Math 61DM Homework # 2
Due at TA session on Friday, October 5. Show your work.

1. Let \( V \) be the vector space of polynomials of degree at most 5, with coefficients in a field \( \mathbb{F} \). Let \( U \) be the subspace of \( V \) consisting of polynomials of the form \( az^5 + bz + c \) with \( a, b, c \in \mathbb{F} \). Find a subspace \( W \) such that every element \( v \in V \) can be written in one and only one way as the sum of an element in \( U \) and another element in \( W \).

2. Prove that there exists a quadratic polynomial \( ax^2 + bx + c \) whose graph passes through the points \((0, 1), (1, 0), (2, 3)\). Is such polynomial unique?

3. Find a single homogeneous linear equation with unknowns \( x_1, x_2, x_3 \) such that the solution set is the span of the 2 vectors \((1, 1, 1), (1, 2, 0)\).

4. Let \( V \) be a vector space and suppose \( S = \{v_1, \ldots, v_k\} \) is a finite set of linearly dependent vectors in \( V \). Prove that there is a proper subset of \( S \) whose span is equal to the span of \( S \). By a proper subset, we mean a subset \( T \subseteq S \) such that \( T \neq S \).

5. Use Gaussian elimination in \( \mathbb{R} \) to show that the solution set of the homogeneous system
\[
x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\
x_1 + 4x_2 + 3x_3 + 2x_4 = 0 \\
2x_1 + 5x_2 + 6x_3 + 7x_4 = 0 \\
x_1 + 3x_3 + 6x_4 = 0
\]
is a plane through \( \mathbf{0} \) (that is, it is the span of 2 linearly independent vectors, and find 2 linearly independent vectors whose span is the solution space.

6. In parts (a) and (b) we will show that multiplicative inverses exist in \( \mathbb{Z}/p\mathbb{Z} \) for \( p \) prime. So, you should not use this fact in your answers to (a) and (b). However, feel free to use other basic facts about arithmetic in \( \mathbb{Z}/p\mathbb{Z} \).
   (a) Suppose \( a \in \mathbb{Z}/p\mathbb{Z}, a \neq 0 \). For any \( x, y \in \mathbb{Z}/p\mathbb{Z} \) show that if \( ax = ay \) (with multiplication in \( \mathbb{Z}/p\mathbb{Z} \), i.e. modulo \( p \)) then \( x = y \). [Hint: you will need to unwrap the definition of “mod \( p \) multiplication”, and make use of the hypotheses that \( p \) is prime and \( a \neq 0 \).]
   (b) By considering the set \( \{a0, a1, \ldots, a(p - 1)\} \) over \( \mathbb{Z}/p\mathbb{Z} \), or otherwise, show that there exists \( b \in \mathbb{Z}/p\mathbb{Z} \) such that \( ab = 1 \) (again, with multiplication mod \( p \)).
   (c) In the set of integers \( \mathbb{Z} \), solve the system of equations
\[
2x + y = 2 \mod 5, \quad 3x - 2y = 0 \mod 5,
\]
by using Gaussian elimination in \( \mathbb{Z}/(5\mathbb{Z}) \).

7. If \( a, b \in \mathbb{R} \) with \( a < b \), prove:
   (a) There is a rational \( r \in (a, b) \).
   (b) There is an irrational \( c \in (a, b) \). Hint: Use part (a) and Q.2 of HW #1.
   (c) \( (a, b) \) contains infinitely many rationals and infinitely many irrationals.

8. Let \( a_n \) denote the number of nonnegative integers less than \( 3^n \) that do not have two consecutive ones when written in base 3. Equivalently, \( a_n \) is the number of sequences of 0, 1, 2 of length \( n \) which do not have two consecutive 1’s. Find a recurrence equation for \( a_n \) and then solve for \( a_n \) explicitly.

9. How many positive integers up to 2018 are not divisible by 2, 3, 6 or 11?