Mathematics Department Stanford University
Math 61DM Practice Midterm Exam I

Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages

Note: work sheets (the final two pages here) are provided for your convenience, but will not be graded

Q.1	
Q.2	
Q.3	
Q.4	
Q.5	
Total	

Name (Print Clearly): ________________________________

I understand and accept the provisions of the honor code (Signed) ________________________________

1 Let $\square$ denote the 1×1 square and Γ be the 2×2 square with one of the four 1×1 squares removed.

(a): Let $f(n)$ be the number of ways to tile a $2 \times n$ rectangle by Γ. Determine and prove an explicit formula for $f(n)$.

(b): Let $g(n)$ be the number of ways to tile a $2 \times n$ rectangle by Γ and $\square$.

(i) Determine a recursive formula for $g(n)$.

(iii) Explain why there are constants a, b, c such that $g(n) = a\alpha^n + b\beta^n + c\gamma^n$ for all n, where α, β, γ are the solutions to $x^3 - x^2 - 4x - 2 = 0$.

2 Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{pmatrix}$$

be a matrix with entried in $\mathbb{R}$. Using Gaussian elimination, find a basis for the null space $N(A)$. What is the dimension of the column space $C(A)$? (Explain your answer.) Describe how you could find a basis for $C(A)$.

3 Let $v = (a_1, a_2, a_3)^T$ and $w = (b_1, b_2, b_3)^T$ be two column vectors in $\mathbb{R}^3$ (written with respect to the standard basis), and define a function $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(x) = (x \cdot w)v - (x \cdot v)w.$$

(Here $x \cdot w$ is the dot product of x with w.) Explain what it means for T to be a linear transformation, and show that it actually is one. Then calculate the matrix A associated to T with respect to the standard basis e_1, e_2, e_3.

4 (a) What does the oddtown theorem say? Why is it true?

(b) Suppose a town with n citizens has m clubs $C_1, \ldots, C_m$ of one type, and m clubs $D_1, \ldots, D_m$ of another type such that

- for each i, C_i and D_i have an odd number of members in common, and
- if $i > j$, then clubs C_i and D_j have an even number of common members.

Prove that $m \leq n$.

5 Suppose that V is a k-dimensional vector space, and suppose that $w_1, \ldots, w_\ell$ are a set of linearly independent vectors in V. Give a detailed proof that $\ell \leq k$.

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