Math 61DM Homework # 2

Due at TA session on Friday, October 5. Show your work.

1. Let V be the vector space of polynomials of degree at most 5, with coefficients in a field $\mathbb{F}$. Let U be the subspace of V consisting of polynomials of the form $az^5 + bz + c$ with $a, b, c \in \mathbb{F}$. Find a subspace W such that every element $v \in V$ can be written in one and only one way as the sum of an element in U and another element in W.

2. Prove that there exists a quadratic polynomial $ax^2 + bx + c$ whose graph passes through the points $(0, 1)$, $(1, 0)$, $(2, 3)$. Is such polynomial unique?

3. Find a single homogeneous linear equation with unknowns x_1, x_2, x_3 such that the solution set is the span of the 2 vectors $(1, 1, 1), (1, -2, 0)$.

4. Let V be a vector space and suppose $S = \{v_1, \ldots, v_k\}$ is a finite set of linearly dependent vectors in V. Prove that there is a proper subset of S whose span is equal to the span of S. By a proper subset, we mean a subset $T \subset S$ such that $T \neq S$.

5. Use Gaussian elimination in $\mathbb{R}$ to show that the solution set of the homogeneous system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \\ x_1 + 4x_2 + 3x_3 + 2x_4 &= 0 \\ 2x_1 + 5x_2 + 6x_3 + 7x_4 &= 0 \\ x_1 + 3x_3 + 6x_4 &= 0 \end{aligned}$$

is a plane through $\mathbf{0}$ (that is, it is the span of 2 linearly independent vectors, and find 2 linearly independent vectors whose span is the solution space.

6. In parts (a) and (b) we will show that multiplicative inverses exist in $\mathbb{Z}/p\mathbb{Z}$ for p prime. So, you should not use this fact in your answers to (a) and (b). However, feel free to use other basic facts about arithmetic in $\mathbb{Z}/p\mathbb{Z}$.
(a) Suppose $a \in \mathbb{Z}/p\mathbb{Z}$, $a \neq 0$. For any $x, y \in \mathbb{Z}/p\mathbb{Z}$ show that if $ax = ay$ (with multiplication in $\mathbb{Z}/p\mathbb{Z}$, i.e. modulo p) then $x = y$. [Hint: you will need to unwrap the definition of "mod p multiplication", and make use of the hypotheses that p is prime and $a \neq 0$.]
(b) By considering the set $\{a0, a1, \ldots, a(p-1)\}$ over $\mathbb{Z}/p\mathbb{Z}$, or otherwise, show that there exists $b \in \mathbb{Z}/p\mathbb{Z}$ such that $ab = 1$ (again, with multiplication mod p).
(c) In the set of integers $\mathbb{Z}$, solve the system of equations

$$2x + y = 2 \mod 5, \; 3x - 2y = 0 \mod 5,$$

by using Gaussian elimination in $\mathbb{Z}/(5\mathbb{Z})$.

7. If $a, b \in \mathbb{R}$ with $a < b$, prove:

(a) There is a rational $r \in (a, b)$.
(b) There is an irrational $c \in (a, b)$. Hint: Use part (a) and Q.2 of HW #1.
(c) (a, b) contains infinitely many rationals and infinitely many irrationals.

8. Let a_n denote the number of nonnegative integers less than 3^n that do not have two consecutive ones when written in base 3. Equivalently, a_n is the number of sequences of $0, 1, 2$ of length n which do not have two consecutive 1's. Find a recurrence equation for a_n and then solve for a_n explicitly.

9. How many positive integers up to 2018 are not divisible by 2, 3, 6 or 11?