Math 110 Homework # 3.

Due in class on Friday, May 24. Show your work.

1. Suppose that for some positive integer n that $6n+1$, $12n+1$, and $18n+1$ are all prime. Show that the product $(6n+1)(12n+1)(18n+1)$ is a Carmichael number.

2. Let p be an odd prime. Use quadratic reciprocity to characterize for which odd primes p is -3 a quadratic residue mod p.

3. Use quadratic reciprocity to determine if 65 is a quadratic residue mod 127 (you can use the fact that 127 is prime and $65 = 5 \cdot 13$ is the prime factorization of 65.

4. Let a_n be the number of ways of tiling $2 \times n$ board by 1×2 and 2×2 rectangles. For example, $a_1 = 1$ and $a_2 = 3$. Find a recursive formula for a_n.

5. Explain why the following identity holds:

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k}.$$

6. Using the inclusion-exclusion principle, determine the number of positive integers up to 10,000 which are not divisible by 2, 3, 5, or 7.

7. How many 5-smooth positive integers are there up to $1,000$?

7. Use the pigeonhole principle to show that any subset of the first 1000 positive integers of size 751 contains four consecutive integers.