Homework # 1.

Due in class on Monday, October 1.

1. Use the principle of mathematical induction to check that

$$\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4} \text{ for } n = 1, 2, \ldots.$$

The principle of mathematical induction is discussed in the lecture notes, it says that if P_n is a (true or false) proposition for each $n = 1, 2, \ldots$ and if (a) P_1 is true, and (b) for each n we are able to check that P_{n+1} is true whenever P_n is true, then P_n is true for all $n = 1, 2, \ldots$.

2. Prove that (i) $\sqrt{28}$ is irrational, and (ii) If x, y are rational, $x \neq 0$, and z irrational, then $y + xz, y + x/z$ are both irrational. (Recall that x rational means that $\exists$ integers p, q with $q \neq 0$ such that $x = p/q$.) Note: $\sqrt{28}$ is defined to be the positive real number a such that $a^2 = 28$; see Problem 8.

3. By examining the proof of the triangle inequality $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, given in lecture/book (recall that proof began with the identity $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2(\mathbf{x}\cdot\mathbf{y})$), prove that equality holds in the triangle inequality if and only if either at least one of $\mathbf{x}, \mathbf{y}$ is $\mathbf{0}$ or $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ and $\mathbf{y} = \lambda\mathbf{x}$ with $\lambda > 0$.

4. (Another proof of the Cauchy-Schwarz inequality.) Given two vectors $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$, prove the identity

$$\frac{1}{2}\sum_{i,j=1}^{n}(a_i b_j - a_j b_i)^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2.$$

Use this to show that $|\mathbf{a}\cdot\mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|$.

Note: For any given a_{ij}, $i, j = 1, \ldots, n$, the notation $\sum_{i,j=1}^{n} a_{ij}$ means $\sum_{j=1}^{n}(\sum_{i=1}^{n} a_{ij}))$, which is the same as $\sum_{i=1}^{n}(\sum_{j=1}^{n} a_{ij}))$.

5. Using the dot product, prove, for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

(a) The parallelogram law: $\|\mathbf{x}-\mathbf{y}\|^2 + \|\mathbf{x}+\mathbf{y}\|^2 = 2(\|x\|^2 + \|y\|^2)$

(b) The law of cosines: $\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\,\|\mathbf{y}\|\cos\theta$, assuming $\mathbf{x}, \mathbf{y}$ are non-zero and θ is the angle between $\mathbf{x}$ and $\mathbf{y}$ as discussed in lecture.

(c) Give a geometric interpretion of identities (a),(b) (i.e. describe what (a) is saying about the parallelogram determined by $\mathbf{x}, \mathbf{y}$—i.e. *OACB* where $\overrightarrow{OA} = \mathbf{x}$, $\overrightarrow{OB} = \mathbf{y}$, $\overrightarrow{OC} = \mathbf{x}+\mathbf{y}$, and what (b) is saying about the triangle determined by $\mathbf{x}$ and $\mathbf{y}$—i.e. *OAB*, where $\overrightarrow{OA} = \mathbf{x}$, $\overrightarrow{OB} = \mathbf{y}$).

6. Suppose V is a vector space over a field $\mathbb{F}$.

(a) Show that if X, Y are subspaces of V then so are

$X \cap Y = \{v : v \in X \text{ and } v \in Y\}$ and $X + Y = \{v \in V : \exists x \in X,\ y \in Y \text{ s.t. } v = x + y\}$.

(b) Show that if $a_{ij} \in \mathbb{F}$, $1 \leq i \leq m$, $1 \leq j \leq n$, are fixed, then

$$W = \{x = (x_1, \ldots, x_n) \in \mathbb{F}^n : \ \forall i \in \{1, \ldots, m\} \ \sum_{j=1}^{n} a_{ij} x_j = 0\}$$

is a subspace of $\mathbb{F}^n$.

(c) Give an example of a vector space V, and subspaces X, Y, such that

$$X \cup Y = \{v \in V : \ v \in X \text{ or } v \in Y\}$$

is not a subspace of V.

7. Give a bijective proof that $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

8. Let a_n be the number of ways of tiling a $2 \times n$ board by 1×2 rectangles and 2×2 squares. For example, $a_1 = 1$ and $a_2 = 3$.

(a) Find a recursive formula for a_n.

(b) Find an explicit formula for a_n.