Mathematics Department Stanford University
Math 61DM Homework 4

Due at TA section, Friday Oct. 19

Reminder: Midterm 1, Tue Oct 16, 7pm

1. Find a basis for the nullspace and the column space of the linear transformation described by each of the following matrices:

   (i) $A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & 2 & 4 \end{pmatrix}$

   (ii) $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$

2. Let U, V and W be finite dimensional vector spaces over a field F and suppose that $S\colon U \to V$ and $T\colon V \to W$ are linear transformations. Prove that $\dim N(T \circ S) \leq \dim N(T) + \dim N(S)$.

3. Let S and T be linear transformations on a finite dimensional vector space V over F. Suppose that $T \circ S = \mathrm{id}$. Prove that $S \circ T = \mathrm{id}$.

4. Let V and W be 6-dimensional subspaces of $\mathbb{R}^{10}$. Prove that $V \cap W$ contains more than just the zero vector.

5. Let $V = \{a_0 + a_1X + a_2X^2 + a_3X^3 \colon a_i \in \mathbb{R}\}$ be the vector space of real polynomials of degree at most 3, and consider the linear transformation $T\colon V \to V$, sending a polynomial $p \in V$ to the polynomial $T(p)$ defined by $(T(p))(X) = p(X+1)$. So e.g. $T(1+3X) = 1+3(X+1) = 4+3X$. [You need not write down a proof that this is a linear transformation, although you could check for yourself why this is true.]

   Compute the matrix for T with respect to the basis $\{1, X, X^2, X^3\}$ for V.

6. Let G be a graph with $n = |V|$ vertices and $m = |E|$ edges, and let $A = A(G)$ denote its adjacency matrix. For a graph $G = (V, E)$ and a vertex $x \in V$, the *degree* of x is the number of neighbours it has, $d(x) = \#\{y \in V \colon xy \in E\}$. For a matrix B, its *grand sum* $\mathrm{su}(B)$ is defined as the sum of the entries of B.

   (i) Prove that $\mathrm{su}(A) = \sum_{x \in V} d(x) = 2m$.

   (ii) Prove that $\mathrm{su}(A^2) = \sum_{x \in V} d(x)^2$.

   (iii) Give a combinatorial interpretation of the quantity $\mathrm{su}(A^3)$ (i.e. relate this to counting some kind of object in G).

   (iv) Let δ denote the minimum degree and Δ denote the maximum degree of the vertices of G. Show that for any positive integer k,

   $$n\delta^k \leq \mathrm{su}(A^k) \leq n\Delta^k\,.$$

7. Show that a code C corrects t errors if and only if $d(C) \geq 2t + 1$.