E[\boldsymbol{X}] &= \boldsymbol{\mu} \\ \text{cov}(\boldsymbol{X}) &= \boldsymbol{\Sigma} \end{aligned}$$

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So that,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \text{var}(X_1) & \text{cov}(X_1, X_2) & \ldots & \text{cov}(X_1, X_N) \\ \text{cov}(X_2, X_1) & \text{var}(X_2) & \ldots & \text{cov}(X_2, X_N) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(X_N, X_1) & \text{cov}(X_N, X_2) & \ldots & \text{var}(X_N) \end{pmatrix}$$

Multivariate Normal Distribution

Consider the (bivariate) special case where $\boldsymbol{\mu} = (0, 0)$ and

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$$f(x_1, x_2) = (2\pi)^{-2/2} 1^{-1/2} \exp\left(-\frac{1}{2}\left((\boldsymbol{x} - \mathbf{0})' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\boldsymbol{x} - \mathbf{0})\right)\right)$$

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$\rightsquigarrow$ product of univariate standard normally distributed random variables

Standard Multivariate Normal

Definition

Suppose $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_N)$ *is*

$$\boldsymbol{Z} \sim \textit{Multivariate Normal}(\mathbf{0}, \boldsymbol{I}_N).$$

Then we will call $\boldsymbol{Z}$ *the standard multivariate normal.*

Independence and Multivariate Normal

Proposition

Suppose X and Y are independent. Then

$$cov(X, Y) = 0$$

Proof.

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$$\text{cov}(X, Y) = E[XY] - E[X]E[Y]$$

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Calculating $E[XY]$

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$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y)dxdy$$

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$$\begin{aligned} E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy \end{aligned}$$

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&= E[X]E[Y]
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Then $\text{cov}(X, Y) = 0$.

□

- More generally if X and Y are independent, $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$ for functions $g : \Re \rightarrow \Re$ and $h : \Re \rightarrow \Re$.

Zero covariance does not generally imply Independent

Suppose $X \in \{-1, 1\}$ with $P(X = 1) = P(X = -1) = 1/2$.
Suppose $Y \in \{-1, 0, 1\}$ with $Y = 0$ if $X = -1$ and
$P(Y = 1) = P(Y = -1)$ if $X = 1$.

$$
\begin{aligned}
E[XY] &= \sum_{i \in \{-1,1\}} \sum_{j \in \{-1,0,1\}} ij P(X = i, Y = j) \\
&= -1 \times 0 \times P(X = -1, Y = 0) + 1 \times 1 \times P(X = 1, Y = 1) \\
&\quad -1 \times 1 \times P(X = 1, Y = -1) \\
&= 0 + P(X = 1, Y = 1) - P(X = 1, Y = -1) \\
&= 0.25 - 0.25 = 0 \\
E[X] &= 0 \\
E[Y] &= 0
\end{aligned}
$$

Proposition

Suppose $\boldsymbol{X} \sim$ *Multivariate Normal*$(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. *where* $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$.
If $cov(X_i, X_j) = 0$, *then* X_i *and* X_j *are independent*

Iterated Expectations

Proposition

Suppose X and Y are random variables. Then

$$E[X] = E[E[X|Y]]$$

- Inner Expectation is $E[X|Y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$.
- Outer expectation is over y.

Iterated Expectations

Proof.

□

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$$E[E[X|Y]] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X|Y}(x|y) f_Y(y) dx dy$$

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Proof.

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□

Iterated Expectations

Definition

Suppose Y is a continuous random variable with $Y \in [0,1]$ and pdf of Y given by

$$f(y) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y^{\alpha_1 - 1}(1-y)^{\alpha_2 - 1}$$

Then we will say Y is a Beta distribution with parameters α_1 and α_2. Equivalently,

$$Y \sim Beta(\alpha_1, \alpha_2)$$

- Beta is a distribution on proportions
- Beta is a special case of the Dirichlet distribution
- $E[Y] = \frac{\alpha_1}{\alpha_1 + \alpha_2}$

Iterated Expectations

Suppose

$$\begin{aligned} \pi &\sim \text{Beta}(\alpha_1, \alpha_2) \\ Y|\pi, n &\sim \text{Binomial}(n, \pi) \end{aligned}$$

What is $E[Y]$?

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$$E[Y] = E[E[Y|\pi]]$$

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$$\begin{aligned} E[Y] &= E[E[Y|\pi]] \\ &= \int_{-\infty}^{\infty} \sum_{j=0}^{N} \binom{N}{j} j p(j|\pi) f(\pi) d\pi \end{aligned}$$

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Change of Coordinates

Proposition

Suppose X is a random variable and $Y = g(X)$, where $g : \Re \rightarrow \Re$ that is a monotonic function.
Define $g^{-1} : \Re \rightarrow \Re$ such that $g^{-1}(g(X)) = X$ and is differentiable. Then,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{\partial g^{-1}(y)}{\partial y} \right| \textit{ if } y = g(x) \textit{ for some } x \\ &= 0 \textit{ otherwise} \end{aligned}$$

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Suppose $g(\cdot)$ is monotonically increasing (WLOG)

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Now differentiating to get the pdf

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$$\begin{aligned} \frac{\partial F_Y(y)}{\partial y} &= \frac{\partial F_X(g^{-1}(y))}{\partial y} \\ &= f_X(g^{-1}(y)) \frac{\partial g^{-1}(y)}{\partial y} \end{aligned}$$

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Now differentiating to get the pdf

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\frac{\partial F_Y(y)}{\partial y} &= \frac{\partial F_X(g^{-1}(y))}{\partial y} \\
&= f_X(g^{-1}(y)) \frac{\partial g^{-1}(y)}{\partial y}
\end{aligned}$$

Then this is a pdf because $\frac{\partial g^{-1}(Y)}{\partial y} > 0$.

□

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We've used this to derive many of the pdfs

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$$

We've used this to derive many of the pdfs

- Normal distribution
- Chi-Squared Distribution

Moment Generating Functions

Definition

Suppose X is a random variable with pdf f. Define,

$$E[X^n] = \int_{-\infty}^{\infty} x^n f(x) dx$$

We will call X^n the n^{th} moment of X

- By this definition $var(X) =$ Second Moment $-$ First Moment^2
- We are assuming that the integral converges

Moment Generating Functions

Proposition

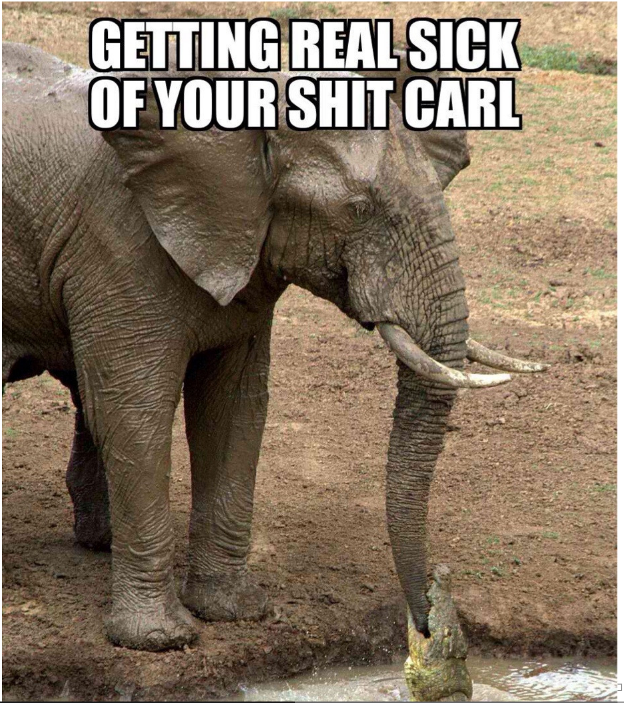
Suppose X is a random variable with pdf $f(x)$. Call $M(t) = E[e^{tX}]$,

$$\begin{aligned} M(t) &= E[e^{tX}] \\ &= \int_{-\infty}^{\infty} e^{tx} f(x) dx \end{aligned}$$

We will call $M(t)$ the moment generating function, because:

$$\frac{\partial^n M(t)}{\partial^n t}|_0 = E[X^n]$$

(Assuming that we can interchange derivative and integral)


GETTING REAL SICK
OF YOUR SHIT CARL

Moment Generating Functions

Proof.

Recall the Taylor Expansion of e^{tX} at 0,

Moment Generating Functions

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$$e^{tX} = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \ldots$$

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Then,

$$E[e^{tX}] = 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \ldots$$

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Differentiate once:

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Then,

$$E[e^{tX}] = 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \ldots$$

Differentiate once:

$$\frac{\partial M(t)}{\partial t} = 0 + E[X] + \frac{2t}{2!}E[X^2] + \ldots$$

$$M^{'}(0) = 0 + E[X] + 0 + 0 \ldots$$

□

Proof.

Differentiate n times

□

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Differentiate n times

$$\frac{\partial^n M(t)}{\partial^n t} = 0 + 0 + 0 + \ldots + \frac{n \times n-1 \times \ldots 2 \times t^0 E[X^n]}{n!} + \frac{n! t E[X^{n+1}]}{(n+1)!} + \ldots$$

Proof.

Differentiate n times

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$$= \frac{n! E[X^n]}{n!} + \frac{n! t E[X^{n+1}]}{(n+1)!} + \ldots$$

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Evaluated at 0, yields $M^n(0) = E[X^n]$

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- If two random variables, X and Y have the same moment generating functions, then $F_X(x) = F_Y(y)$ for almost all x.

The Moments of the Normal Distribution

Suppose $Z \sim N(0, 1)$.

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$tx - \frac{1}{2}x^2 = -\frac{1}{2}\left((x-t)^2 - t^2\right)$

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$$E[e^{tX}] = \frac{1}{\sqrt{2\pi}} e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2} dx$$

The Moments of the Normal Distribution

Suppose $Z \sim N(0,1)$.

$$E[e^{tX}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2} dx$$

$tx - \frac{1}{2}x^2 = -\frac{1}{2}\left((x-t)^2 - t^2\right)$

$$\begin{aligned} E[e^{tX}] &= \frac{1}{\sqrt{2\pi}} e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2} dx \\ &= e^{\frac{t^2}{2}} \end{aligned}$$

Extracting Moments of the Normal Distribution

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$$M^{'}(0) = E[X] = e^{t^2/2}t|_0 = 0$$

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M^{'}(0) &= E[X] = e^{t^2/2}t|_0 = 0 \\
M^{''}(0) &= E[X^2] = e^{t^2/2}(t^2+1)|_0 = 1
\end{aligned}
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M^{'''}(0) &= E[X^3] = e^{t^2/2}t(t^2+3)|_0 = 0
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M^{''''}(0) &= E[X^4] = e^{t^2/2}(t^4+6t^2+3)|_0 = 3
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M^{5}(0) &= E[X^5] = e^{t^2/2}t(t^4+10t^2+15)|_0 = 0
\end{aligned}
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M^5(0) &= E[X^5] = e^{t^2/2} t(t^4+10t^2+15)|_0 = 0 \\
M^6(0) &= E[X^6] = e^{t^2/2}(t^6+15t^4+45t^2+15)|_0 = 15
\end{aligned}
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Extracting Moments of the Normal Distribution

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\end{aligned}
$$

Proposition

Suppose X_i are a sequence of independent random variables. Define

$$Y = \sum_{i=1}^{N} X_i$$

Then

$$M_Y(t) = \prod_{i=1}^{N} M_{X_i}(t)$$

Proof.

$$M_Y(t) = E[e^{tY}]$$

□

Proof.

$$
\begin{aligned}
M_Y(t) &= E[e^{tY}] \\
&= E[e^{t\sum_{i=1}^{N} X_i}]
\end{aligned}
$$

□

Proof.

$$\begin{aligned} M_Y(t) &= E[e^{tY}] \\ &= E[e^{t\sum_{i=1}^{N} X_i}] \\ &= E[e^{tX_1 + tX_2 + \ldots tX_N}] \end{aligned}$$

Proof.

$$\begin{aligned} M_Y(t) &= E[e^{tY}] \\ &= E[e^{t\sum_{i=1}^{N} X_i}] \\ &= E[e^{tX_1 + tX_2 + \ldots tX_N}] \\ &= E[e^{tX_1}]E[e^{tX_2}]\ldots E[e^{tX_N}] \text{ (by independence)} \end{aligned}$$

Proof.

$$
\begin{aligned}
M_Y(t) &= E[e^{tY}] \\
&= E[e^{t\sum_{i=1}^{N} X_i}] \\
&= E[e^{tX_1 + tX_2 + \ldots tX_N}] \\
&= E[e^{tX_1}]E[e^{tX_2}]\ldots E[e^{tX_N}] \text{ (by independence)} \\
&= \prod_{i=1}^{N} E[e^{tX_i}]
\end{aligned}
$$

□

Tomorrow:
Sequences of Random Variables