Math 235a Homework # 1.

Due in class on May 13.

1. Prove that every graph on n vertices contains a path with three edges, or its connected components are triangles or stars. Deduce that any graph on n vertices with more than n edges contains a path with three edges.

2. Prove that every graph on n vertices with minimum degree greater than $n/2$ contains a cycle of length n.

3. Prove that every graph on n vertices contains an ϵ-regular pair of subsets each of order at least δn with $\delta = 2^{-\epsilon^{-O(1)}}$.

4. A subset $X \subset V(G)$ of vertices of a graph is *ϵ-regular* if the pair (X, X) is ϵ-regular.
Prove that for each $\epsilon > 0$ there is $\delta > 0$ such that every graph on n vertices contains a subset of vertices on at least δn vertices which is ϵ-regular.

Note: It is open as to whether or not we can take δ to be single-exponential, i.e., $\delta = 2^{-\epsilon^{-O(1)}}$. The best known bound is double-exponential.

5. Prove that every 3-uniform hypergraph on n vertices in which no six vertices contains three edges has $o(n^2)$ edges.

6. Prove that for each $\epsilon > 0$ and graph H there is $\delta > 0$ such that every graph on n vertices contains an induced copy of H or has a vertex subset of size at least δn with edge density at most ϵ or at least $1 - \epsilon$.

Note: It is open as to whether or not we can take δ to be polynomial, i.e., $\delta = \epsilon^{C(H)}$ for some $C(H)$ depending only on H. A polynomial bound would imply the Erdős-Hajnal conjecture, that every induced H-free graph on n vertices contains a clique or independent set of order $n^{c(H)}$ for some $c(H) > 0$.

7. Prove that for any $\epsilon > 0$ there is $\delta > 0$ such that if a graph on n vertices has at least ϵn^2 edges, then it contains a d-regular subgraph with $d \geq \delta n$.