MATH 61DM Homework 1 Solutions

Exercise 1

Use the principle of mathematical induction to check that

$$\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4} \text{ for } n = 1, 2, \ldots .$$

We first show that the above equation is true in the case that n=1.

When $n = 1$,

$$\sum_{j=1}^{n} j^3 = 1^3 = 1.$$

Similarly, setting $n = 1$ again,

$$\frac{n^2(n+1)^2}{4} = \frac{1^2 \cdot 2^2}{4} = 1.$$

Hence we have shown the above equation is true when $n = 1$. This completes the base case.

We now show that when $\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$ is true for $n = k$, then the same equation is true when $n = k + 1$.

We assume that

$$\sum_{j=1}^{k} j^3 = \frac{k^2(k+1)^2}{4},$$

which implies that

$$\sum_{j=1}^{k+1} j^3 = (k+1)^3 + \sum_{j=1}^{k} j^3 = (k+1)^3 + \frac{k^2(k+1)^2}{4}.$$

By expanding and factoring these algebraic expressions, we get the following:

$$\begin{aligned}\sum_{j=1}^{k+1} j^3 &= (k+1)^3 + \frac{k^2(k+1)^2}{4} \\ &= \frac{4(k+1)^3}{4} + \frac{k^2(k+1)^2}{4} \\ &= \frac{4(k+1)^3 + k^2(k+1)^2}{4} \\ &= \frac{(k+1)^2(4(k+1)+k^2)}{4} \\ &= \frac{(k+1)^2(4+4k+k^2)}{4}. \\ \sum_{j=1}^{k+1} j^3 &= \frac{(k+1)^2(k+2)^2}{4}.\end{aligned}$$

In fact, this is what we desired to show: $\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$ for $n = k+1$.

This completes the inductive step. We may now conclude that $\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$ for $n = 1, 2, \ldots$.

Exercise 2

Prove that (i) $\sqrt{28}$ is irrational, and (ii) If x, y are rational, $x \neq 0$, and z irrational, then $y + xz, y + x/z$ are both irrational.

(i) We will show $\sqrt{28}$ is irrational.

For the sake of contradiction, assume $\sqrt{28}$ is rational; i.e., there exist $p, q \in \mathbb{Z}$ with $q \neq 0$ so that $\sqrt{28} = p/q$.

We may further assume that the greatest common divisor of p, q is 1; we may see this by noting that if some $d > 1$ divides both p and q, then we may replace p with $p' = p/d$ and q with $q' = q/d$. If again p' and q' have some $d' > 1$ that divides them both, we may repeat the process. We cannot continue indefinitely because at each stage the absolute value of the new denominator q' is less than or equal to $|q| - 1$ (because $d > 1$). If the process goes on forever, then eventually the absolute value of the denominator would be less than or equal to 0, which is impossible since the denominator cannot be equal to 0.

Hence we assume that p and q have no factors besides 1 in common.

If $\sqrt{28} = \frac{p}{q}$, then by squaring both sides and simplifying we see that $28q^2 = p^2$. This means that $7 \cdot (4q^2) = p^2$. Since $4q^2$ is an integer, 7 must divide p^2. Since 7 is prime, for any $a, b \in \mathbb{Z}$ with 7 dividing ab, either 7 divides a or 7 divides b. Hence 7 must divide p since 7 divides p^2.

Thus $p = 7r$ for some $r \in \mathbb{Z}$. Then $28q^2 = (7r)^2 = 49r^2$; dividing by 7, we see that $4q^2 = 7r^2$.

Since r^2 is an integer, we see that 7 divides $4q^2$. Since 7 is prime, this means that either 7 divides 4, which is false, or 7 divides q^2. Hence 7 must divide q^2, which, since 7 is prime, means that 7 divides q.

Hence p, q are both divisible by 7, contradicting our assumption that the greatest common divisor of p, q was 1.

Therefore $\sqrt{28}$ cannot be written as p/q for $p, q \in \mathbb{Z}, q \neq 0$, so $\sqrt{28}$ is irrational.

(ii) We will show that if x, y are rational, $x \neq 0$, and z is irrational, then $y + xz$ and $y + x/z$ are both irrational.

First we will show that $y + xz$ is irrational. For sake of contradiction, assume that $s = y + xz$ is rational.

Since $s = y + xz$, we see that $s - y = xz$. Since s is rational and y is rational, $s - y$ is rational. Hence xz is rational.

Let $xz = t$. Then t is rational, and x is rational, so $t/x = z$ is rational. This gives a contradiction, since we assumed z was irrational.

Hence $y + xz$ must be irrational.

Now we show $y + x/z$ is irrational. For sake of contradiction, assume that $r = y + x/z$ is rational.

Since $r = y + x/z$, we see that $\frac{x}{r-y} = z$. Since r and y are rational, $r - y$ is rational; since $r - y$ and x are rational, $\frac{x}{r-y} = z$ is rational. his gives a contradiction, since we assumed z was irrational.

Hence $y + x/z$ must be irrational.

Exercise 3

By examining the proof of the triangle inequality $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, given in lecture/book, prove that equality holds in the triangle inequality if and only if either at least one of $\mathbf{x}, \mathbf{y}$ is $\mathbf{0}$ or $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ and $\mathbf{y} = \lambda \mathbf{x}$ with $\lambda > 0$.

We will first show that if equality holds in the triangle inequality, then at least one of $\mathbf{x}, \mathbf{y}$ is $\mathbf{0}$ or $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ and $\mathbf{y} = \lambda \mathbf{x}$ with $\lambda > 0$.

We assume that $||\mathbf{x}+\mathbf{y}|| = ||x|| + ||y||$. Squaring both sides and using the identity from the book ($||\mathbf{x}+\mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2\mathbf{x}\cdot\mathbf{y}$), we see that

$$(||\mathbf{x}|| + ||\mathbf{y}||)^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2||\mathbf{x}||||\mathbf{y}|| = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2\mathbf{x}\cdot\mathbf{y}.$$

Further simplifying, we see that $||\mathbf{x}||||\mathbf{y}|| = \mathbf{x}\cdot\mathbf{y}$.

Now, what we wish to show is that if both $\mathbf{x},\mathbf{y} \neq \mathbf{0}$, then $\mathbf{y} = \lambda\mathbf{x}$ with $\lambda > 0$.

As in Simon Chapter 1 Section 2 Equation 2.5, we note that $|\mathbf{x}\cdot\mathbf{y}| \leq ||\mathbf{x}||||\mathbf{y}||$, with equality only when $\mathbf{y} = \mathbf{0}$ or $\mathbf{x} = ||\mathbf{y}||^{-2}(\mathbf{x}\cdot\mathbf{y})\mathbf{y}$. Since by nature of absolute value, $\mathbf{x}\cdot\mathbf{y} \leq |\mathbf{x}\cdot\mathbf{y}|$ with equality only when $\mathbf{x}\cdot\mathbf{y} \geq 0$, we see that we may only have $\mathbf{x}\cdot\mathbf{y} = ||\mathbf{x}||||\mathbf{y}||$ when $\mathbf{y} = \mathbf{0}$ or when $\mathbf{x}\cdot\mathbf{y} \geq 0$ and $\mathbf{x} = ||\mathbf{y}||^{-2}(\mathbf{x}\cdot\mathbf{y})\mathbf{y}$.

Note that if $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{x}\cdot\mathbf{y} = 0$, then automatically $\mathbf{x} = ||\mathbf{y}||^{-2}(\mathbf{x}\cdot\mathbf{y})\mathbf{y}$ implies that $\mathbf{x} = \mathbf{0}$.

Therefore if equality holds in the triangle inequality, either at least one of $\mathbf{x},\mathbf{y}$ is $\mathbf{0}$, or $\mathbf{x}\cdot\mathbf{y} > 0$ and $\mathbf{x} = ||\mathbf{y}||^{-2}(\mathbf{x}\cdot\mathbf{y})\mathbf{y}$.

If $\mathbf{x}\cdot\mathbf{y} > 0$ and $\mathbf{x},\mathbf{y} \neq \mathbf{0}$, then $\mathbf{x} = ||\mathbf{y}||^{-2}(\mathbf{x}\cdot\mathbf{y})\mathbf{y}$ implies that $\mathbf{y} = \frac{||\mathbf{y}||^2}{\mathbf{x}\cdot\mathbf{y}}\mathbf{x}$, with $\frac{||\mathbf{y}||^2}{\mathbf{x}\cdot\mathbf{y}} > 0$ because $||\mathbf{y}||^2 > 0$ and $\mathbf{x}\cdot\mathbf{y} > 0$.

Then by letting $\lambda = \frac{||\mathbf{y}||^2}{\mathbf{x}\cdot\mathbf{y}}$ when necessary, we see that if equality holds in the triangle inequality, then at least one of $\mathbf{x},\mathbf{y}$ is $\mathbf{0}$ or $\mathbf{x},\mathbf{y} \neq \mathbf{0}$ and $\mathbf{y} = \lambda\mathbf{x}$ with $\lambda > 0$.

We now assume that at least one of $\mathbf{x},\mathbf{y}$ is $\mathbf{0}$ or $\mathbf{x},\mathbf{y} \neq \mathbf{0}$ and $\mathbf{y} = \lambda\mathbf{x}$ with $\lambda > 0$.

If at least one of $\mathbf{x},\mathbf{y}$ is $\mathbf{0}$, then we may assume without loss of generality that $\mathbf{x} = 0$. Then

$$||\mathbf{x}+\mathbf{y}|| = ||\mathbf{0}+\mathbf{y}|| = ||y|| = ||\mathbf{0}|| + ||y|| = ||x|| + ||y||.$$

Therefore if at least one of $\mathbf{x},\mathbf{y}$ is $\mathbf{0}$, then equality holds in the triangle inequality.

Hence we may now assume that $\mathbf{x},\mathbf{y} \neq \mathbf{0}$ and $\mathbf{y} = \lambda\mathbf{x}$ with $\lambda > 0$.

Then

$$||\mathbf{x}+\mathbf{y}|| = ||\mathbf{x}+\lambda\mathbf{x}|| = (\lambda+1)||\mathbf{x}||,$$

since $\lambda + 1 > 0$. Similarly since $\lambda > 0$, we see that

$$||\mathbf{x}|| + ||\mathbf{y}|| = ||\mathbf{x}|| + ||\lambda\mathbf{x}|| = ||\mathbf{x}|| + \lambda||\mathbf{x}|| = (\lambda+1)||\mathbf{x}||.$$

Therefore

$$(\lambda+1)||\mathbf{x}|| = ||\mathbf{x}+\mathbf{y}|| = ||\mathbf{x}|| + ||\mathbf{y}||,$$

which is what we desired to show.

Exercise 4

Given two vectors $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$, prove the identity

$$\frac{1}{2}\sum_{i,j=1}^{n}(a_i b_j - a_j b_i)^2 = ||\mathbf{a}||^2||\mathbf{b}||^2 - (\mathbf{a}\cdot\mathbf{b})^2.$$

Use this to show that $|\mathbf{a}\cdot\mathbf{b}| \leq ||\mathbf{a}||||\mathbf{b}||$.

We wish to show that for two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ that $\frac{1}{2}\sum_{i,j=1}^{n}(a_i b_j - a_j b_i)^2 = ||\mathbf{a}||^2||\mathbf{b}||^2 - (\mathbf{a}\cdot\mathbf{b})^2$.

Using the definition of the magnitude of a vector and of the dot product, we get the following:

$$||\mathbf{a}||^2||\mathbf{b}||^2 - (\mathbf{a}\cdot\mathbf{b})^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2$$

$$||\mathbf{a}||^2||\mathbf{b}||^2 - (\mathbf{a}\cdot\mathbf{b})^2 = \left(\sum_{i,j=1}^{n} a_i^2 b_j^2\right) - \left(\sum_{i,j=1}^{n} a_i a_j b_i b_j\right)$$

$$||\mathbf{a}||^2||\mathbf{b}||^2 - (\mathbf{a}\cdot\mathbf{b})^2 = \sum_{i,j=1}^{n}(a_i^2 b_j^2 - a_i a_j b_i b_j)$$

We next split the sum into two halves and reindex.

$$||\mathbf{a}||^2||\mathbf{b}||^2 - (\mathbf{a}\cdot\mathbf{b})^2 = \frac{1}{2}\left(\sum_{i,j=1}^{n} a_i^2 b_j^2 - a_i a_j b_i b_j\right) + \frac{1}{2}\left(\sum_{k,\ell=1}^{n} a_k^2 b_\ell^2 - a_k a_\ell b_k b_\ell\right).$$

Then, by reindexing again, we see that

$$||\mathbf{a}||^2||\mathbf{b}||^2 - (\mathbf{a}\cdot\mathbf{b})^2 = \frac{1}{2}\left(\sum_{i,j=1}^{n} a_i^2 b_j^2 - a_i a_j b_i b_j\right) + \frac{1}{2}\left(\sum_{j,i=1}^{n} a_j^2 b_i^2 - a_i a_j b_i b_j\right).$$

$$\begin{aligned}
||\mathbf{a}||^2||\mathbf{b}||^2 - (\mathbf{a}\cdot\mathbf{b})^2 &= \frac{1}{2}\left(\sum_{i,j=1}^{n} a_i^2 b_j^2 - a_i a_j b_i b_j + a_j^2 b_i^2 - a_i a_j b_i b_j\right) \\
&= \frac{1}{2}\left(\sum_{i,j=1}^{n} a_i^2 b_j^2 - 2a_i a_j b_i b_j + a_j^2 b_i^2\right) \\
&= \frac{1}{2}\sum_{i,j=1}^{n}(a_i b_j - a_j b_i)^2
\end{aligned}$$

which is what we desired to show.

Note that, since for any $\mathbf{a}, \mathbf{b}$ and for any i, j we must have $(a_i b_j - a_j b_i)^2 \geq 0$, then $\frac{1}{2}\sum_{i,j=1}^{n}(a_i b_j - a_j b_i)^2 \geq 0$. Hence

$$||\mathbf{a}||^2||\mathbf{b}||^2 - (\mathbf{a}\cdot\mathbf{b})^2 \geq 0.$$

This means that $||\mathbf{a}||^2||\mathbf{b}||^2 \geq (\mathbf{a}\cdot\mathbf{b})^2$; we may take positive square roots of both sides to see that $|||\mathbf{a}||||\mathbf{b}||| = ||\mathbf{a}||||\mathbf{b}|| \geq |\mathbf{a}\cdot\mathbf{b}|$.

Exercise 5

Using the dot product, prove, for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

(a) The parallelogram law: $||\mathbf{x}-\mathbf{y}||^2 + ||\mathbf{x}+\mathbf{y}||^2 = 2(||x||^2 + ||\mathbf{y}||^2)$,

(b) The law of cosines: $||\mathbf{x}-\mathbf{y}||^2 = ||\mathbf{x}||^2 + ||y||^2 - 2||x||||y||\cos\theta$, assuming $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ and θ is the angle between $\mathbf{x}, \mathbf{y}$,

(c) Give a geometric interpretation of identities (a),(b).

(a) We wish to show that for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ that $||\mathbf{x}-\mathbf{y}||^2+||\mathbf{x}+\mathbf{y}||^2 = 2(||x||^2+||\mathbf{y}||^2)$.

We note that, as in Simon Chapter 1 Section 2 Equation 2.2, for any $t \in \mathbb{R}$, we must have $||\mathbf{x}+t\mathbf{y}||^2 = t^2||y||^2 + 2t\mathbf{x}\cdot\mathbf{y} + ||\mathbf{x}||^2$.

Then by setting $t = 1, -1$, we see that:

$$\begin{aligned} ||\mathbf{x}-\mathbf{y}||^2 + ||\mathbf{x}+\mathbf{y}||^2 &= (-1)^2||y||^2 + 2(-1)\mathbf{x}\cdot\mathbf{y} + ||\mathbf{x}||^2 + 1^2||y||^2 + 2\cdot 1\cdot\mathbf{x}\cdot\mathbf{y} + ||\mathbf{x}||^2 \\ &= ||y||^2 - 2\mathbf{x}\cdot\mathbf{y} + ||\mathbf{x}||^2 + ||y||^2 + 2\mathbf{x}\cdot\mathbf{y} + ||\mathbf{x}||^2 \\ &= 2(||\mathbf{y}||^2 + ||\mathbf{x}||^2) \end{aligned}$$

which is what we desired to show.

(b) We wish to show that for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \mathbf{x}, \mathbf{y} \neq \mathbf{0}$ that $||\mathbf{x}-\mathbf{y}||^2 = ||\mathbf{x}||^2 + ||y||^2 - 2||x||||y||\cos\theta$, for θ the angle between $\mathbf{x}$ and $\mathbf{y}$.

Again by Simon Chapter 1 Section 2 Equation 2.2, we see that

$$||\mathbf{x}-\mathbf{y}||^2 = ||\mathbf{y}||^2 - 2\mathbf{x}\cdot\mathbf{y} + ||\mathbf{x}||^2.$$

Now, by Simon Chapter 1 Section 2 Equation 2.8, $\mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}||||\mathbf{y}|| \cos\theta$ for θ the angle between $\mathbf{x}$ and $\mathbf{y}$. Combining this with the above, we see that

$$||\mathbf{x} - \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||y||^2 - 2||x||||y|| \cos\theta,$$

as we desired to show.

(c) Consider the parallelogram $OACB$ where $OA = \mathbf{x}, OB = \mathbf{y}, OC = \mathbf{x} + \mathbf{y}$. Then $BA = \mathbf{x} - \mathbf{y}$ is the other diagonal; hence (a) is saying that the sum of the squares of the lengths of the diagonals is equal to the sum of the lengths of the squares of all four sides of the parallelogram.

Now consider (b): this identity says that the length of BA is in fact the sum of the squares of the lengths of OA and OB minus twice the product of the lengths of OA and OB times the cosine of the angle between OA and OB.

Exercise 6

Suppose V is a vector space over a field $\mathbb{F}$.

(a) Show that if X, Y are subspaces of V then so are

$X \cap Y = \{v : v \in X \text{ and } v \in Y\}$ and $X + Y = \{v \in V : \exists x \in X, y \in Y \text{ s.t. } v = x + y\}$.

(b) Show that if $a_{ij} \in \mathbb{F}, 1 \leq i \leq m, 1 \leq j \leq n$, are fixed, then

$$W = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}^n : \forall i \in \{1, \ldots, m\} \sum_{j=1}^{n} a_{ij} x_j = 0\}$$

is a subspace of $\mathbb{F}^n$.

(c) Give an example of a vector space V and subspaces X, Y such that

$$X \cup Y = \{v \in V : v \in X \text{ or } v \in Y\}$$

is not a subspace of V.

(a) We wish to show that if X, Y are subspaces of V, then so is $X \cap Y$.

First we show it contains the zero vector $\mathbf{0}$. Since X, Y are subspaces of V, $\mathbf{0} \in X$ and $\mathbf{0} \in Y$. Hence $\mathbf{0} \in X \cap Y$ as desired.

We now show that $X \cap Y$ is closed under scalar multiplication. Assume $\mathbf{w} \in X \cap Y$ and $\lambda \in \mathbb{F}$; we wish to show that $\lambda\mathbf{w} \in X \cap Y$.

Since $\mathbf{w} \in X \cap Y$, we must have $\mathbf{w} \in X$ and $\mathbf{w} \in Y$. Then since X, Y are subspaces of V, $\lambda\mathbf{w} \in X$ and $\lambda\mathbf{w} \in Y$; this means that $\lambda\mathbf{w} \in X \cap Y$ as we desired to show.

We now show that $X \cap Y$ is closed under addition. Assume $\mathbf{w_1} \in X \cap Y$ and $\mathbf{w_2} \in X \cap Y$; we wish to show that $\mathbf{w_1} + \mathbf{w_2} \in X \cap Y$.

Since $\mathbf{w_1}, \mathbf{w_2} \in X \cap Y$, we must have $\mathbf{w_1}, \mathbf{w_2} \in X$ and $\mathbf{w_1}, \mathbf{w_2} \in Y$. Then since X, Y are subspaces of V, $\mathbf{w_1} + \mathbf{w_2} \in X$ and $\mathbf{w_1} + \mathbf{w_2} \in Y$; this means that $\mathbf{w_1} + \mathbf{w_2} \in X \cap Y$ as we desired to show.

Hence $X \cap Y$ is a subspace of V.

Now we show that if X, Y are subspaces of V, then so is $X + Y$.

First we show it contains the zero vector $\mathbf{0}$. Since X, Y are subspaces of V, $\mathbf{0} \in X$ and $\mathbf{0} \in Y$. Hence $\mathbf{0} = \mathbf{0} + \mathbf{0} \in X + Y$ as desired.

We now show that $X + Y$ is closed under scalar multiplication. Assume $\mathbf{w} \in X \cap Y$ and $\lambda \in \mathbb{F}$; we wish to show that $\lambda\mathbf{w} \in X \cap Y$.

Since $\mathbf{w} \in X + Y$, we must have $\mathbf{w} = \mathbf{x} + \mathbf{y}$ for some $\mathbf{x} \in X$ and $\mathbf{y} \in Y$. Then since X, Y are subspaces of V, $\lambda\mathbf{x} \in X$ and $\lambda\mathbf{y} \in Y$; this means that $\lambda\mathbf{w} = \lambda\mathbf{x} + \lambda\mathbf{y} \in X + Y$ as we desired to show.

We now show that $X + Y$ is closed under addition. Assume $\mathbf{w_1} \in X + Y$ and $\mathbf{w_2} \in X + Y$; we wish to show that $\mathbf{w_1} + \mathbf{w_2} \in X + Y$.

Since $\mathbf{w_1}, \mathbf{w_2} \in X + Y$, we must have $\mathbf{w_1} = \mathbf{x}_1 + \mathbf{y}_1$ and $\mathbf{w_2} = \mathbf{x}_2 + \mathbf{y}_2$ for some $\mathbf{x}_1, \mathbf{x}_2 \in X, \mathbf{y}_1, \mathbf{y}_2 \in Y$. Then since X, Y are subspaces of V, $\mathbf{x}_1 + \mathbf{x}_2 \in X$ and $\mathbf{y_1} + \mathbf{y_2} \in Y$; this means that $\mathbf{w_1} + \mathbf{w_2} = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{y}_1 + \mathbf{y}_2 \in X \cap Y$ as we desired to show.

Hence $X + Y$ is a subspace of V.

(b) We wish to show that if $a_{ij} \in \mathbb{F}, 1 \leq i \leq m, 1 \leq j \leq n$, are fixed, then

$$W = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}^n : \forall i \in \{1, \ldots, m\} \sum_{j=1}^{n} a_{ij} x_j = 0\}$$

is a subspace of $\mathbb{F}^n$.

First we show it contains the zero vector $\mathbf{0} = (0, \ldots, 0)$. Note that regardless of the values of the a_{ij}, for all i from 1 to m, we must have $\sum_{j=1}^{n} a_{ij} \cdot 0 = 0$. This means that $\mathbf{0} = (0, \ldots, 0) \in W$.

We now show that W is closed under scalar multiplication. Assume $\mathbf{x} = (x_1, \ldots, x_n) \in W$ and $\lambda \in R$. Then for all $i \in \{1, \ldots, m\}$ we have $\sum_{j=1}^{n} a_{ij} x_j = 0$ by definition of W.

Then

$$\lambda \sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} a_{ij} (\lambda x_j) = \lambda \cdot 0 = 0.$$

Then since $\lambda\mathbf{x} = (\lambda x_1, \ldots, \lambda x_n)$, we see that $\lambda\mathbf{x} \in W$ as desired.

We now show that W is closed under addition. Assume $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are both in W. We wish to show that $\mathbf{x} + \mathbf{y} \in W$.

Since $\mathbf{x}, \mathbf{y} \in W$, we see that for any $i \in \{1, \ldots, m\}$, we have $\sum_{j=1}^n a_{ij}x_j = \sum_{j=1}^n a_{ij}y_j = 0$. Then

$$\left(\sum_{j=1}^n a_{ij}x_j\right) + \left(\sum_{j=1}^n a_{ij}y_j\right) = 0 + 0 = 0,$$

so by combining two sums into one, we get

$$\sum_{j=1}^n (a_{ij}x_j + a_{ij}y_j) = \sum_{j=1}^n a_{ij}(x_j + y_j) = 0.$$

Since $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n)$, we have shown $\mathbf{x} + \mathbf{y} \in W$ as desired.

Hence W is a subspace of $\mathbb{F}^n$ as we desired to show.

(c) Consider the example $V = \mathbb{F}^2$, with $X = \{\mathbf{x} = (x_1, x_2) \in V : x_2 = 0\}$ and $Y = \{\mathbf{y} = (y_1, y_2) \in V : y_1 = 0\}$.

Consider $(x_1, 0), (x_1', 0) \in X, (0, y_2), (0, y_2') \in Y$, and $\lambda \in F$. Note that $\mathbf{0} = (0, 0) \in X$ and $\mathbf{0} = (0, 0) \in Y$.

We also see that $\lambda(x_1, 0) = (\lambda x_1, 0) \in X$ and $\lambda(0, y_2) = (0, \lambda y_2) \in Y$, so X and Y are closed under scalar multiplication.

Furthermore, $(x_1, 0) + (x_1', 0) = (x_1 + x_1', 0) \in X$ and $(0, y_2) + (0, y_2') = (0, y_2 + y_2') \in Y$, so X, Y are closed under addition. Hence X, Y are subspaces of V.

Now choose any $a \in \mathbb{F}$ such that $a \neq 0$. Then $(a, 0) \in X$ and $(0, a) \in Y$, so $(a, 0), (0, a) \in X \cup Y$ by definition of union. However, $(a, 0) + (0, a) = (a, a) \notin X$ and $(a, a) \notin Y$, so $(a, a) \notin X \cup Y$. Hence $X \cup Y$ is not closed under addition, so it is not a subspace of V.

Exercise 7

Give a bijective proof that $2^n = \sum_{k=0}^n \binom{n}{k}$.

In order to show that $2^n = \sum_{k=0}^n \binom{n}{k}$, we will construct two sets A and B, with $|A| = 2^n$ and $|B| = \sum_{k=0}^n \binom{n}{k}$. Then we will give a bijection between these sets, showing that they have the same cardinality; this will imply that $2^n = \sum_{k=0}^n \binom{n}{k}$.

Let $A = \{a = (a_1, \ldots, a_n) : \forall i \in \{1, \ldots, n\}, a_i \in \{0, 1\}\}$; A is the set of ordered n-tuples with all elements either 0 or 1. There are 2^n such tuples, since there are two possibilities for each element, and so $|A| = 2^n$.

Now let $B = \{S : S \subseteq \{1, \ldots, n\}\}$; B is the set of all subsets of $\{1, \ldots, n\}$. Since $\binom{n}{k}$ is the number of subsets of $\{1, \ldots, n\}$ of size k, then the total number of subsets of $\{1, \ldots, n\}$ is $\sum_{k=0}^n \binom{n}{k}$. Thus $|B| = \sum_{k=0}^n \binom{n}{k}$.

Now we define a function $f : A \to B$ by $f(a_1, \ldots, a_n) = \{i : a_i = 1\}$; that is, the element $a = (a_1, \ldots, a_n)$ of A is mapped to the subset of $\{1, \ldots, n\}$ which contains i for which $a_i = 1$ and does not contain i for which $a_i = 0$.

We will show that this function is injective and surjective.

Assume we have $a, a' \in A$ such that $f(a) = f(a')$. Then for each $i \in \{1, \ldots, n\}$, the fact that $i \in f(a)$ if and only if $i \in f(a')$ implies that in fact $a_i = 1$ if and only if $a'_i = 1$, by the definition of f. Since each a_i and a'_i is either 1 or 0, this implies that in fact $a_i = a'_i$ for all i, so $a = a'$. Hence f is injective.

Now take some $S \in B$; by the definition of B, $S \subseteq \{1, \ldots, n\}$. We define $s = (s_1, \ldots, s_n) \in A$ by $s_i = 1$ if $i \in S$ and 0 if $i \notin S$ for all $i \in \{1, \ldots, n\}$. Then $f(s) = S$ by the definition of f, so in fact f is surjective.

Hence f is bijective, so $|A| = |B|$; this means that $2^n = \sum_{k=0}^{n} \binom{n}{k}$ as we desired to show.

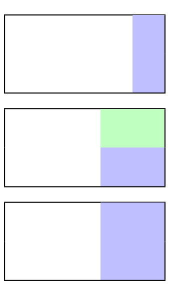
Exercise 8

Let a_n be the number of ways of tiling a $2 \times n$ board by 1×2 rectangles and 2×2 squares. For example, $a_1 = 1$ and $a_2 = 3$.

(a) Find a recursive formula for a_n.

(b) Find an explicit formula for a_n.

(a) Consider the following pictures:



These represent the three possibilities for the final blocks of the tiling: either the last block is a single 1×2 rectangle, or two 1×2 rectangles, or a single 2×2 square.

Therefore, for $n > 2$, the number a_n of such tilings can be found by counting up how many tilings there are in each of the three categories above.

In the first case, with a single 1×2 block at the end, the number of such tilings will be a_{n-1}, since the rest of the $2 \times n$ board may be tiled however we want (and what is left except the final block is a $2 \times n - 1$ board).

Similarly, in the second and third cases both, the number of such tilings will be a_{n-2} each.

This gives us the recursive formula $a_n = a_{n-1} + 2a_{n-2}$ for $n > 2$, with $a_1 = 1$ and $a_2 = 3$.

(b) We now wish to find an explicit formula for the a_n. We will use the method of generating functions.

Define

$$A(x) = \sum_{n=1}^{\infty} a_n x^n = x + 3x^2 + \sum_{n=3}^{\infty} a_n x^n.$$

Then by the recurrence $a_n = a_{n-1} + 2a_{n-2}$ for $n > 2$, we make the following manipulations:

$$\begin{aligned}
A(x) &= x + 3x^2 + \sum_{n=3}^{\infty} a_n x^n \\
&= x + 3x^2 + \sum_{n=3}^{\infty} (a_{n-1} + 2a_{n-2}) x^n \\
&= x + 3x^2 + \left(\sum_{n=3}^{\infty} a_{n-1} x^n \right) + \left(\sum_{n=3}^{\infty} 2a_{n-2} x^n \right) \\
&= x + 3x^2 + \left(\sum_{n=2}^{\infty} a_n x^{n+1} \right) + \left(\sum_{n=1}^{\infty} 2a_n x^{n+2} \right) \\
&= x + 3x^2 + x(A(x) - a_1 x) + 2x^2 A(x) \\
A(x) &= x + 3x^2 + xA(x) - x^2 + 2x^2 A(x) \\
A(x) - xA(x) - 2x^2 A(x) &= x + 2x^2 \\
A(x)(1 - x - 2x^2) &= x + 2x^2 \\
A(x) &= \frac{x + 2x^2}{1 - x - 2x^2}
\end{aligned}$$

We note that $1-x-2x^2$ factors as $(1-2x)(1+x)$. Then via partial fraction decomposition, we see that

$$\frac{x + 2x^2}{(1-2x)(1+x)} = x \cdot \left(\frac{-1/3}{1+x} + \frac{4/3}{1-2x} \right).$$

Now we use known expressions for the generating function of geometric series to see that:

$$
\begin{aligned}
A(x) &= \frac{x+2x^2}{1-x-2x^2} \\
&= x \cdot \left(\frac{-1/3}{1+x} + \frac{4/3}{1-2x} \right) \\
&= x \cdot \left(\frac{-1/3}{1+x} \right) + x \cdot \left(\frac{4/3}{1-2x} \right) \\
&= x \cdot \left(\sum_{n=0}^{\infty} \frac{-1}{3} \cdot (-1)^n x^n \right) + x \cdot \left(\sum_{n=0}^{\infty} \frac{4}{3} \cdot 2^n x^n \right) \\
&= \left(\sum_{n=0}^{\infty} \frac{1}{3} \cdot (-1)^{n+1} x^{n+1} \right) + \left(\sum_{n=0}^{\infty} \frac{2}{3} \cdot 2^{n+1} x^{n+1} \right) \\
&= \left(\sum_{n=1}^{\infty} \frac{1}{3} \cdot (-1)^n x^n \right) + \left(\sum_{n=1}^{\infty} \frac{2}{3} \cdot 2^n x^n \right) \\
A(x) &= \sum_{n=1}^{\infty} \left(\frac{1}{3} \cdot (-1)^n + \frac{2}{3} \cdot 2^n \right) x^n
\end{aligned}
$$

Then since $A(x) = \sum_{n=1}^{\infty} a_n x^n$, we may equate coefficients of x^n on both sides of the above equation to see that

$$
a_n = \frac{1}{3} \cdot (-1)^n + \frac{2}{3} \cdot 2^n.
$$