Algebra problems
Math 61DM Homework 7

Due in lecture, Monday Nov. 12

1. Use 2 methods, (i) by computing $\det A$ and $\operatorname{adj} A$, (ii) using elementary row operations to reduce $A|I$ to $I|B$, to calculate the inverse of the 3×3 matrix $\begin{pmatrix} 1 & 4 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 1 \end{pmatrix}$.

2. Let $\Delta = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}$, called a "Vandermonde determinant". Show that Δ is the product of all possible factors $x_j - x_i$ with $1 \le i < j \le n$; that is, $\Delta = \Pi_{1 \le i < j \le n}(x_j - x_i)$.

Hint: Consider using row operations to reduce the n-variable case to the $(n-1)$-variable case.

3. Suppose A, B are $n \times n$ matrices and $AB = I$. Prove that then $\det A \neq 0$ and $B = (\det A)^{-1} \operatorname{adj} A$. (In particular $AB = I \Rightarrow BA = I$ and B is the unique inverse $(\det A)^{-1} \operatorname{adj} A$).

4. Let $A_1, \ldots, A_m$ be distinct subsets of $[n]$. Assume that their pairwise symmetric differences have only two sizes (so there are a and b such that $|A_i \triangle A_j| = a$ or b for all $1 \le i < j \le m$). Prove that $m \le 1 + \frac{n(n+1)}{2}$. Give an example of such a family of size $m = 1 + \frac{n(n-1)}{2}$.

The symmetric difference $A \triangle B$ of two sets A, B is the set $(A \setminus B) \cup (B \setminus A)$ of elements in A or B, but not both.

5. Let S be a subset of $\mathbb{F}_3^n$ and suppose that for every pair of distinct vectors $u, v \in S$ there is an index i for which $v_i \equiv u_i + 1 \pmod 3$. Show that $|S| \le 2^n$.

Here u_i is the i-th coordinate of u, and v_i is the i-th coordinate of v. Also, the above property (that $v_i \equiv u_i + 1 \pmod 3$ for some i) holds for all $2\binom{|S|}{2}$ ordered pairs (u, v) with $u, v \in S$ distinct.

6. A spherical 2-distance set S is a collection of points in $\mathbb{R}^n$ such that each has distance one from the origin and the pairwise distances of points in S take on only two values. Show that $|S| \le n(n+3)/2$.

You can refer to M15 in explaining your answer.

Remark: There are beautiful geometric examples showing this bound is tight for $n = 2, 6, 22$.

7. For p prime and $n = 4p$, prove that every coloring of the points of $\mathbb{R}^n$ with at most 1.1^n colors has two points of Euclidean distance one with the same color.

Hint: you can use results/ideas from M17 and M18.