Math 61DM Homework # 3

Due at TA session on Friday, October 12.

1. Let F be a field, $m \geq 0$ a fixed nonnegative integer and let

$$V = \{a_0 + a_1 x + \cdots + a_m x^m \colon a_0, \ldots, a_m \in F\}$$

be the vector space consisting of all polynomials over F of degree at most m. Suppose that $p_1, p_2, \ldots, p_{m+1} \in V$ are polynomials such that $p_j(1) = 0$ for all j. Prove that the vectors $p_1, p_2, \ldots, p_{m+1}$ are linearly dependent.

2. Let V be a finite dimensional vector space of a field F and suppose that $U_1, \ldots, U_m$ are subspaces of V. Define

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \colon u_j \in U_j \text{ for all } 1 \leq j \leq m\}.$$

Prove that $\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m$.

3. Determine all values of $\lambda \in \mathbb{R}$ such that the collection of vectors

$$(\lambda, 1, 1),\ (1, \lambda, 1),\ (1, 1, \lambda)$$

in $\mathbb{R}^3$ is linearly dependent.

4. Suppose that V is a vector space over F, that $v_1, \ldots, v_k \in V$ are linearly independent, and that $u \in V$ is any other vector. Show that

$$\dim \operatorname{span}(v_1 + u, v_2 + u, \ldots, v_k + u) \geq k - 1.$$

Can this dimension be equal to $k - 1$?

5. Let p be a prime and $\mathbb{F}_p$ be the finite field with p elements. Prove that every subspace of $\mathbb{F}_p^n$ of dimension d has exactly p^d elements.

6. Let p be a prime. Suppose a town has n residents that form m clubs with the following rules:
(i) The number of members of each club is not a multiple of p.
(ii) The number of common members in each pair of distinct clubs is a multiple of p.
Prove that $m \leq n$.

7. Suppose a town has n residents that form m clubs with the following rules:
(i) The number of members of each club is not a multiple of 15.
(ii) The number of common members in each pair of distinct clubs is a multiple of 15.
Prove that $m \leq 2n$.

Hint: Try to reduce it to the previous problem with $p = 3, 5$.

8. In this problem, you will prove the following identity holds for each positive integer n:

$$\frac{n}{n+1} = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r+1} \binom{n}{r}.$$

First multiply both sides by $(n+1)!$ and explain why the left hand side would then count the number of permutations π of $1, 2, \ldots, n+1$ for which $\pi(n+1) \neq n+1$. Use the the inclusion-exclusion principle to argue then the right hand side also counts the same set of permutations. To do this, explain why a permutation π of $1, 2, \ldots, n+1$ satisfies $\pi(n+1) \neq n+1$ if and only if $\pi(n+1) < \pi(s)$ for at least one $s \in \{1, \ldots, n\}$. For $1 \leq s \leq n$, let A_s be the set of permutations π of $1, \ldots, n+1$ which satisfy $\pi(n+1) < \pi(s)$. Argue that for any $S \subset \{1, \ldots, n\}$ of size $r > 0$, we have $|\bigcap_{i \in S} A_i| = (n+1)!/(r+1)$.