Mathematics Department Stanford University
Math 61DM Homework 6

Due at Lecture, Monday Nov. 5

1. (a) If S is the 2 dimensional subspace of $\mathbb{R}^4$ spanned by the vectors $(1,1,0,0)^{\mathrm{T}}, (0,0,1,1)^{\mathrm{T}}$, find an orthonormal basis for S and find the matrix of the orthogonal projection of $\mathbb{R}^4$ onto S.

(b) If S is the subspace of $\mathbb{R}^4$ spanned by the vectors $(1,0,0,1)^{\mathrm{T}}, (1,1,0,0)^{\mathrm{T}}, (0,0,1,1)^{\mathrm{T}}$, find an orthonormal basis for S, and find the matrix of the orthogonal projection of $\mathbb{R}^4$ onto S.

2. Showing all row operations, calculate the determinant of $\begin{pmatrix} 10 & 11 & 12 & 13 & 426 \\ 2000 & 2001 & 2002 & 2003 & 421 \\ 2 & 2 & 1 & 0 & 419 \\ 100 & 101 & 101 & 102 & 2000 \\ 2003 & 2004 & 2005 & 2006 & 421 \end{pmatrix}$.

3. This problem asks you to prove the factor theorem.
(i) Suppose $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ is a degree d polynomial (so $a_d \neq 0$) with coefficients in a field $\mathbb{F}$. Prove that the number of zeros (elements $x \in \mathbb{F}$ with $f(x) = 0$) is at most d.
(ii) Conversely, if $E \subset \mathbb{F}$ with $|E| \leq d$, then there is a degree d polynomial that vanishes on E, i.e., $f(x) = 0$ for all $x \in E$.

4. Suppose $f(x) = a_d x^d + a_{d-1}x^{d_1} + \cdots + a_1 x + a_0$ is a degree d polynomial with coefficients in $\mathbb{R}$. Prove that $f(x)$ has a nonzero multiple in which all the exponents are prime numbers.

For instance, such a multiple of $f(x) = x^2 - x + 5$ is $x^5 + 4x^3 + 5x^2 = (x^3 + x^2)(x^2 - x + 5)$. You may assume that that there are infinitely many prime numbers.

5. Let σ be the permutation which sends the ordered 5-tuple $(1\,2\,3\,4\,5)$ to $(5\,4\,3\,2\,1)$.
(a) Prove that if we write σ as the product of adjacent transpositions, then the minimum possible number of such adjacent transpositions is 10.
(b) Prove that if we write σ as the product of transpositions, then the minimum possible number of such transpositions is 2.

6. Recall a *monomial* in n variables $x_1, \ldots, x_n$ of degree d is a polynomial of the simple form $x_1^{a_1} \cdots x_n^{a_n}$ with $a_1 + \cdots + a_n = d$.
(a) Prove that the number of distinct monomials in n variables of degree d is $\binom{d+n-1}{n-1} = \binom{d+n-1}{d}$.
(b) Prove the dimension of the space of polynomials over a field F in n variables of degree at most d is $\binom{d+n}{n} = \binom{d+n}{d}$. (Hint: relate the number of monomials of degree $\leq d$ with the number of monomials of degree d in $n+1$ variables.)

7. Suppose A and B are subsets of $\mathbb{R}^3$ such that the distances between all members of A and all members of B are equal. Assume $|A| \leq |B|$. Explain why $|A| \leq 2$. Your explanation may be geometric in nature.