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CONTRACTING WITH EXTERNALITIES\*

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The paper studies contracting between one principal and  $N$  agents in the presence of multilateral externalities. When the principal commits to publicly observed bilateral contracts, inefficiencies arise due to the externalities on agents' reservation utilities. In contrast, when the principal's offers are privately observed, inefficiencies are due to the externalities at efficient outcomes. When the principal can condition her trade with each agent on others' messages, she implements an efficient outcome, while threatening deviators with the harshest possible punishment. However, in the presence of noise that goes to zero more slowly than  $N$  goes to infinity, asymptotically agents become nonpivotal, and inefficiency obtains.

INTRODUCTION

Many contracting situations involve multilateral externalities. To give a few examples, a shareholder tendering his shares to a superior corporate raider has a positive externality on other shareholders [Grossman and Hart 1980], a creditor exchanging debt for equity in a distressed firm has a positive externality on the firm's other creditors [Gertner and Scharfstein 1991], a buyer of a VCR has a positive "network" externality on owners of compatible VCRs [Katz and Shapiro 1986b], a merger of competing firms has a positive externality on other competing firms

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[Lewis 1983], a private contributor to a public good has a positive externality on other consumers of the good [Bergstrom, Blume, and Varian 1986], a buyer signing an exclusive dealing contract that hinders competition imposes a negative externality on other buyers [Rasmusen, Ramseyer, and Wiley 1991], a downstream firm purchasing an intermediate input from a manufacturer imposes a negative externality on competing firms [Hart and Tirole 1990; Katz and Shapiro 1986a], a principal designing an incentive scheme in a common agency setting imposes an externality on other principals dealing with the same agent [Pauly 1974; Bernheim and Whinston 1986]. In all these situations it has been shown that even when all agents participate in contracting, it may fail to internalize externalities and may yield inefficient outcomes. However, the connections among existing models, and the general nature of arising inefficiencies, have not been well understood.

This paper develops and studies a general model of contracting with externalities which unifies the above examples. In the model, outlined in Section I, one party (the principal) makes contract offers to  $N$  other parties (agents). The utility of each agent depends on all agents' trades with the principal. This model incorporates many existing models, described in Section II, as special cases.

Section III studies a contracting game in which the principal commits to a set of publicly observed bilateral contract offers. Since the principal's profit can be written as the difference between total surplus and the sum of agents' reservation utilities, contracting distortions are due to the principal's incentive to reduce these reservation utilities. Therefore, inefficiencies arise whenever externalities on *nontraders* (i.e., on agents' reservation utilities) are present.

To identify the effect of the principal's rent-extraction motive on the contracting outcome, I assume that the total surplus depends only on the aggregate trade (the sum of all agents' trades), and that all agents' trades are measured in identical increments. Under these assumptions, which are satisfied in almost all applications, the aggregate trade is socially insufficient or excessive depending on whether the externalities on nontraders are positive or negative. Using the techniques of monotone comparative statics [Topkis 1998; Milgrom and Shannon 1994], I establish this and subsequent results in considerable generality, thus unifying and systematizing existing results in specific applications.

Section IV studies a contracting game in which the principal's offer to each agent is privately observed by this agent. As the principal is now unable to commit to compensate an agent for the externalities imposed on him, she has an incentive to deviate from an efficient trade profile whenever externalities are present at this trade profile. In particular, under the same assumptions as before, the aggregate trade is insufficient or excessive depending on whether the externalities *at the efficient trade profiles* are positive or negative. The outcomes of private contracting are compared with those of public contracting in Section V. Not surprisingly, the comparison hinges on the relative magnitudes of the externalities on agents who trade with the principal and those on nontraders.

In the rest of the paper I return to the assumption that the principal is able to make public commitments, but allow her to commit to a mechanism in which her trade with each agent may be contingent on other agents' messages. Examples of such mechanisms studied in the literature include auctions [Katz and Shapiro 1986a] and conditional bids [Bagnoli and Lipman 1988]. Section VI studies conditions under which the results of Section III generalize to such contracting situations. It shows that if the principal is restricted to choose from a family of mechanisms in which agents' participation constraints bind, she again has an incentive to distort the outcome to reduce agents' reservation utilities. On the other hand, if the principal's choice of mechanism is not restricted, she optimally offers the agents an efficient trade profile, while threatening any agent who rejects the mechanism with the harshest possible punishment. In this way, the principal maximizes total surplus and minimizes agents' reservation utilities at the same time.

The principal's fully optimal mechanism makes each agent pivotal, which seems implausible when the number of agents is large. Some studies instead assume that small agents are nonpivotal, i.e., take the aggregate trade as given, which generally yields inefficient contracting outcomes. However, this assumption has generated much controversy in the setting of takeovers. Section VII clarifies the issue, by providing two sets of conditions under which Grossman and Hart's [1980] assumption that small shareholders are nonpivotal is justified. The first result shows that if the principal is restricted to use mechanisms in which small agents have a small effect on the aggregate trade (which include bilateral contracts), then asymptotically a nonpivotal outcome

obtains, *provided that agents' payoffs are continuous in the aggregate trade*. Therefore, a raider's ability to appropriate some of the takeover surplus with bilateral contracting discovered by Bagnoli and Lipman [1988] and Holmstrom and Nalebuff [1990] is due to their assumption that the firm's value is discontinuous in the raider's stake.

The second result of Section VII shows that even when agents' payoffs are discontinuous or the principal can use discontinuous mechanisms, such as conditional bids [Bagnoli and Lipman 1988], small agents will still be asymptotically nonpivotal as long as there is some noise in the execution of the mechanism. Specifically, I suppose that each agent is exogenously unable to respond to the principal's offer with a small probability  $\epsilon_N$  (for example, his acceptance message may be lost in the mail). As  $N \rightarrow \infty$  and  $\epsilon_N \rightarrow 0$  in such a way that  $N\epsilon_N \rightarrow \infty$ , asymptotically a nonpivotal outcome obtains. This result, which supports and extends the conjecture of Grossman and Hart [1980], demonstrates that contracting inefficiencies with a large number of agents are robust to the introduction of general contracting mechanisms, given small frictions in the execution of these mechanisms.

### I. THE MODEL

Consider a model in which one party, "the principal," can contract with  $N$  other parties, "agents."<sup>1</sup> (With a slight abuse of notation,  $N$  will represent the *set* as well as the *number* of agents.) The principal's "trade" with each agent  $i$  is denoted by  $x_i \in \mathcal{X}_i$ , where  $\mathcal{X}_i$  is a compact subset of the set  $\mathfrak{R}_+$  of nonnegative real numbers, with  $0 \in \mathcal{X}_i$ . Let the vector  $x = (x_1, \dots, x_N) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_N$  denote the agents' trade profile. Externalities among agents arise because each agent's utility depends not only on his own trade with the principal, but also on other agents' trades. Namely, each agent  $i$ 's payoff is  $u_i(x) - t_i$ , and the principal's payoff is  $f(x) + \sum_i t_i$ , where  $t_i \in \mathfrak{R}$  is the monetary transfer from agent  $i$  to the principal. The default ("no trade") point for each agent  $i$  is  $t_i = x_i = 0$ .

In some applications described in the next section, the principal "sells"  $x_i$  to agent  $i$ , in which case both  $u_i(x)$  and  $[-f(x)]$  (the principal's cost of producing  $x$ ) are increasing in  $x_i$ , and one

1. For lack of better unifying terminology, these labels are used to reflect the paper's focus on games in which the "principal" offers contracts to "agents."

can expect agent  $i$ 's payment  $t_i$  to be positive. In other applications, the principal “buys”  $x_i$  from agent  $i$ , in which case  $[-u_i(x)]$  and  $f(x)$  (the principal's net benefit of  $x$ ) are increasing in  $x_i$ , and one can expect  $t_i$  to be negative. This distinction between buying and selling will prove immaterial for the results to follow. What *will* be important is the dependence of an agent's utility on *other agents'* trades; i.e., whether externalities are positive or negative.

For future reference, let  $\mathfrak{M}^*$  denote the set of trade profiles maximizing the total surplus of the  $N + 1$  parties:

$$(1) \quad \mathfrak{M}^* = \arg \max_{x \in \mathfrak{X}_1 \times \dots \times \mathfrak{X}_N} f(x) + \sum_i u_i(x).$$

The set  $\mathfrak{M}^*$  will serve as a benchmark against which contracting outcomes are compared, and the contracting parties' failure to maximize their joint surplus will be referred to as “inefficiency.”<sup>2</sup>

Many of this paper's results will use additional structure of the parties' payoffs and trade domains. Here I state some assumptions that will prove useful. In the next section I will point out which assumptions are satisfied in specific applications.

CONDITION W.  $f(x) + \sum_i u_i(x) = W(\sum_i x_i)$ .

In words, Condition W (for “welfare”) says that the total surplus is a function  $W(X)$  of the aggregate trade  $X = \sum_i x_i$  only. In most applications, Condition W will hold by virtue of the parties' payoffs satisfying the following “linearity” condition:

CONDITION L.  $f(x) = F(X)$  and  $u_i(x) = x_i \alpha(X) + \beta_i(X)$  for all  $i \in N$ , where  $X = \sum_j x_j$ .

Indeed, under Condition L we have  $W(X) = F(X) + X \alpha(X) + \sum_i \beta_i(X)$ .

CONDITION D. Either  $\mathfrak{X}_i = [0, \bar{x}_i]$  or  $\mathfrak{X}_i = \{kz : k = 0, 1, \dots, \bar{k}_i\}$  for all  $i \in N$ ,

Condition D (for “domains”) says that all agents' trades are measured in the same increments, which could be either infinitesimal or finite.

2. In some economic applications, the contracting outcome affects the welfare of some parties who do not participate in contracting (e.g., final consumers in vertical contracting [Hart and Tirole 1990; Katz and Shapiro 1986a] and mergers for monopoly [Lewis 1983]). In such situations our notion of “efficiency” does not have a normative appeal.

CONDITION S.  $x_i \in [0,1]$  for all  $i$ . Furthermore, for any trade profile  $x \in [0,1]^N$  and any permutation  $\pi$  of  $N$ , letting  $x_\pi$  denote the permuted trade profile  $(x_{\pi(1)}, \dots, x_{\pi(N)})$ , we have  $f(x_\pi) = f(x)$ , and  $u_i(x_\pi) = u_{\pi(i)}(x)$  for all  $i$ .

The second part of Condition S (for “symmetry”) states that agents are identical, in the sense that the parties’ payoffs are symmetric with respect to permutations of agents. Note that Condition S implies all the other conditions.<sup>3</sup>

## II. APPLICATIONS

*A1: Vertical Contracting* [Hart and Tirole 1990; O’Brien and Schaffer 1992; McAfee and Schwartz 1994; Rey and Tirole 1996; Katz and Shapiro 1986a; Kamien, Oren, and Tauman 1992]. The principal supplies an intermediate good to  $N$  agents (downstream firms), who then produce substitute consumer goods.  $x_i \geq 0$  is firm  $i$ ’s purchase of the intermediate good, and  $t_i$  is its payment to the supplier. Due to downstream competition, each firm  $i$ ’s utility  $u_i(x_i, x_{-i})$  is decreasing in other firms’ purchases  $x_{-i}$ .

In the more specialized models of Hart and Tirole [1990] and Rey and Tirole [1996], downstream firms produce a homogeneous final good, transforming each unit of the intermediate good into a unit of the final good at a cost  $c$ . After purchasing their inputs  $x_i$ , the firms play the Bertrand-Edgeworth game of downstream price competition with capacity constraints. Assuming that all purchased inputs are utilized in equilibrium,<sup>4</sup> and letting  $P(\cdot)$  denote the inverse demand function for the final good, each firm  $i$ ’s profit is given by  $u_i(x) = [P(X) - c]x_i$ , where  $X = \sum_j x_j$ . Therefore, the parties’ payoffs satisfy Conditions L and W.

While the first four papers assume that a downstream firm cannot produce without using the principal’s input, and therefore  $u_i(0, x_{-i}) \equiv 0$ , Katz and Shapiro [1986a] and Kamien, Oren, and Tauman [1992] study models in which each downstream firm has access to an inferior technology which does not use the principal’s input. In these models,  $u_i(0, x_{-i})$  can be positive and can depend on  $x_{-i}$  (the importance of this will be shown in Section III). The two

3. Indeed, the imposed symmetry among agents implies that the total surplus and the principal’s profit are fully determined by the number  $X = \sum_j x_j$  of agents who have  $x_j = 1$ . Similarly, by considering all permutations  $\pi$  that hold  $i$  fixed, it is easy to see that agent  $i$ ’s utility can be written as  $U_i(x_i, X)$ . In addition, Condition S implies that the function  $U_i(\cdot, \cdot)$  is the same for all agents. Condition L then follows since any function of  $x_i \in [0,1]$  is linear in  $x_i$ . Condition D also follows trivially.

4. See Tirole [1988, Ch. 5] for more detail.

papers assume that the intermediate good is a fixed input (a license to use the principal's patent), and that downstream firms are identical. Then Condition S holds, and with it all the other conditions.

*A2: Exclusive Dealing* [Rasmusen, Ramseyer, and Wiley 1991; Segal and Whinston forthcoming]. The principal is an incumbent monopolist who offers exclusive dealing contracts to  $N$  identical buyers (agents). The contract obliges a buyer not to buy from a rival seller. Let  $x_i \in [0,1]$  indicate whether buyer  $i$  signs such a contract, and  $(-t_i)$  be the compensation paid to him by the incumbent. After observing the number  $X$  of signers, a potential entrant decides whether to enter. Due to the entrant's economies of scale, the probability of entry,  $\rho(X)$ , is a nonincreasing function of  $X$ . In the case of no entry, in the second stage the incumbent makes the monopoly profit  $\pi^m$  on each buyer by charging him the monopoly price  $p^m$ . In the case of entry, the entrant and incumbent compete for the buyers who have not signed in the first stage, and the incumbent, whose marginal cost is higher than the entrant's, makes no profit on these buyers. The incumbent still charges  $p^m$  to the buyers who have signed exclusives, and earns  $\pi^m$  on each of them.

Since all buyers are identical, Condition S holds, and with it all the other conditions. The incumbent's net profit can be written as  $f(x) = [\rho(X)X + (1 - \rho(X))N]\pi^m$ . Normalizing each buyer's surplus under price  $p^m$  to zero, and letting  $b$  denote his surplus under the competitive price, his utility can be written as  $u_i(x) = (1 - x_i)\rho(X)b$ . Since this utility is nonincreasing in  $X$ , by signing an exclusive contract, a buyer imposes a negative externality on other buyers.

*A3: Selling an Indivisible Object* [Jehiel and Moldovanu 1996; Jehiel, Moldovanu, and Stachetti 1996]. The principal has one unit of an indivisible good for sale to the agents. (This could, for example, be a nuclear weapon, patent, or asset to be sold to one of several competing nations or firms.) Let  $x_i \in [0,1]$  indicate whether agent  $i$  receives this good, and  $t_i$  denote the agent's payment to the principal. The principal's inability to sell more than one unit can be modeled by setting  $f(x) = -K$ , with  $K$  very large, whenever  $\sum_j x_j \geq 2$ . Letting  $\gamma_{ij}$  denote the utility of agent  $i$  if agent  $j$  gets the good, we can write  $u_i(x) = \sum_j \gamma_{ij}x_j$ . The literature studies situations with "identity-dependent" externalities, in which the utility  $\gamma_{ij}$  of agent  $i$  depends on which agent  $j \neq i$  receives the good. Such situations do not satisfy Condition W.



*A4: Common Insurance* [Pauly 1974]. The principal is a risk-averse individual who contracts with  $N$  risk-neutral insurance firms (agents). There are two possible outcomes: the “accident” state, in which the individual suffers a monetary loss  $a > 0$ , and the “no accident” state, in which she suffers no loss. The individual’s insurance contract with each firm  $i$  specifies a premium  $(-t_i)$  and a compensation  $x_i \geq 0$  in the “accident” state.

Externalities among insurance firms are due to the individual’s moral hazard. For example, suppose that the individual chooses the probability  $\rho$  of accident, at a private monetary cost  $c(\rho)$ . If her preferences satisfy Constant Absolute Risk Aversion,<sup>5</sup> her certainty equivalent is

$$v(\rho, X) - c(\rho) + \sum_i t_i \quad \text{with } v(\rho, X) = -\frac{1}{r} \log [1 - \rho + \rho e^{-r(X-a)}],$$

where  $X = \sum_i x_i$  and  $r > 0$  is her coefficient of absolute risk aversion. Letting  $\rho^*(X)$  denote the individual’s optimal choice of  $\rho$ , the parties’ expected utilities net of lump-sum transfers are given by  $f(x) = v(\rho^*(X), X) - c(\rho^*(X))$  and  $u_i(x) = -\rho^*(X)x_i$ . These payoffs satisfy Conditions L and W. Since  $\rho^*(X)$  is nondecreasing in  $X$ ,<sup>6</sup> by increasing the individual’s insurance and thereby raising the probability of accident, each company imposes a negative externality on other companies.

*A5: Common Agency* [Bernheim and Whinston 1986].<sup>7</sup> Modify the previous model in two respects. First, the individual herself suffers no monetary loss in either state (i.e.,  $a = 0$ ). Second, in one of the states, called the “good” state, each firm  $i$  receives a benefit  $b_i$ . As in the previous application, the individual’s contract with each firm  $i$  specifies a lump-sum payment  $t_i$  to the individual and a “bonus” payment  $x_i$  in the “good” state. Unlike in the previous application, however, the motivation for contracting is not to insure the individual, but to make her choose an action that is more desirable for the firms.

The parties’ expected utilities net of lump-sum transfers are

5. Pauly considers more general risk preferences, which may exhibit wealth effects (in particular, the individual’s optimal choice of  $\rho$  may depend on  $\sum_i t_i$ ). Wealth effects are neglected in this paper.

6. This can be seen, for example, by observing that the individual’s utility is strictly supermodular in  $(\rho, X)$ , and applying the Monotone Selection Theorem [Milgrom and Shannon 1994].

7. Common agency models in which asymmetric information is present at the contracting stage (see, e.g., Martimort and Stole [1994]) fall outside of this paper’s framework.



given by  $f(x) = v(\rho^*(X), X) - c(\rho^*(X))$  and  $u_i(x) = \rho^*(X)[b_i - x_i]$ . These payoffs satisfy Conditions L and W. Assume that  $b_i \geq 0$  for all  $i$ , and restrict attention to contracts with  $x_i \in [0, b_i]$  (higher bonuses will not arise in equilibrium). Then by increasing its bonus and thereby raising the probability of the good state, each firm has a positive externality on other firms.<sup>8</sup>

*A6: Takeovers* [Grossman and Hart 1980; Bagnoli and Lipman 1988; Holmstrom and Nalebuff 1992; Burkart, Gromb, and Panunzi, 1998]. The principal is a corporate raider, who makes a tender offer to  $N$  shareholders (agents).  $x_i \geq 0$  is the number of shares tendered by shareholder  $i$ , and  $(-t_i)$  is the raider's payment to this shareholder. Let  $v(X)$  denote the expected value of the firm's public shares as a function of  $X$ , the total number of shares tendered. Finally, let  $c(X)$  denote the raider's "transaction cost" of acquiring  $X$  shares (it could also be negative, reflecting her private benefit from controlling  $X$  shares). Then the parties' utilities net of monetary transfers are given by  $f(x) = Xv(X) - c(X)$  and  $u_i(x) = (\bar{x}_i - x_i)v(X)$ , where  $\bar{x}_i$  is shareholder  $i$ 's endowment of shares. These payoffs satisfy Conditions L and W.

Say that the raider is *superior* if  $v(X)$  is nondecreasing in  $X$ , which may be due to the raider's greater ability or incentive to enhance the firm's value when she holds a larger stake in the firm. In this case, a tendering shareholder has a positive externality on other shareholders. On the other hand, say that the raider is *inferior* if  $v(X)$  is nonincreasing in  $X$ , which may be due to the raider's greater ability to "loot" the firm's assets or "freeze out" other shareholders when she owns a larger stake in the firm. In this case, a tendering shareholder imposes a negative externality on other shareholders.

*A7: Debt Workouts* [Gertner and Scharfstein 1991]. The principal is the collective of shareholders of a financially distressed firm, which offers a debt-equity swap to the firm's creditors (agents), assumed to be of equal seniority. Let  $\bar{x}_i$  denote the face value of debt initially held by creditor  $i$ , and suppose that the swap results in the creditor forgiving an amount  $x_i \in [0, \bar{x}_i]$  of debt in exchange for an equity stake  $s_i$  in the firm. Let  $v(X)$  denote the expected value of the firm's equity, and  $d(X)$  denote the expected

8. The model of Bernheim and Whinston [1986] is substantially more general. First, the individual's risk preferences may exhibit wealth effects (see footnote 5). Second, there may be more than two possible outcomes, which requires considering contracts of more than two dimensions. Third, firms' preferences over outcomes may diverge, e.g., we may have  $b_i < 0$  for some firms, in which case contracting may have negative, as well as positive, externalities.

value of a \$1 face value of debt, as functions of the total amount  $X$  of debt tendered. Then we can write the shareholders' payoff as  $(1 - \sum_i s_i)v(X)$ , and each creditor  $i$ 's payoff as  $(\bar{x}_i - x_i)d(X) + s_i v(X)$ . If offers are made publicly, each creditor  $i$  knows the value  $t_i = s_i v(X)$  of the equity stake offered to him. In this notation, the parties' payoffs satisfy Conditions L and W.

The function  $d(X)$  can be expected to be increasing, since a less leveraged firm (i) has a greater debt repayment ratio in each state of the world, *ceteris paribus*, and (ii) is less prone to engage in rent-seeking activities, such as risky investments or costly bankruptcy. Thus, a creditor who exchanges debt for equity has a positive externality on other creditors.<sup>9</sup>

*A8: Acquisition for Monopoly* [Lewis 1983; Kamien and Zang 1990; Krishna 1993]. The principal makes acquisition offers to  $N$  capacity owners (agents). These capacities can be used to produce a homogeneous consumer good. Let  $x_i \in [0,1]$  indicate whether owner  $i$  sells his capacity to the principal, and let  $(-t_i)$  be the principal's payment to this owner. If the capacities are identical, the model satisfies Condition S, and with it all the other conditions. An owner who sells his capacity cannot produce—hence  $u_i(1, x_{-i}) \equiv 0$ . Since acquisitions increase the market's concentration and the resulting market price, the profit  $u_i(0, x_{-i})$  of an owner who does not sell his capacity is increasing in other owners' sales  $x_{-i}$ .

*A9: Network Externalities* [Katz and Shapiro 1986b]. The principal is the seller of a good for which each agent (buyer) has a unit demand.  $x_i \in [0,1]$  denotes buyer  $i$ 's purchase of the good, and  $t_i$  denotes his payment to the seller. Since all buyers are assumed to be identical, Condition S is satisfied, and with it all the other conditions. When the seller's good exhibits "network externalities," the utility  $u(1, x_{-i})$  of each consumer  $i$  of the good is increasing in other buyers' purchases  $x_{-i}$ . A consumer who does not buy the seller's good uses an old substitute technology, which we assume here to be "unsponsored," i.e., supplied by a competitive market. If the old technology also exhibits network effects, its users' utility,  $u(0, x_{-i})$ , is decreasing in  $x_{-i}$ . Thus, a consumer who

9. Gertner and Scharfstein [1991] also consider exchanges of debt for cash or senior debt. These exchanges cannot be captured by our simple model, since *both* of the goods exchanged involve externalities on other creditors. For example, a creditor has a positive externality on other creditors by forgiving his debt, but he imposes a negative externality on them by accepting cash or senior debt in exchange. Gertner and Scharfstein find that the net external effect of such exchanges may be negative.

buys from the seller has a positive externality on other consumers who buy, but a negative externality on those who do not buy.

*A10: Pure Public Good/Bad* [Bergstrom, Blume, and Varian 1986; Neeman 1997]. The principal is a provider of a public good/bad, who can contract with  $N$  consumers of the good (agents). Let  $x_i \geq 0$  be the amount of the public good/bad “contributed by agent  $i$ ,” and let  $t_i$  be agent  $i$ 's payment to the provider. Then each consumer  $i$ 's utility net of monetary transfers is a function  $v_i(X)$  of the total provision  $X$  of the public good/bad. The model satisfies Conditions L and W. In the public good model of Bergstrom, Blume, and Varian,  $v_i(X)$  is increasing in  $X$ ; i.e., each consumer's contribution has a positive externality on other consumers. Neeman considers a formally similar model of “public bads,” in which  $v_i(X)$  is decreasing in  $X$ . His examples include vote trading (voters sell their votes to the principal who implements an inefficient policy) and “yellow dog” employment contracts (workers agree not to join labor unions which would increase their joint bargaining strength).

Note that all the applications satisfy Conditions L, W, and D, with the exception of A1 (Vertical Contracting) in the general case of differentiated final goods, and A3 (Indivisible Object).

### III. BILATERAL CONTRACTING WITH PUBLIC OFFERS

This section analyzes the following two-stage game: in the first stage the principal commits to a set  $\{(x_i, t_i)\}_{i \in N}$  of publicly observable bilateral contract offers to agents. In the second stage agents simultaneously decide whether to accept or reject their respective offers. I will study the principal's preferred Subgame-Perfect Nash Equilibria (SPNE) of the game.<sup>10</sup>

Since the principal can always offer  $(x_i, t_i) = (0, 0)$ , without loss of generality we can restrict attention to equilibria in which every agent accepts his offer. This play constitutes a second-stage Nash equilibrium if and only if the following participation constraints are satisfied:

$$(2) \quad u_i(x) - t_i \geq u_i(0, x_{-i}) \text{ for all } i \in N.$$

10. Segal [1998] shows that the game may have multiple SPNE, and the equilibrium preferred by the agents may differ from that preferred by the principal. However, this section's qualitative results are preserved when agents coordinate on their preferred equilibrium.

The right-hand side of the inequality represents the reservation utility of agent  $i$ , i.e., the utility he would obtain by rejecting his offer, provided that everyone else accepts. In the principal's preferred SPNE, all agents' participation constraints must bind (otherwise her profit could be increased by raising some transfers without upsetting the constraints). Expressing transfers from the binding constraints and substituting them in the principal's objective function, the set of her profit-maximizing trade profiles can be defined as

$$(3) \quad \mathfrak{M} = \arg \max_{x \in \mathfrak{X}_1 \times \dots \times \mathfrak{X}_N} f(x) + \sum_i u_i(x) - \sum_i u_i(0, x_{-i}).$$

The principal's objective function differs from the total surplus in (1) by its last term, which is the sum of the agents' reservation utilities. If each agent's reservation utility does not depend on other agents' trades (in which case we will say that there are no externalities on nontraders), then the profit-maximization program (3) is equivalent to the surplus-maximization program (1), and we have

**PROPOSITION 1.** If  $u_i(0, x_{-i})$  does not depend on  $x_{-i} \in \mathfrak{X}_{-i}$  for all  $i$ , then  $\mathfrak{M} = \mathfrak{M}^*$ .

Intuitively, when externalities on nontraders are absent, and the principal can commit to compensate traders for the externalities imposed on them, she internalizes these externalities and implements efficient outcomes. Externalities on nontraders are absent in application A1 (Vertical Contracting) in the absence of a substitute technology, application A4 (Common Insurance), and application A9 (Network Externalities) when the old technology exhibits no network effects. In these situations the principal's commitment to public offers yields efficient outcomes. Proposition 1 also demonstrates the role of excludability in the provision of Public Goods (A10). By excluding noncontributors, a public good provider eliminates externalities on nontraders and provides an efficient level of the good despite the remaining externalities on contributors.<sup>11</sup>

When externalities on nontraders are present, on the other hand, the principal has an incentive to distort the trade profile  $x$  to

11. The possibility of efficient provision of an excludable public good has also been demonstrated by Moldovanu [1996] in a symmetric-information setting and by Maskin [1994] in a private-information setting.

reduce the sum of agents' reservation utilities. The direction of this distortion will depend on the sign of the externalities:

DEFINITION 1. Externalities on nontraders are positive [negative] if  $u_i(0, x_{-i})$  is nondecreasing [nonincreasing] in  $x_{-i} \in \mathcal{X}_{-i}$  for all  $i$ .

Intuition suggests that with positive [negative] externalities on nontraders, the principal will optimally reduce agents' reservation utilities by trading too little [too much] from the social viewpoint. However, this intuition is not correct in general:

EXAMPLE 1. Let  $N = 2$ ,  $\mathcal{X}_1 = \mathcal{X}_2 = [0, 20]$ ,  $u_i(x_i, x_{-i}) = a_i x_i + b_i x_{-i} - c_i x_i^2 + d_i x_i x_{-i}$  for  $i = 1, 2$ , and  $f(x_1, x_2) \equiv 0$ . Consider the parameterized program,

$$\max_{x_1, x_2 \in [0, 20]} f(x_1, x_2) + \sum_{i=1,2} u_i(x_i, x_{-i}) - z \sum_{i=1,2} u_i(0, x_{-i}).$$

Here  $z = 0$  corresponds to the surplus maximization program (1), and  $z = 1$  corresponds to the principal's profit-maximization program (3). Assume that  $c_i > 0$  for  $i = 1, 2$ , and  $\Delta \equiv 4c_1 c_2 - (d_1 + d_2)^2 > 0$ , so that the objective function is strictly concave. Then, assuming that the solution is interior (which will be true for parameter values suggested below), it is characterized by the following first-order conditions:

$$x_i = \frac{a_i + b_{-i} + dx_{-i} - zb_{-i}}{2c_i} \quad \text{for } i = 1, 2,$$

where  $d = d_1 + d_2$ . Suppose that externalities on nontraders are positive, which here means that  $b_i > 0$  for  $i = 1, 2$ . The first-order conditions then show that, in accordance with intuition, the principal has an incentive to distort  $x_i$  downward *given*  $x_{-i}$ . However, the principal's optimal choice of  $x_i$  depends on  $x_{-i}$ , which will also in general differ from its efficient level. In particular, the above first-order condition shows that when  $d < 0$ , and the principal's choice of  $x_{-i}$  is below its efficient level, she has an incentive to increase  $x_i$ . This indirect effect may outweigh the direct effect, in which case the principal's choice of  $x_i$  *exceeds* the efficient level. Moreover, there exist cases in which the aggregate trade  $X(z) = x_1(z) + x_2(z)$  is increased by moving from  $z = 0$  to  $z = 1$ . To see this, compute  $X'(z) = [-2(b_1 c_1 + b_2 c_2) - (b_1 + b_2)d]/\Delta$ , and consider the following parameter values:  $c_1 = 1$ ,  $c_2 = 9$ ,

$d = -5$ ,  $b_1 = 5$ ,  $b_2 = 1$ ,  $a_1 = 14$ ,  $a_2 = 43$ . For these values, all the above assumptions are satisfied, and  $X'(z) = \frac{2}{11} > 0$ , i.e., the principal implements a socially excessive aggregate trade, despite positive externalities on nontraders.

The example demonstrates that interactions between different dimensions of the principal's maximization program may preclude a definitive comparison with efficient outcomes. It turns out, however, that under Conditions W and D, a comparison can be obtained by reducing the principal's problem to the one-dimensional problem of choosing the aggregate trade  $X$ . Indeed, under Condition W, which requires that the total surplus be a function  $W(X)$  of the aggregate trade  $X$  only, we can use the "aggregation method" of Milgrom and Shannon [1994] to rewrite the principal's profit-maximization program (3) as<sup>12</sup>

$$(4) \quad \max_{X \in \Sigma_i \mathfrak{X}_i} W(X) - R(X),$$

where  $R(X) = \min_{x \in \mathfrak{X} \times \dots \times \mathfrak{X}_N} [\sum_i u_i(0, x_{-i}) : \sum_i x_i = X]$  is the minimum sum of agents' reservation utilities that is consistent with the aggregate trade  $X$ .<sup>13</sup>

A key step in comparing efficient aggregate trades with the principal's profit-maximizing aggregate trades is given by the following lemma:

**LEMMA 1.** If Condition D holds and externalities on nontraders are positive [negative], then  $R(X)$  is nondecreasing [nonincreasing] on its domain.

*Proof.* Consider the case of positive externalities on nontraders. Take any  $X'$ ,  $X \in \Sigma_i \mathfrak{X}_i$ , with  $X' \leq X$ . Take  $x \in \mathfrak{X}_1 \times \dots \times \mathfrak{X}_N$  such that  $\sum_i x_i = X$  and  $R(X) = \sum_i u_i(0, x_{-i})$ . Under Condition D there exists  $x' \in \mathfrak{X}_1 \times \dots \times \mathfrak{X}_N$  such that  $x' \leq x$  and  $\sum_i x'_i = X'$ . With positive externalities on nontraders, this implies that  $R(X') \leq \sum_i u_i(0, x'_{-i}) \leq \sum_i u_i(0, x_{-i}) = R(X)$  whenever  $R(X')$  is defined. The proof for the case of negative externalities is similar. ■

The proof relies on the fact that under Condition D, the principal can always reduce or increase the aggregate trade while weakly reducing or increasing all agents' trades at once. For example, with positive externalities, this implies that the princi-

12. Where  $\Sigma_i \mathfrak{X}_i = \{\sum_i x_i : x \in \mathfrak{X}_1 \times \dots \times \mathfrak{X}_N\}$ .

13. When the functions  $u_i(\cdot)$  are not continuous, the minimum may not exist for some aggregate trade  $X$ . However, note that such aggregate trade cannot arise at a profit-maximizing outcome.

pal can always reduce the aggregate trade while reducing all agents' reservation utilities, and therefore  $R(\cdot)$  is nondecreasing. Note the importance of Condition D for this result. For example, suppose that  $N = 2$ , and that agents 1 and 2 are only allowed to trade in multiples of 3 and 5, respectively. Then  $X = 10$  can be implemented by trading with agent 2 only. In order to reduce  $X$  to 8, the principal would have to increase her trade with agent 1 to three units. With positive externalities this increases agent 2's reservation utility, and possibly the sum of agents' reservation utilities.

Once the dependence of the sum of agents' reservation utilities on the aggregate trade  $X$  is known, the effect of the principal's rent-seeking incentive on  $X$  is immediate. To formulate the result, let  $M^* = \{\sum_i x_i : x \in \mathcal{M}^*\}$  be the set of efficient aggregate trades, and  $M = \{\sum_i x_i : x \in \mathcal{M}\}$  be the set of the principal's profit-maximizing aggregate trades. The assumptions required to ensure that the sets  $M$  and  $M^*$  are single-valued (such as strict concavity of the objective function and convexity of the feasible set) would not be natural in many applications. Instead of imposing such assumptions, I compare the two sets using the *strong (induced) set order* (see, e.g., Milgrom and Shannon [1994] and Topkis [1998]). Namely, for two sets  $A, B$ , we will say that  $A \leq B$  if whenever  $a \in A, b \in B$ , and  $a \geq b$ , we must also have  $a \in B$  and  $b \in A$ . Note that when  $A, B \subset \mathfrak{R}$ ,  $A \leq B$  if and only if  $A \setminus B$  lies below  $A \cap B$ , which in turn lies below  $B \setminus A$ . Armed with this concept, we have

**PROPOSITION 2.** If Conditions W and D hold, then with positive [negative] externalities on nontraders,  $M \leq [\geq] M^*$ .

*Proof.* Consider the parameterized program  $\max_{x \in \Sigma_{\vec{x}_i}} W(X) - zR(X)$ , where  $z = 0$  corresponds to the surplus-maximization program, and  $z = 1$  corresponds to the principal's profit-maximization program. Lemma 1 implies that with positive (negative) externalities on nontraders, the objective function is supermodular in  $(-X, z)$  [in  $(X, z)$ ]. The result follows by Topkis' Monotonicity Theorem [Topkis 1998, Theorem 2.8.1]. ■

A glance at the applications in Section II shows that Proposition 2 unifies many existing inefficiency results. For instances of positive externalities on nontraders, it predicts that an individual's compensation scheme in a common agency situation is flatter than the second-best compensation scheme (A5), that takeovers



by superior raiders and debt-equity swaps are less likely to occur than is socially optimal (A6 and A7), that acquisitions for monopoly may not occur even though they increase industry profits (A8), and that a public good may be privately underprovided (A10). For instances of negative externalities on nontraders, the proposition predicts that an intermediate good manufacturer may sell more than the vertical profit-maximizing quantity when a substitute is available (A1), that socially inefficient exclusion may occur (A2), that takeovers by inferior raiders are more likely to occur than socially optimal (A6), that a sponsored technology may be excessively adopted when the competing unsponsored technology exhibits network effects (A9), and that a public bad may be overprovided (A10).<sup>14</sup>

#### IV. BILATERAL CONTRACTING WITH PRIVATE OFFERS

Some of the literature described in Section II studies contracting games in which the principal does not have as much commitment power as assumed in the previous section. Thus, in the context of Network Externalities (A9), Katz and Shapiro [1986b] study a game in which the principal approaches two different groups of agents in two periods, and cannot commit to the second period's price in the first period. The literature on Vertical Contracting (A1) studies a game in which the principal makes offers to all agents simultaneously, but each agent only observes his own offer. In the context of Common Agency (A5), Bernheim and Whinston [1986] study a game in which it is the agents (firms) who make simultaneous offers to the principal (a risk-averse individual). In these situations, even in the absence of externalities on nontraders, the principal's inability to commit to compensate traders for the externalities imposed on them may give rise to inefficient outcomes:

**EXAMPLE 2.** Consider the setting of Vertical Contracting (A1) with two downstream firms (agents) producing a homogeneous good, the inverse demand for which is given by  $P(X) = 1 - X$ . Let  $\mathcal{X}_1 = \mathcal{X}_2 = [0, 1]$ . For simplicity let all costs be zero, so that  $f(x) \equiv 0$  and  $u_i(x_1, x_2) = x_i P(x_1 + x_2)$  for  $i = 1, 2$ . Since externalities on nontraders are absent, Proposition 1 implies

14. While suggestive of distortions, the comparison in Proposition 2 is weak; i.e., it does not rule out the possibility that contracting outcomes are efficient. Numerous examples of strict inefficiencies can be found in the literature described in Section II.

that the principal's commitment to bilateral offers would yield efficient (vertical profit-maximizing) outcomes, here given by  $\mathcal{M}^* = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 = 1/2\}$ .

If the principal cannot publicly commit to her bilateral offers, however, vertically efficient outcomes can no longer be sustained. For example, suppose that the principal offers an efficient trade profile  $(x_1^*, x_2^*) \in \mathcal{M}^*$  with  $x_1^* > 0$ . Once agent 1 has accepted his offer, the principal and agent 2 have an incentive to renegotiate to a trade  $x_2$  that maximizes their bilateral surplus,  $f(x_1^*, x_2) + u_2(x_1^*, x_2) = x_2(1 - x_1^* - x_2)$ . Their optimal choice is  $x_2 = (1 - x_1^*)/2 > 1/2 - x_1^* = x_2^*$ . Intuitively, because  $x_2$  has a negative externality on agent 1 who has already purchased  $x_1^*$ , the principal and agent 2 together have an incentive to trade excessively from the viewpoint of joint vertical profits.

In order to understand the nature of the inefficiency arising when the principal is unable to commit, this section analyzes a particular contracting game, which has been considered in the Vertical Contracting setting.<sup>15</sup> The game consists of two stages: in the first stage, the principal makes each agent  $i$  an offer  $(x_i, t_i)$ , which is privately observed by the agent. In the second stage, agents simultaneously decide whether to accept or reject.<sup>16</sup>

Each agent's acceptance decision in this game depends on his beliefs about offers extended to other agents. In a Perfect Bayesian Equilibrium, arbitrary beliefs can be assigned following the principal's out-of-equilibrium offers, which gives rise to an enormous multiplicity of equilibria. To make a more precise prediction, I follow the Vertical Contracting literature by restricting agents to hold so-called "passive beliefs" [McAfee and Schwartz, 1994]: even after observing an unexpected offer from the principal, an agent believes that other agents face their equilibrium offers.

Consider the principal's incentive to deviate from an equilib-

15. While I expect that this section's qualitative results would generalize to other bargaining games in which the principal lacks full commitment power, a comprehensive study of such games is left to future research.

16. The literature on vertical contracting (A1) usually considers a more complicated contracting game, in which the principal offers each agent  $i$  a tariff  $t_i(x_i)$  (often restricted to be a two-part tariff), and upon accepting this tariff, the agent chooses his trade  $x_i$ . If this choice is made without observing other agents' tariffs (the *ex post unobservability* case of McAfee and Schwartz [1994]), this game yields the same equilibrium outcomes as the game I study.

Our model of Debt Workouts (A7) assumed that each creditor knows the expected value of the equity he is offered. This assumption is not legitimate when the creditor does not observe the offers extended to other creditors. Therefore, this section's analysis will not be valid for this application.

rium outcome  $(\hat{x}, \hat{t})$ . Since she can always offer  $(x_i, t_i) = (0, 0)$ , just as in the previous section, without loss of generality we can restrict attention to deviations in which all agents accept their offers. If agent  $i$  holds passive beliefs, he accepts an offer  $(x_i, t_i)$  if and only if  $u_i(x_i, \hat{x}_{-i}) - t_i \geq u_i(0, \hat{x}_{-i})$ . The principal's optimal deviation should maximize her profit subject to these participation constraints:

$$\max_{x \in \mathbb{R}_1 \times \dots \times \mathbb{R}_N, t \in \mathbb{R}^N} f(x) + \sum_i t_i$$

subject to  $u_i(x_i, \hat{x}_{-i}) - t_i \geq u_i(0, \hat{x}_{-i})$  for all  $i \in N$ .

$(\hat{x}, \hat{t})$  is an equilibrium outcome if and only if the principal does not want to deviate from it; i.e., it solves this program. In particular, note that the outcome must satisfy the same participation constraints (2) as in the commitment program, and thus the principal can never do better here than in the commitment case. Moreover, she is likely to suffer from her lack of commitment, due to the additional requirement that  $\hat{x}$  be her best response to agents' beliefs.<sup>17</sup>

All participation constraints in the above program must bind, since otherwise the principal could profitably deviate by increasing transfers for some agents. Expressing transfers from the binding constraints and substituting them in the objective function, and taking into account that the principal takes agents' reservation utilities  $u_i(0, \hat{x}_{-i})$  as given, we find that trade profile  $\hat{x}$  can be sustained in equilibrium if and only if it satisfies the following condition:<sup>18</sup>

$$(5) \quad \hat{x} \in \arg \max_{x \in \mathbb{R}_1 \times \dots \times \mathbb{R}_N} f(x) + \sum_i u_i(x_i, \hat{x}_{-i}).$$

Let  $\mathfrak{C}$  denote the set of such trade profiles. The nonemptiness of  $\mathfrak{C}$  is only ensured under additional assumptions, which are discussed in Appendix 2. All of this section's results will be vacuous (but formally correct) when  $\mathfrak{C}$  is empty.

17. By the same logic, a player prefers being a Stackelberg leader to moving simultaneously with other players.

18. The condition implies (but is stronger than) "pairwise proofness" [McAfee and Schwartz 1994], which requires that for each agent  $i$ ,  $\hat{x}_i \in \arg \max_{x_i \in \mathbb{R}_i} f(x_i, \hat{x}_{-i}) + u_i(x_i, \hat{x}_{-i})$ . Indeed, "pairwise proofness" only ensures that the principal cannot profitably deviate by changing her offer to a single agent, and does not check the profitability of multiagent deviations. See footnote 20 to Example 3 below, and Rey and Verge [1997].

Example 2 demonstrates that in the presence of externalities at an efficient trade profile, the principal, who is unable to commit to compensate agents for these externalities, has an incentive to deviate from this trade profile. Conversely, it turns out that when externalities are absent at an efficient trade profile, private contracting produces efficient outcomes, regardless of any externalities that might exist at other trade profiles:

PROPOSITION 3. If there exists  $x^* \in \mathfrak{M}^*$  such that  $u_i(x_i^*, x_{-i})$  does not depend on  $x_{-i} \in \mathfrak{X}_{-i}$  for all  $i$ , then  $\mathfrak{E} \subset \mathfrak{M}^*$ .

*Proof.* For any  $\hat{x} \in \mathfrak{E}$ , the equilibrium condition (5) implies that

$$f(\hat{x}) + \sum_i u_i(\hat{x}) \geq f(x^*) + \sum_i u_i(x_i^*, \hat{x}_{-i}) = f(x^*) + \sum_i u_i(x^*).$$

Therefore,  $\hat{x} \in \mathfrak{M}^*$ . ■

In applications A5 (Common Agency), A6 (Takeovers), and A8 (Acquisition for Monopoly), externalities are absent on agents who trade the maximum amount (“sell out”). In certain cases, this outcome is efficient: in Common Agency with a risk-neutral individual, in Takeovers with a superior raider and low bidding costs, in Acquisition for Monopoly when monopoly maximizes industry profits. For these cases, Proposition 3 establishes that all equilibrium outcomes are efficient.<sup>19</sup>

EXAMPLE 3. Consider the setting of Takeovers (A6) where the firm’s value  $v(X)$  is strictly increasing in  $X$ , and  $c(X) \equiv 0$  (i.e., bidding costs are absent). Then at the unique efficient outcome, all shares are sold to the raider:  $x^* = (\bar{x}_1, \dots, \bar{x}_N)$ . At this trade profile there are no externalities on traders:  $u_i(\bar{x}_i, x_{-i}) = 0$  for all  $x_{-i}$ . Therefore, according to Proposition 3,  $x^*$  is the only candidate equilibrium outcome. It is easy to check that indeed  $x^* \in \mathfrak{E}$ .<sup>20</sup>

On the other hand, when externalities are present at all efficient trade profiles, they must distort the contracting outcome.

19. In the Common Agency setting, this result parallels Theorem 2 of Bernheim and Whinston [1986], established for a contracting game in which firms make simultaneous offers to the individual.

20. However, note that with a sufficiently large fixed bidding cost,  $x^*$  is not an equilibrium outcome (even though it is “pairwise proof” [McAfee and Schwartz 1994]), since the raider is better off not acquiring any shares at all. Hence, in this case no equilibrium exists.

To identify the direction of distortion, consider the following definition:

**DEFINITION 2.** Externalities on efficient traders are positive [negative] if for all  $x^* \in \mathcal{M}^*$  and each agent  $i$ ,  $u_i(x_i^*, x_{-i})$  is nondecreasing [nonincreasing] in  $x_{-i} \in \mathcal{X}_{-i}$ .

One is tempted to conjecture that with positive [negative] externalities on efficient traders, the equilibrium trade is lower [higher] than socially optimal. However, just as in the case of public offers, this conjecture can be undermined by interactions of different dimensions of the principal's problem (see Example 1). A definitive comparison again requires Conditions W and D. Letting  $E = \{\sum_i x_i : x \in \mathcal{C}\}$  denote the set of equilibrium aggregate trades, we have

**PROPOSITION 4.** If Conditions W and D hold, then with positive [negative] externalities on efficient traders,  $E \cup M^* \leq [\geq] M^*$ .

*Proof.* Consider the case of negative externalities on efficient traders. Suppose that  $X^* \in M^*$  and  $\hat{X} \in E \cup M^*$ , and that  $(\hat{x}_1, \dots, \hat{x}_N) \in \mathcal{C}$ , with  $\sum_i \hat{x}_i = \hat{X} \leq X^*$ . Since  $X^* \in E \cup M^*$  trivially, the strong set order comparison only requires proving that  $\hat{X} \in M^*$ .

Under Condition D there exists  $x^* \in \mathcal{X}_1 \times \dots \times \mathcal{X}_N$  such that  $X^* = \sum_i x_i^*$  and  $x^* \geq \hat{x}$ . Then we can write

$$\begin{aligned} W(\hat{X}) &= f(\hat{x}) + \sum_i u_i(\hat{x}_i, \hat{x}_{-i}) \geq f(x^*) \\ &+ \sum_i u_i(x_i^*, \hat{x}_{-i}) \geq f(X^*) + \sum_i u_i(x_i^*, x_{-i}^*) = W(X^*). \end{aligned}$$

The first inequality obtains from the equilibrium condition (5), the second from the fact that externalities on efficient traders are negative, and the last equality from Condition W. Therefore,  $\hat{X} \in M^*$ , which implies the result. The proof for the case of positive externalities on efficient traders is similar. ■

The established comparison between the sets  $E$  and  $M^*$  is somewhat weaker than the strong set order. For example, with positive externalities on efficient traders, the comparison means that  $E \setminus M^*$  lies below  $M^*$ , but allows some elements of  $E \cap M^*$  to lie above  $M^* \setminus E$ . When  $M^* \neq \mathcal{Q}$ , the comparison implies, in particular, that  $\sup E \leq \sup M^*$ .

V. COMPARISON BETWEEN PUBLIC AND PRIVATE CONTRACTING

As shown in the two preceding sections, with publicly observed offers contracting distortions stem from the externalities on nontraders, while with privately observed offers they stem from the externalities on efficient traders. In the particular case in which the externality imposed on an agent by changing other agents' trades does not depend on his own trade, the two contracting regimes yield the same outcomes:

**PROPOSITION 5.** *If for each agent  $i$  and all  $x_{-i}, x'_{-i} \in \mathcal{X}_{-i}$ ,  $u_i(x_i, x'_{-i}) - u_i(x_i, x_{-i})$  does not depend on  $x_i \in \mathcal{X}_i$ , then  $\mathfrak{C} = \mathfrak{M}$ .*

*Proof.* By (5),  $\hat{x} \in \mathfrak{C}$  if and only if

$$f(\hat{x}) + \sum_i u_i(\hat{x}_i, \hat{x}_{-i}) \geq f(x) + \sum_i u_i(x_i, \hat{x}_{-i}) \quad \text{for all } x \in \mathcal{X}_1 \times \dots \times \mathcal{X}_N.$$

By the proposition's assumption, we can write  $u_i(x_i, \hat{x}_{-i}) = u_i(x_i, x_{-i}) + u_i(0, \hat{x}_{-i}) - u_i(0, x_{-i})$ . Substituting in the above inequality and subtracting  $\sum_i u_i(0, \hat{x}_{-i})$  from both sides yields

$$\begin{aligned} f(\hat{x}) + \sum_i [u_i(\hat{x}_i, \hat{x}_{-i}) - u_i(0, \hat{x}_{-i})] \\ \geq f(x) + \sum_i [u_i(x_i, x_{-i}) - u_i(0, x_{-i})] \quad \text{for all } x \in \mathcal{X}_1 \times \dots \times \mathcal{X}_N. \end{aligned}$$

By (3) this holds if and only if  $\hat{x} \in \mathfrak{M}$ . ■

The proposition's assumption, which is equivalent to the additive separability of  $u_i(x_i, x_{-i})$  in  $x_i$  and  $x_{-i}$ , is quite strong: out of all the applications described in Section II, it is only satisfied in application A10 (Pure Public Good/Bad) with linear benefit functions  $v_i(X)$ . In order to study the general case where the externalities on nontraders differ from those on efficient traders,<sup>21</sup> consider the following definition:

**DEFINITION 3.** Externalities are increasing [decreasing] if for each agent  $i$ ,  $u_i(x_i, x_{-i})$  has increasing differences in  $(x_i, x_{-i})$  [ $(-x_i, x_{-i})$ ] [Topkis 1998; Milgrom and Shannon 1994], i.e., for all  $x_{-i}, x'_{-i} \in \mathcal{X}_{-i}$  with  $x'_{-i} \geq x_{-i}$ ,  $u_i(x_i, x'_{-i}) - u_i(x_i, x_{-i})$  is nondecreasing (nonincreasing) in  $x_i \in \mathcal{X}_i$ .

21. These externalities may even have opposite signs: for example, in the context of Network Externalities (A9), assuming that full adoption of the new technology is efficient, the externalities on efficient traders are positive, while the externalities on nontraders are negative.

In words, with increasing [decreasing] externalities, the externality imposed on agent  $i$  by increasing other agents' trades is more [less] positive when he trades more.<sup>22</sup> It is easy to verify that externalities are increasing in applications A9 (Network Externalities) and A2 (Exclusive Dealing), and they are decreasing in applications A4 (Common Insurance), and A8 (Acquisition for Monopoly). In application A1 (Vertical Contracting), externalities must be decreasing in the absence of a substitute technology, but may be increasing in other cases (see Katz and Shapiro [1986a]). In application A6 (Takeovers), externalities are decreasing [increasing] when the raider is superior [inferior] to the incumbent management and  $\bar{x}_i = \{0, \bar{x}_i\}$  for all  $i$ . In application A10 (Pure Public Good/Bad), externalities are increasing [decreasing] when the functions  $v_i(\cdot)$  are convex [concave].

Intuition suggests that with increasing (decreasing) externalities, the principal's incentive to trade with private offers falls short of [exceeds] that with public offers. Unfortunately, just as with similar intuitions in the previous sections, this conjecture is not generally true because of the interaction between different dimensions of the principal's multidimensional program. Moreover, unlike in previous sections, even Conditions W and D do not ensure a definitive comparison here. The problem is that even when the allocation of a given aggregate trade  $X$  does not affect total surplus, it may still affect the principal's profit with both public and private offers. For this reason, a definitive comparison can only be established under the stronger Condition S, which ensures that the allocation of a given  $X$  among agents is always irrelevant:

**PROPOSITION 6.** When Condition S holds and externalities are increasing [decreasing],  $E \cup M \leq [\geq] M$ .

*Proof.* Under Condition S we can write  $u_i(x) = U(x_i, \sum_j x_j)$  and  $f(x) = F(\sum_j x_j)$ , and

$$M = \arg \max_{x \in \{0, \dots, M\}} F(X) + X[U(1, X) - U(0, X - 1)].$$

Consider the case of increasing externalities. Suppose that  $X \in M$

22. An equivalent formulation of this property is that agent  $i$ 's willingness to increase his trade from  $x_i$  to  $x'_i \geq x_i$ ,  $u_i(x'_i, x_{-i}) - u_i(x_i, x_{-i})$ , is nondecreasing [nonincreasing] in other agents' trades  $x_{-i}$ . As shown in Segal [1998], this property determines whether the public contracting game of Section III has multiple equilibria (see footnote 10).



and  $\hat{X} \in E$ , and  $X \leq \hat{X}$ . Since we trivially have  $X \in E \cup M$ , to establish the strong set order comparison, we only need to prove that  $\hat{X} \in M$ . Since agents' participation constraints bind in the private-offer equilibrium  $\hat{X}$ , each agent  $i$  with  $x_i = 1$  pays  $t_i = U(1, \hat{X}) - U(0, \hat{X} - 1)$ . Consider a deviation from the equilibrium in which the principal offers  $x_i = t_i = 0$  to  $\hat{X} - X$  agents who previously had  $x_i = 1$ . Since the deviation must be unprofitable, we have

$$\begin{aligned} F(\hat{X}) + \hat{X}[U(1, \hat{X}) - U(0, \hat{X} - 1)] \\ \geq F(X) + X[U(1, \hat{X}) - U(0, \hat{X} - 1)] \\ \geq F(X) + X[U(1, X) - U(0, X - 1)], \end{aligned}$$

where the second inequality follows from the condition of increasing externalities. Therefore,  $\hat{X} \in M$ . The proof for decreasing externalities is similar, except that we consider the principal's deviation to offer  $x_i = 1$  to  $X - \hat{X}$  agents who previously had  $x_i = 0$ . ■

Another way to compare the contracting outcomes with private and public offers is to focus on situations in which  $N$  is large. As shown in the working paper version [Segal 1997], when Condition L holds with differentiable functions  $F(\cdot)$  and  $\alpha(\cdot)$  and a continuous function  $\beta(\cdot)$ , the above comparison between  $M$  and  $E$  holds in the asymptotic setting described in Section VII below when  $N$  is sufficiently large.

What do these comparisons imply for the relative efficiency of public and private contracting outcomes? The implications are unambiguous when the total surplus  $W(X)$  is quasi concave in  $X$ . Under this assumption, which is reasonable in all applications listed in Section II,  $W(X)$  is always (weakly) increased by moving  $X$  closer to its efficient value. Therefore, our comparative statics results imply that when the externalities on traders are of the same sign, but of greater [smaller] absolute value, than those on nontraders, private contracting is less [more] efficient than public contracting (for a formal proof, see Segal [1997, Subsection 5.3]).

The former case includes settings in which externalities are negative and decreasing, such as Vertical Contracting (A1) without substitute technologies and Common Insurance (A4), and settings in which externalities are positive and increasing, such as Network Externalities (A9) when the old technology exhibits no network effects. In fact, in these three settings there are no

externalities on nontraders, and inefficiencies can be ascribed to the principal's inability to commit. This point has been made in the respective literatures.

Perhaps more surprisingly, the principal's inability to commit can *raise* the total surplus of the contracting parties. This happens when the externalities on traders are of the same sign, but of smaller magnitude, than those on nontraders. This includes settings in which externalities are positive and decreasing, such as Takeovers (A6) with a superior raider and Acquisition for Monopoly (A8), and settings in which externalities are negative and increasing, such as Takeovers (A6) with an inferior raider and Exclusive Dealing (A2). In these cases, welfare would be enhanced by legal restrictions on the principal's commitment.<sup>23</sup>

## VI. GENERAL COMMITMENT MECHANISMS

In this section I return to the assumption that the principal is able to make public commitments, but generalize the analysis of Section III by allowing the principal to commit to mechanisms in which one agent's trade can be made contingent on other agents' messages. For example, in the context of Vertical Contracting (A1), Katz and Shapiro [1986a] and Kamien, Oren, and Tauman [1992] study an auction in which the seller commits to sell  $\hat{X}$  units of the good to the highest bidders. Unlike with bilateral contracting, whether buyer  $i$  obtains the good now depends on other buyers' bids. In the context of Takeovers (A6), Bagnoli and Lipman [1988] study conditional bids, in which the raider commits to purchase exactly  $\hat{X}$  shares at a certain price if at least  $\hat{X}$  shares are tendered by stockholders, and to buy no shares otherwise. Unlike with bilateral contracting, the number of

23. In the context of takeovers with a superior raider, a similar observation has been made by Harrington and Prokop [1993], who study multiperiod bidding by a raider. Their numerical simulations show that the raider's inability to commit to a bid increases the likelihood of takeover and expected total surplus. An analogy can be found in the "Coase conjecture," which states that a durable good monopolist who is unable to commit sells a larger quantity (see, e.g., Tirole [1988, Chapter 1]). Harrington and Prokop also note that the raider's inability to commit reduces her surplus, and so may prevent her from recouping the sunk cost of a takeover. In this case, social welfare is reduced, and our policy recommendation is reversed.

Our policy recommendations may also be reversed when the contracting outcome affects the welfare of some parties who do not participate in contracting, such as final consumers in Vertical Contracting (A1) and Acquisition for Monopoly (A9).

shares sold by shareholder  $i$  now depends on other shareholders' tenders.

*A. Characterization and Examples*

A general mechanism (game form) used by the principal can be described as  $\Gamma = \langle S_1, \dots, S_N, g(\cdot) \rangle$ , where  $S_i$  is agent  $i$ 's message space, and  $g: S_1 \times \dots \times S_N \rightarrow \mathcal{X}_1 \times \dots \times \mathcal{X}_N \times \mathfrak{R}^N$  is the outcome function, which prescribes the trade and transfer profiles for any message profile. The fact that participation in the mechanism is voluntary can be reflected by endowing each agent  $i$  with a special "reject" message  $s_i = 0$ , guaranteeing him the bundle  $(x_i, t_i) = (0, 0)$ :

DEFINITION 4. A mechanism  $\Gamma = \langle S_1, \dots, S_N, g(\cdot) \rangle$  is voluntary if  $0 \in S_i$  for all  $i$ , and  $g_i(s) = (0, 0)$  whenever  $s_i = 0$ .

As in Section III, I assume that the principal can induce agents to play any given Nash equilibrium of the mechanism, which gives rise to the following implementation concept:

DEFINITION 5. An allocation  $(\bar{x}, \bar{t}) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_N \times \mathfrak{R}^N$  is implementable if there exist a voluntary mechanism  $\Gamma = \langle S_1, \dots, S_N, g(\cdot) \rangle$  and a Nash equilibrium play  $s$  of  $\Gamma$  such that  $g(s) = (\bar{x}, \bar{t})$ .

In the spirit of the revelation principle, we can focus on a special class of voluntary mechanisms, to be called *direct* mechanisms, in which each agent has only two possible messages, "reject" ( $s_i = 0$ ), and "accept" ( $s_i = 1$ ). (The difference from the standard implementation setting, described, e.g., in Mas-Colell, Whinston, and Green [1995, Chapter 23], is that even though agents have no private information to report, they must be endowed with a message giving them the option not to participate in the mechanism.) Every play  $s \in \{0, 1\}^N$  in such a mechanism can be represented with the corresponding "acceptance set"  $A(s) = \{i \in N: s_i = 1\}$ . Thus, a direct mechanism can be described by set functions  $\mathbf{x}: 2^N \rightarrow \mathcal{X}_1 \times \dots \times \mathcal{X}_N$  and  $\mathbf{t}: 2^N \rightarrow \mathfrak{R}^N$ , so that  $g(s) = (x(A(s)), t(A(s)))$ .<sup>24</sup> In words,  $x(A)$  is the trade profile prescribed when the set of accepting agents is  $A$ , and  $t(A)$  is the transfer profile in this situation.

DEFINITION 6. An allocation  $(\bar{x}, \bar{t}) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_N \times \mathfrak{R}^N$  is directly implementable if there exists a direct mechanism  $(\mathbf{x}, \mathbf{t})$  in

24. Boldface type will be used to denote functions on  $2^N$ , e.g.,  $\mathbf{x}: 2^N \rightarrow \mathcal{X}_1 \times \dots \times \mathcal{X}_N$  and normal type to denote their values; e.g.,  $x(1, 2) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_N$ .

which  $A = N$  is a Nash equilibrium play and  $(x(N), t(N)) = (\bar{x}, \bar{t})$ .

Since a direct mechanism  $(\mathbf{x}, \mathbf{t})$  must be voluntary, we must have  $x_i(A) = t_i(A) = 0$  whenever  $i \notin A$ , and the requirement that  $A = N$  be a Nash equilibrium play of  $(\mathbf{x}, \mathbf{t})$  can be represented with the following participation constraints:

$$(6) \quad u_i(x(N)) - t_i(N) \geq u_i(0, x_{-i}(N \setminus i)) \quad \text{for all } i \in N.$$

The modified revelation principle can now be formulated as follows:

**PROPOSITION 7.** An allocation  $(\bar{x}, \bar{t}) \in \mathfrak{X}_1 \times \dots \times \mathfrak{X}_N \times \mathfrak{R}^N$  is implementable if and only if it is directly implementable.

*Proof.* If  $(\bar{x}, \bar{t})$  is implementable, then there exists a voluntary mechanism  $\Gamma = \langle S_1, \dots, S_N, g(\cdot) \rangle$  and a message profile  $\bar{s} = (\bar{s}_1, \dots, \bar{s}_N) \in S_1 \times \dots \times S_N$  such that (i)  $g(\bar{s}) = (\bar{x}, \bar{t})$ , and (ii)  $\bar{s}$  is a Nash equilibrium of  $\Gamma$ , i.e., letting  $g(s) \equiv (x(s), t(s))$ ,

$$u_i(x(\bar{s})) - t_i(\bar{s}) \geq u_i(x(s'_i, \bar{s}_{-i})) - t_i(s'_i, \bar{s}_{-i})$$

for all  $i \in N$ , and all  $s'_i \in S_i$ .

For any  $A \subset N$ , and any  $i \in N$ , define

$$\tilde{s}_i(A) = \begin{cases} 0, & i \notin A, \\ \bar{s}_i, & i \in A. \end{cases}$$

The above inequalities imply that the direct mechanism  $(\tilde{x}(A), \tilde{t}(A)) \equiv g(\tilde{s}(A))$  satisfies participation constraints (6). Since  $(\tilde{x}(N), \tilde{t}(N)) = g(\tilde{s}(N)) = g(\bar{s}) = (\bar{x}, \bar{t})$ , the allocation is directly implementable. ■

Proposition 7 allows us to restrict attention to direct mechanisms, so long as we are not concerned with the existence of “undesirable” equilibria.<sup>25</sup> Then participation constraints (6) imply that only equilibrium transfers  $t(N)$ , and only trades  $x(A)$  for  $|A| \geq N - 1$  are relevant for implementation. Here are some mechanisms considered in the literature, with their correspond-

25. Just as in the standard implementation setting, more complex mechanisms, including possibly multistage mechanisms, may be useful for eliminating undesirable equilibria. For example, in the setting of Exclusive Dealing (A2), Segal and Whinston [forthcoming] find that exclusion at zero cost may be the unique subgame-perfect equilibrium outcome of a sequential acceptance game, but the corresponding direct mechanism, which is a simultaneous acceptance game, has another equilibrium, preferred by all buyers, in which all of them reject exclusive contracts.

ing direct mechanisms (for definiteness assume that the principal buys  $x_i$  from agents):

*Bilateral Contracting with Public Offers* (studied in Section III). A set  $\{(\hat{x}_i, \hat{t}_i)\}_{i \in N}$  of bilateral contract offers is equivalent to a direct mechanism

$$(x_i(A), t_i(A)) = \begin{cases} (\hat{x}_i, \hat{t}_i) & \text{if } i \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Note that by adjusting transfers  $\hat{t}_i$ , the principal always optimally makes all agents' participation constraints bind.

*Auctions* [Katz and Shapiro 1986a; Kamien, Oren, and Tausman 1992]. The principal commits to buy a quantity  $\hat{X} \in \sum_i \hat{x}_i$  via an auction. (To ensure that  $\hat{X}$  can always be allocated among participating agents, assume Condition D.) Taking into account that the principal cannot buy more than the total endowment of the participating bidders, the corresponding direct mechanism satisfies  $\sum_i x_i(A) = \min\{\hat{X}, \sum_{i \in A} \hat{x}_i\}$  for all  $A \subset N$ . Recall that only acceptance sets  $A$  with  $|A| \geq N - 1$  are relevant for implementation, and we have  $\sum_i x_i(A) = \hat{X}$  for such sets as long as  $\hat{X} \leq \sum_{j \neq i} \hat{x}_j$  for all  $i$ . In this case each agent takes the aggregate trade  $\hat{X}$  as given when making his acceptance decision. Then under Condition L each agent is willing to sell each unit of the good for exactly  $\alpha(\hat{X})$ . Thus, in any deterministic auction all trades must take place at this price. In particular, this implies that all agents' participation constraints bind.

*Any-and-all Bids* [Bagnoli and Lipman 1988; Holmstrom and Nalebuff 1992; Katz and Shapiro 1986a]. The principal offers the same price  $p$  to every agent for each unit of the good. Letting  $\hat{x}_i \in \arg \max_{x_i \in \hat{x}_i} [u_i(x_i, \hat{x}_{-i}) + px_i]$  denote the equilibrium tender of agent  $i$ , the corresponding direct mechanism is given by

$$(x_i(A), t_i(A)) = \begin{cases} (\hat{x}_i - p\hat{x}_i) & \text{if } i \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Note that every such bid is a bilateral mechanism, since each agent's trade depends only on his own message.

Under Condition S the principal will optimally set  $p$  to make all agents with  $\hat{x}_i = 1$  indifferent between selling and not. Thus, all agents' participation constraints will bind. More generally, however, when agents are heterogeneous or different agents trade different positive amounts in equilibrium, the principal may not be able to make all agents' participation constraints bind at once.

*Conditional Bids* [Bagnoli and Lipman 1988]. The principal sets a price  $p$  at which all agents can tender their goods. If fewer than  $\hat{X}$  units are tendered, no trade takes place. If more than  $\hat{X}$  units are tendered, then the principal purchases only  $\hat{X}$  units, using some rationing rule. Suppose that in equilibrium exactly  $\hat{X}$  units are tendered, and let  $\hat{x}_i$  denote the equilibrium tender of agent  $i$ . Then the corresponding direct mechanism is given by

$$(x_i(A), t_i(A)) = \begin{cases} (\hat{x}_i - p\hat{x}_i) & \text{if } A = N, \\ 0 & \text{otherwise.} \end{cases}$$

By the same logic as in the previous example, the principal will optimally make all agents' participation constraints bind under Condition S, but not necessarily in the general case.

### B. Restricted Mechanisms

This subsection identifies conditions under which Section III's analysis of bilateral contracting can be extended to situations where the principal can choose from a different, but still restricted, family of mechanisms. The first such condition is that all agents' participation constraints bind in equilibrium:

**DEFINITION 7.** A direct mechanism  $(\mathbf{x}, \mathbf{t})$  is binding if all agents' participation constraints (6) bind.

This condition would be satisfied, for example, when the principal can charge fixed fees for participation in the mechanism. However, note that some of the mechanisms described in the previous subsection do not satisfy this property when agents are heterogeneous or trade different positive amounts in equilibrium.

In a binding direct mechanism  $(\mathbf{x}, \mathbf{t})$ , the equilibrium transfers  $t(N)$  can be obtained from the trade component  $\mathbf{x}$  using the binding participation constraints (6). Since all other transfers are irrelevant for Nash implementation, binding mechanisms can be fully described by their trade components. Let  $\mathfrak{D}$  denote the set of all binding direct mechanisms:

$$\mathfrak{D} = \{\mathbf{x} \in (\mathfrak{X}_1 \times \dots \times \mathfrak{X}_N)^{2^N} : x_i(A) = 0 \text{ whenever } i \notin A\}.$$

We assume that the principal is restricted to choose from a subset ("family")  $\mathfrak{F} \subset \mathfrak{D}$  of such mechanisms. The set  $\mathfrak{M}_{\mathfrak{F}}^*$  of mechanisms

that maximize total surplus within the family  $\tilde{\mathcal{F}}$  can be defined as<sup>26</sup>

$$(7) \quad \mathcal{M}_{\tilde{\mathcal{F}}}^* = \arg \max_{\mathbf{x} \in \tilde{\mathcal{F}}} f(x(N)) + \sum_i u_i(x(N)).$$

On the other hand, if the principal chooses a binding mechanism from  $\tilde{\mathcal{F}}$  to maximize her profit, the set  $\mathcal{M}_{\tilde{\mathcal{F}}}$  of her optimal mechanisms can be defined as

$$(8) \quad \mathcal{M}_{\tilde{\mathcal{F}}} = \arg \max_{\mathbf{x} \in \tilde{\mathcal{F}}} f(x(N)) + \sum_i u_i(x(N)) - \sum_i u_i(0, x_{-i}(N \setminus i)).$$

It is clear that in the absence of externalities on nontraders, the two above programs coincide, and we obtain a generalization of Proposition 1:

**PROPOSITION 8.** If  $u_i(0, x_{-i})$  does not depend on  $x_{-i} \in \mathcal{X}_{-i}$  for all  $i$ , then for any family  $\tilde{\mathcal{F}}$  of direct binding mechanisms,  $\mathcal{M}_{\tilde{\mathcal{F}}} = \mathcal{M}_{\tilde{\mathcal{F}}}^*$ .

For example, in the context of Vertical Contracting (A1), Kamien, Oren, and Tauman [1992] find that when the intermediate input (patent) provides a sufficiently large productivity improvement so that the downstream firms who do not use it are driven out of the market, the principal's optimal auction yields a vertically efficient outcome. The same result has been independently obtained by Hart and Tirole [1990], O'Brien and Schaffer [1992], and McAfee and Schwartz [1994] in models where the principal uses bilateral contracts. Proposition 8 demonstrates that due to the absence of externalities on nontraders in this situation, the principal would make an efficient choice from *any* family of binding mechanisms.

When externalities on nontraders are present, on the other hand, the principal's rent extraction motive affects her choice of mechanism. With positive [negative] externalities on nontraders, the principal reduces agent  $i$ 's reservation utility by reducing (increasing) the trade profile  $x_{-i}(N \setminus i)$  following his rejection.<sup>27</sup>

26. When the family  $\tilde{\mathcal{F}}$  satisfies the "full range" condition  $\tilde{\mathcal{F}}(N) = \mathcal{X}_1 \times \dots \times \mathcal{X}_N$ , we have  $\mathcal{M}_{\tilde{\mathcal{F}}}^* = \mathcal{M}^*$ .

27. In particular, this explains the finding of Katz and Shapiro [1986a] that in the setting of Vertical Contracting (A1), the principal prefers auctions to any-and-all bids. Indeed, if an agent deviates by not participating in an auction, the trade that he forsakes goes to other agents. With negative externalities on nontraders, given an equilibrium outcome  $x(N)$ , the deviator is thus punished more severely than under an any-and-all bid, in which other agents' trades are not affected by the



Our objective here, however, is to examine the effect of the principal's rent-extraction motive on the *equilibrium* trade. For this purpose, recall that the feature of bilateral contracting responsible for the comparative statics result of Section III is the principal's ability, under Condition D, to increase (decrease) the equilibrium aggregate trade while increasing (decreasing) all out-of-equilibrium trades at once. Under the assumption that agents' reservation utilities depend only on the *aggregate* trade, it suffices to impose this requirement on aggregate out-of-equilibrium trades only. Formally, define the "aggregate representation"  $\mathbf{X} = \sum_i \mathbf{x}_i$  of a direct mechanism  $\mathbf{x}$  by  $X(A) = \sum_i x_i(A)$  for all  $A \subset N$ , and the aggregate representation  $\Sigma_{\mathcal{F}}$  of a family  $\mathcal{F}$  of mechanisms by  $\Sigma_{\mathcal{F}} = \{\sum_i \mathbf{x}_i : \mathbf{x} \in \mathcal{F}\}$ . Then the desired property can be formulated as follows:

**DEFINITION 8.** Let  $\mathcal{F}$  be a family of mechanisms. For any  $\bar{X} \in \sum_i \bar{x}_i$ , define  $\Sigma_{\mathcal{F}}|\bar{X} = \{\mathbf{X} \in \Sigma_{\mathcal{F}} : X(N) = \bar{X}\}$ . We will say that  $\mathcal{F}$  is ascending<sup>28</sup> if for any  $\bar{X}, \bar{Y} \in \sum_i \bar{x}_i$  such that  $\bar{X} \leq \bar{Y}$ ,

- (i) for any  $\mathbf{X} \in \Sigma_{\mathcal{F}}|\bar{X}$  there exists  $\mathbf{Y} \in \Sigma_{\mathcal{F}}|\bar{Y}$  such that  $X(A) \leq Y(A)$  whenever  $|A| = N - 1$ ;
- (ii) for any  $\mathbf{Y} \in \Sigma_{\mathcal{F}}|\bar{Y}$  there exists  $\mathbf{X} \in \Sigma_{\mathcal{F}}|\bar{X}$  such that  $X(A) \leq Y(A)$  whenever  $|A| = N - 1$ .

Under Condition D, this property is satisfied by all the mechanisms described in this section. Letting  $M_{\mathcal{F}}^* = \{X(N) : \mathbf{X} \in \Sigma \mathcal{M}_{\mathcal{F}}^*\}$  be the set of equilibrium aggregate trades in mechanisms that maximize equilibrium total surplus,<sup>29</sup> and  $M_{\mathcal{F}} = \{X(N) : \mathbf{X} \in \Sigma \mathcal{M}_{\mathcal{F}}\}$  be the set of the principal's profit-maximizing equilibrium aggregate trades, the comparative statics result of Section III can be extended as follows:

**PROPOSITION 9.** Suppose that Condition W holds, that  $u_i(0, x_{-i}) = \beta_i(\sum_{j \neq i} x_j)$  for all  $i$  and all  $x_{-i} \in \mathcal{X}_{-i}$ , and that  $\mathcal{F}$  is an ascending family of binding direct mechanisms. Then with positive [negative] externalities on nontraders,  $M_{\mathcal{F}} \leq [\geq] M_{\mathcal{F}}^*$ .

deviation; i.e.,  $x_{-i}(N \setminus i) = x_{-i}(N)$ . Note that with positive externalities, the principal's preference would be reversed.

28. This property would be necessarily satisfied if  $\Sigma_{\mathcal{F}}$  were a lattice (see, e.g., Milgrom and Shannon [1994]), but it is substantially more general. For example, when  $\mathcal{F}$  is the family of bilateral contracting mechanisms,  $\Sigma_{\mathcal{F}}$  is ascending, but not a lattice (this can already be seen for  $N = 2$ ).

29. When the family  $\mathcal{F}$  satisfies the "full range" condition  $(\Sigma_{\mathcal{F}})(N) = \sum_i \bar{x}_i$ , we have  $M_{\mathcal{F}}^* = M^*$ .

*Proof.* Consider the parameterized program  $\max_{\bar{x} \in \Sigma_{\tilde{\gamma}}(N)} W(\bar{X}) - zR(\bar{X})$ , where  $R(\bar{X}) = \min_{x \in \Sigma_{\tilde{\gamma}}(\bar{X})} \sum_i \beta_i(x(N \setminus i))$  is the minimum sum of agents' reservation utilities consistent with the equilibrium aggregate trade  $\bar{X}$ ,  $z = 0$  corresponds to the surplus-maximization program (7), and  $z = 1$  corresponds to the principal's profit-maximization program (8). When  $\tilde{\gamma}$  is ascending and the externalities on nontraders are positive [negative],  $R(\bar{X})$  is nondecreasing [nonincreasing] in  $\bar{X}$ , and the objective function is supermodular in  $(-\bar{X}, z)$  [in  $(\bar{X}, z)$ ]. The result follows by Topkis' Monotonicity Theorem. ■

In addition to Condition W the proposition assumes that each agent's reservation utility depends only on other agents' aggregate trade, which could be ensured, e.g., with Condition L. Under these assumptions the proposition demonstrates that the relation between the direction of distortion and the sign of externality on nontraders is not specific to bilateral contracting; rather, it holds quite generally when the principal can commit to a mechanism from an ascending family of binding mechanisms. For example, when the principal uses auctions (as in Katz and Shapiro [1986a] and Kamien, Oren, and Tauman [1992]), the arising distortion is of the same sign as when the principal uses bilateral contracts.

### C. Fully Optimal Mechanisms

When the principal can use arbitrary mechanisms, i.e.,  $\tilde{\gamma} = \mathcal{D}$ , we have

**PROPOSITION 10.**  $\mathcal{M}_{\mathcal{D}} = \mathcal{M}_{\mathcal{D}}(N) \times \prod_{i=1}^N \mathcal{M}_{\mathcal{D}}(N \setminus i) \times \prod_{A \subset N, |A| < N-1} \mathcal{M}_{\mathcal{D}}(A)$ ,

where

$$\begin{aligned} \mathcal{M}_{\mathcal{D}}(N) &= \mathcal{M}^*, \\ \mathcal{M}_{\mathcal{D}}(N \setminus i) &= \arg \min_{x_{-i} \in \mathcal{X}_{-i}} u_i(0, x_{-i}) && \text{for all } i \in N, \text{ and} \\ \mathcal{M}_{\mathcal{D}}(A) &= \mathcal{D}(A) && \text{for all } A \subset N \\ &&& \text{with } |A| < N - 1. \end{aligned}$$

*Proof.* Follows from the additive separability of the objective function in the profit-maximization program (8) in  $x(N)$  and  $x(N \setminus i)$  for all  $i$ , and the fact that  $\mathcal{D} = \prod_{A \subset N} \mathcal{D}(A)$ . ■

**COROLLARY 1.** With positive [negative] externalities on nontraders,

$$\mathcal{M}_{\mathcal{D}} \supset \{x \in \mathcal{D} : x(N) \in \mathcal{M}^*, x_j(N \setminus j) = 0 [= \max \mathcal{X}_j] \quad \text{when } j \neq \hat{j}.$$

According to these results, if the principal can choose any mechanism from  $\mathfrak{D}$ , she maximizes her profit by implementing an efficient trade profile  $x(N)$  in equilibrium, while at the same time minimizing each agent  $i$ 's rent by choosing the harshest punishment  $x(N \setminus i)$  following his deviation.<sup>30</sup> With positive externalities on nontraders, the harshest punishment for agent  $i$  is  $x_j(N \setminus i) = 0$  for all  $j$ . This punishment can be implemented with an offer that requires unanimous acceptance: if at least one agent rejects, no trade takes place. (In the context of Takeovers (A6) with a superior raider, this coincides with the conditional bid suggested by Bagnoli and Lipman [1988].) With negative externalities on nontraders, the harshest punishment is  $x_j(N \setminus i) = \max \bar{x}_j$  for all  $j \neq i$ . In words, the deviator is punished by implementing the maximum possible trades with all other agents. (Mechanisms of this kind have been suggested by Katz and Shapiro [1986a] and Kamien, Oren, and Tauman [1992] in the context of Vertical Contracting (A1), and by Jehiel, Moldovanu, and Stachetti [1996] in the context of Indivisible Object (A3).) In both cases, the separation of rent extraction and surplus maximization results in efficiency.<sup>31</sup>

## VII. ASYMPTOTIC NONPIVOTALNESS

The fully optimal mechanism derived in Proposition 10 seems unrealistic in environments with a large number of agents, where small agents may be expected to take the aggregate trade  $X$  as given in making their decisions, i.e., to be *nonpivotal*. Many papers, including Gertner and Scharfstein [1991] in the context of Debt Workouts (A7), Katz and Shapiro [1986b] in the context of Network Externalities (A9), and Grossman and Hart [1980] in the context of Takeovers (A6), indeed assume that small agents are nonpivotal, and find that this generally gives rise to inefficient contracting outcomes. The objective of this section is to identify conditions under which the assumption of nonpivotalness is

30. One might wonder whether these optimal mechanisms have other Nash equilibria that are preferred by the agents. In fact, it is easy to ensure that "all accept" is the unique Nash equilibrium, by choosing the out-of-equilibrium payments  $t_i(A)$  for  $A \neq N$  to be large enough so that each agent strictly prefers to accept if he expects at least one other agent to reject. Then acceptance becomes a weakly dominant strategy, and it can be made strictly dominant by slightly raising the equilibrium payments  $t_i(N)$ .

31. Observe that these results do not contradict Proposition 9, which only offers a weak comparison with the first-best, and does not rule out the possibility that an efficient outcome arises.

justified, and, therefore, inefficiency obtains with a large number of agents.<sup>32</sup>

To take a concrete example, consider the setting of Takeovers, where much controversy about pivotalness has been generated. Suppose for definiteness that the raider is superior; i.e., the firm's value  $v(X)$  is nondecreasing in  $X$ . If the raider uses a direct mechanism whose aggregate representation is  $\mathbf{X}$  and in which shareholders' participation constraints bind, (8) allows us to express her profit as

$$\begin{aligned} W(X(N)) - \sum_i u_i(0, x_{-i}(N \setminus i)) \\ = \sum_i [v(X(N)) - v(X(N \setminus i))] \bar{x}_i - c(X(N)). \end{aligned}$$

Say that shareholder  $i$  is nonpivotal if  $v(X(N \setminus i)) = v(X(N))$ , and pivotal if  $v(X(N \setminus i)) = v(0)$ . Then the above expression demonstrates that *the raider fully appropriates the appreciation in the holdings of pivotal shareholders, and does not appropriate any appreciation in the holdings of nonpivotal shareholders*. In particular, as first observed by Grossman and Hart [1980], if all shareholders are nonpivotal, the raider does not appropriate any of the firm's value improvement. Then, with positive takeover costs, she optimally sets  $X = 0$ ; i.e., a takeover does not take place, regardless of its social efficiency.

In response to this observation, Bagnoli and Lipman [1988] and Holmstrom and Nalebuff [1992] pointed out that for any finite  $N$ , the raider is able to make shareholders pivotal and appropriate some or all of the firm's value improvement. For example, in accordance with our Corollary 1, Bagnoli and Lipman suggest that the raider make a bid to buy shares at the firm's initial value  $v(0)$  conditional on all shareholders' tendering all their shares. This conditional bid ensures that each shareholder is pivotal, which makes the raider the residual claimant of total surplus and induces her to acquire an efficient stake  $X$  in the firm.

It has also been observed in the literature that the raider may be able to make small shareholders pivotal *even with any-and-all bids*. Specifically, suppose that, as in most of the literature, the

32. I will restrict attention to settings where the principal can commit to a publicly observed mechanism. For a nonpivotalness result with privately observed offers, see Segal [1997, subsection 5.2].

firm's value is given by

$$v(X) = \begin{cases} \bar{v} & \text{when } X \geq 0.5, \\ \underline{v} & \text{otherwise.} \end{cases}$$

Then the raider can make each tendering shareholder pivotal by ensuring that  $X = 0.5$  in equilibrium. For example, suppose that the raider bids  $\underline{v}$ . If each shareholder holds one indivisible share, and  $N$  is even, there exists an equilibrium in which exactly  $N/2$  shareholders tender.<sup>33</sup> In this equilibrium each tendering shareholder is pivotal, and the raider appropriates 50 percent of the firm's value improvement. But the raider can do even better when individual shareholdings are divisible. For example, if each shareholder holds an even number of shares, the raider can take over by bidding  $\underline{v} - \Delta v$ . Even though this bid is *below* the firm's initial value, there exists an equilibrium in which each shareholder tenders 50 percent of his holdings. In this equilibrium each shareholder knows that in the event of a takeover, for which he is pivotal, his loss on the shares tendered is exactly offset by the appreciation of his remaining shares. Since in this equilibrium all shareholders' participation constraints bind, and *each* shareholder is pivotal, the raider appropriates 100 percent of the firm's value improvement, and implements the efficient takeover decision regardless of  $N$ .<sup>34</sup>

The above examples suggest that the principal may be able to make small agents pivotal in two ways: (i) by exploiting discontinuities in agents' payoffs, or (ii) by using mechanisms that respond discontinuously to small agents' deviations, such as those described in Proposition 10. In the remainder of this section I present two sets of conditions that rule out such situations, and ensure that small agents are asymptotically nonpivotal.

33. I assume that shareholders can be induced to coordinate on such an equilibrium. Holmstrom and Nalebuff [1992] assume instead that such coordination is infeasible, and restrict attention to *symmetric* mixed-strategy equilibria, in which the principal generally does worse than in the equilibrium we describe. However, Holmstrom and Nalebuff identify a "focal" symmetric equilibrium in which, as shares become infinitely divisible, randomization disappears and the raider captures 50 percent of the firm's value improvement.

34. Bagnoli and Lipman only study indivisible shareholdings, and Holmstrom and Nalebuff do not consider bids below the firm's current value. For these reasons, in neither paper can the raider appropriate more than 50 percent of the firm's value improvement with bilateral offers.

*A. The Asymptotic Setting and Nonpivotal Outcomes*

Consider a sequence of environments with  $N$  identical agents,  $N = 1, 2, \dots$ , which satisfy Condition L. Let the utility function of each agent  $i$  in the environment with  $N$  agents be  $x_i \alpha(X) + (1/N)\beta(X)$ , so that the total surplus function  $W(X) = X\alpha(X) + \beta(X)$  is independent of  $N$ . Assume also that the trade domain of each agent is  $\mathfrak{X}_i = \mathfrak{X}/N$ , where  $\mathfrak{X}$  is a compact subset of  $\mathfrak{R}_+$ , so that the maximum aggregate trade is independent of  $N$ . The set of feasible aggregate trades with  $N$  agents is  $\Sigma_i \mathfrak{X}_i = {}^N\mathfrak{X}/N$ , with the notation

$${}^J\mathfrak{X} = \underbrace{\mathfrak{X} + \dots + \mathfrak{X}}_{J \text{ times}}.$$

Asymptotically, we will allow all aggregate trades from the convex hull of  $\mathfrak{X}$ , denoted by  $\bar{\mathfrak{X}}$ . Observe that  ${}^N\mathfrak{X}/N \subset \bar{\mathfrak{X}}$  for all  $N$ .

To define nonpivotal outcomes, observe that when agent  $i$  takes the aggregate trade  $X$  as given, i.e.,  $X(N) = X(N \setminus i) = X$ , his participation constraint (6) can be written as  $x_i \alpha(X) - t_i \geq 0$ . Expressing transfers from the binding participation constraints and substituting them into the principal's profit, the profit can be written as  $\pi_\infty(X) = F(X) + X\alpha(X)$ . The set of the principal's optimal nonpivotal aggregate trades can then be defined as  $M_\infty = \arg \max_{X \in \bar{\mathfrak{X}}} \pi_\infty(X)$ . Note that the principal exactly implements these nonpivotal outcomes with a finite  $N$  when she is restricted to use auctions. Hence, according to Proposition 9, these outcomes are inefficient due to the externalities on nontraders.

Since  $\bar{\mathfrak{X}}$  is a compact set, a sufficient condition for  $M_\infty$  to be nonempty is for  $\pi_\infty(\cdot)$  to be upper semi-continuous.<sup>35</sup> This condition is more frequently met in applications than the continuity of  $\pi_\infty(\cdot)$ . For example, in the Takeover setting,  $\pi_\infty(X) = -c(X)$ . In the often studied case where the raider has a fixed bidding cost, this function is not continuous, but it is upper semi-continuous, and the set of nonpivotal outcomes is  $M_\infty = \{0\}$ .

*B. Nonpivotalness with Continuous Mechanisms and Payoffs*

This subsection shows that nonpivotal outcomes must obtain asymptotically if the agents' payoffs are continuous and the

35. A function  $g(\cdot)$  is upper semi-continuous at  $x_0$  if for any  $\epsilon > 0$  there exists a neighborhood of  $x_0$  in which  $g(x) < g(x_0) + \epsilon$ . For example, such a function is obtained by taking a continuous function and increasing its value at  $x_0$ .

principal is restricted to use continuous mechanisms (as defined below). Consider the principal's profit-maximization program (8) in the asymptotic setting with  $N$  agents, supposing that the principal chooses from a mechanism family  $\tilde{\mathcal{Y}}^N$ . Note that Condition L allows us to express the program in terms of the mechanisms' aggregate trades only. The set  $M_N = \{X(N) : \mathbf{X} \in \Sigma \mathcal{W}_{\tilde{\mathcal{Y}}^N}\}$  of equilibrium aggregate trades in the principal's profit-maximizing mechanisms can then be described as

$$M_N = \arg \max_{X \in \Sigma \tilde{\mathcal{Y}}^N(N)} \pi_N(X),$$

$$\text{with } \pi_N(X) = W(X) - R_N(X) = \pi_\infty(X) + \beta(X) - R_N(X),$$

where  $R_N(X) = \min \{(1/N) \sum_{j=1}^N \beta(X(N \setminus j)) : \mathbf{X} \in \Sigma \tilde{\mathcal{Y}}^N | X\}$  is the minimum sum of agents' reservation utilities consistent with equilibrium aggregate trade  $X$ .

The sequence  $\{\tilde{\mathcal{Y}}^N\}_{N=1}^\infty$  of mechanism families will be required to have two properties. First, the families must be sufficiently rich, so that, if agents do feel nonpivotal, these mechanisms can asymptotically attain the principal's maximum profit in the nonpivotal program:

**DEFINITION 9.** A sequence  $\{\tilde{\mathcal{Y}}^N\}_{N=1}^\infty$  of mechanism families in the asymptotic setting is asymptotically adequate if  $\sup \pi_\infty(\Sigma \tilde{\mathcal{Y}}^N(N)) \rightarrow \sup \pi_\infty(\tilde{\mathcal{X}})$  as  $N \rightarrow \infty$ .

Since all the mechanisms described in this section satisfy the "full range" condition  $\Sigma \tilde{\mathcal{Y}}^N(N) = {}^N\mathcal{X}/N$ , they are asymptotically adequate provided that  $\sup \pi_\infty({}^N\mathcal{X}/N) \rightarrow \sup \pi_\infty(\tilde{\mathcal{X}})$  as  $N \rightarrow \infty$ .<sup>36</sup> The second assumption is that in these mechanisms, a single agent asymptotically has a negligible effect on the aggregate trade:

**DEFINITION 10.** A sequence  $\{\tilde{\mathcal{Y}}^N\}_{N=1}^\infty$  of mechanism families in the asymptotic setting is asymptotically continuous if  $\sup_{\mathbf{X} \in \Sigma \tilde{\mathcal{Y}}^N, j \in N} |X(N) - X(N \setminus j)| \rightarrow 0$  as  $N \rightarrow \infty$ .

This property is satisfied by bilateral contracting,<sup>37</sup> any-and-all bids, and auctions, but not by conditional bids.

36. This condition on agents' trade domains is in turn satisfied, e.g., when  $\mathcal{X}$  is a closed interval (in which case  ${}^N\mathcal{X}/N = \tilde{\mathcal{X}} = \mathcal{X}$  for all  $N$ ), or when  $\pi_\infty(\cdot)$  is continuous from a side. For an example where this condition fails, take  $\mathcal{X} = [0, 1]$  (in which case  $\cup_{N=1}^\infty {}^N\mathcal{X}/N$  is the set of rational numbers in  $[0, 1]$ ), and suppose that  $\pi_\infty(\cdot)$  is maximized at an irrational point, at which it is discontinuous.

37. More generally, it is satisfied by mechanisms in which each agent's message affects at most  $k$  other agents' trades with  $k$  bounded regardless of  $N$ .



In order to study the convergence of  $M_N$  to  $M_\infty$ , we need a notion of distance between two sets. For any two sets  $A, B \subset \mathfrak{R}$ , define  $d(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b|$ —a measure of how far  $A$  extends beyond  $B$ . For example,  $d(A, B) = 0$  whenever  $A \subset B$ . For future reference, also define  $d_+(A, B) = \sup_{a \in A} \inf_{b \in B} (a - b)_+$ —a measure of how far  $A$  extends above  $B$ . With this notation, we obtain

**PROPOSITION 11.** Suppose that the sequence  $\{\mathfrak{D}^N\}_{N=1}^\infty$  of mechanism families in the asymptotic setting is asymptotically adequate and continuous, and  $\beta(\cdot)$  is continuous on  $\bar{\mathfrak{X}}$ . Then (i)  $\sup \pi_N(\Sigma \mathfrak{D}^N(N)) \rightarrow \sup \pi_\infty(\bar{\mathfrak{X}})$ . In addition, if  $\pi_\infty(\cdot)$  is upper semi-continuous on  $\bar{\mathfrak{X}}$ , then (ii)  $d(M_N, M_\infty) \rightarrow 0$ , and (iii)  $d(W(M_N), W(M_\infty)) \rightarrow 0$ .

While complete proofs of this section’s results are given in Appendix 1, here I outline the logic of the proof of Proposition 11. Part (i) of the proposition follows from the continuity of  $\beta(\cdot)$  and the asymptotic continuity of mechanisms. The proof of part (ii), which says that the equilibrium correspondence  $M_N$  is upper hemi-continuous at  $N = \infty$ ,<sup>38</sup> is related to Ausubel and Deneckere’s [1993] extension of Berge’s “theorem of the maximum” to upper semi-continuous objective functions. Part (iii) follows directly from parts (i) and (ii).

Proposition 11 can help resolve the “Grossman-Hart Paradox” inasmuch as real-life takeovers are not discrete events. For example, a raider who acquires 49 percent of the firm might be able and willing to implement most of the value improvements that could be implemented by owning 51 percent of the firm. Even if these improvements are blocked by other shareholders, the raider’s large toehold makes it likely that she will gain full control in the future, anticipation of which drives up the firm’s market value [Shleifer and Vishny 1986; Harrington and Prokop 1993]. Therefore, there are reasons to expect the firm’s value  $v(X)$  to be continuous in the raider’s acquired stake  $X$ , in which case Proposition 11 allows us to treat small shareholders as nonpivotal, provided that the raider uses asymptotically continuous mechanisms.<sup>39</sup>

38. The correspondence need not be lower hemi-continuous: a counterexample can be found in the setting of Takeovers (A6) with no bidding costs (see Segal [1997]).

39. Another implication of Proposition 11 is that when  $\beta(\cdot)$  is continuous and  $N$  is large, the contracting outcome does not depend on which mechanisms the principal uses, so long as they are asymptotically adequate and continuous. In



### *C. Nonpivotalness with Noise*

When the principal is allowed to use arbitrary mechanisms, including discontinuous ones, the previous asymptotic result does not apply. Indeed, according to Proposition 10, the principal then implements an efficient outcome for any  $N$ , while responding to each agent's deviation with the harshest possible punishment. However, this discontinuity of the fully optimal mechanism seems unrealistic in environments in which  $N$  is large and the principal cannot forecast the number of accepting agents precisely. To formalize this intuition, I consider an asymptotic setting in which with some probability  $\epsilon_N > 0$  any given agent is unable to respond to the principal's offer (and his failure to respond is taken as rejection). I assume that agents' abilities to respond, which are independent across agents, are not observed by the principal, who is therefore unable to predict the number of responders precisely. This uncertainty will make it difficult to make agents pivotal.

An agent's inability to respond may or may not be observed by the agent himself. The former case is one of hidden information. For example, the agent may be unable to respond because his telephone is out of order, of which he is aware, but no one else is. (Other examples: the agent has not received the offer, or he has passed away.) The latter is a case of hidden action: an agent's acceptance decision is not perfectly observed by the principal. For example, the agent's acceptance message may be lost in the mail. This subsection's analysis will not depend on which of these interpretations is adopted.

Let  $A \subset N$  denote the random set of agents who are able to respond to the principal's offer. As before, without loss of generality we can restrict attention to direct mechanisms, in which all agents who are able to respond accept in equilibrium.<sup>40</sup> For these strategies to constitute a Nash equilibrium, each agent must prefer to accept when he knows that others accept whenever they can, i.e., the following participation constraints must be satisfied:

$$E_A[u_i(x(A)) - t_i(A)] \geq E_A[u_i(0, x_{-i}(A \setminus i))] \quad \text{for all } i \in N.$$

Note that unlike in the case of certainty studied before, all

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particular, since both auctions and bilateral contracts have these properties, the advantage of the former over the latter which was demonstrated by Katz and Shapiro [1986a] and Kamien, Oren, and Tauman [1992] in the context of Vertical Contracting (A1) (see footnote 27) disappears as  $N \rightarrow \infty$ .

40. By the logic of the modified revelation principle (Proposition 7), there is no need for the principal to offer different acceptance options to an agent, since she can predict which option the agent will choose if he is able to respond.

acceptance sets  $A \subset N$  will be observed in equilibrium with a positive probability. Hence, the values of  $x_i(A)$  and  $t_i(A)$  for all  $A$  are relevant for agent  $i$ 's acceptance decision. The principal will optimally choose transfers  $t_i(A)$  to make all agents' participation constraints bind, from which we can obtain each agent  $i$ 's expected payment to the principal,  $E_A[t_i(A)]$ .<sup>41</sup> Substituting in the principal's objective function, her expected profit can be written as

$$E_A \left[ f(x(A)) + \sum_{i \in A} t_i(A) \right] = E_A \left[ f(x(A)) + \sum_{i \in A} (u_i(x(A)) - u_i(0, x_{-i}(A \setminus i))) \right].$$

In the asymptotic setting of this section, the principal's expected profit can be written as a function of the mechanism's aggregate representation  $\mathbf{X} = \sum_i \mathbf{x}_i$  only:

$$(9) \quad \pi_N(\mathbf{X}) = E_A \left[ F(X(A)) + X(A)\alpha(X(A)) + \frac{1}{N} \sum_{i \in A} [\beta(X(A)) - \beta(X(A \setminus i))] \right].$$

Letting  $\mathfrak{D}_N$  denote the set of all direct mechanisms in the asymptotic setting with  $N$  agents, the set  $\mathbf{M}_N$  of the aggregate representations of the principal's profit-maximizing mechanisms can be defined as  $\mathbf{M}_N = \arg \max_{\mathbf{X} \in \mathfrak{D}_N} \pi_N(\mathbf{X})$ .

Since the acceptance set  $A$  is now random, we can think of  $\mathbf{X}$  as a random variable, and of  $\mathbf{M}_N$  as a random set. We will therefore use the concept of convergence in probability to establish the convergence of  $\mathbf{M}_N$  to the set  $M_\infty$  of nonpivotal aggregate trades.<sup>42</sup>

This section's result will require the domain  $\mathfrak{X}$  to satisfy the following property:

**DEFINITION 11.** The domain  $\mathfrak{X}$  is asymptotically adequate if for some  $\gamma < 1$ ,  $\sup \pi_\infty(\gamma^M \mathfrak{X} / N) \rightarrow \sup \pi_\infty(\bar{\mathfrak{X}})$  as  $N \rightarrow \infty$ .

This assumption ensures that if agents do feel nonpivotal, the principal can asymptotically attain her maximum profit in the nonpivotal program despite possible discontinuities in  $\pi_\infty(\cdot)$  (see

41. Given this expected payment, the principal is indifferent about the choice of payments  $t_i(A)$  for different acceptance sets  $A$ . In particular, she could charge noncontingent participation fees.

42. Recall that a sequence of random variables  $\{Z_N\}_{N=1}^\infty$  converges in probability to  $Z$  (written as  $Z \xrightarrow{p} Z$ ) if for any  $\delta > 0$ ,  $\Pr \|Z_N - Z\| \leq \delta \rightarrow 1$  as  $N \rightarrow \infty$ .

footnote 35) or noise. The assumption is satisfied, in particular, when  $M_\infty \neq \{\max \mathcal{X}\}$ , and either  $\pi_\infty(\cdot)$  is continuous from a side, or  $\mathcal{X}$  is a closed interval (in which case  ${}^N\mathcal{X}/N \equiv \bar{\mathcal{X}} = \mathcal{X}$ ).

Finally, an important condition for this section's result is that noise go to zero more slowly than  $N$  goes to infinity. Intuitively, it is hard to make an agent pivotal if the probability that any given number of responders is realized goes to zero as  $N \rightarrow \infty$ . This can be ensured by assuming that the variance in the number of signers,  $N\epsilon_N(1 - \epsilon_N)$ , goes to infinity. Then we obtain

**PROPOSITION 12.** Suppose that Condition L holds,  $\mathcal{X}$  is asymptotically adequate, and  $\beta(\cdot)$  and  $\pi_\infty(\cdot)$  are bounded on  $\bar{\mathcal{X}}$ . Suppose also that  $N\epsilon_N \rightarrow \infty$  and  $\epsilon_N \rightarrow 0$  as  $N \rightarrow \infty$ . Then (i)  $\sup \pi_N(\mathcal{D}_N) \rightarrow \sup \pi_\infty(\bar{\mathcal{X}})$ . In addition, if  $\pi_\infty(X)$  is upper semi-continuous, then (ii)  $d(\mathbf{M}_N, M_\infty) \xrightarrow{p} 0$ . In addition, if  $\beta(\cdot)$  is upper semi-continuous,<sup>43</sup> then (iii)  $d_+(W(\mathbf{M}_N), W(M_\infty)) \xrightarrow{p} 0$ . Moreover, if  $\beta(\cdot)$  is continuous, then (iv)  $d(W(\mathbf{M}_N), W(M_\infty)) \xrightarrow{p} 0$ .

To develop intuition for the result, consider the principal's incentives to choose aggregate trades  $X(A)$  for various acceptance sets  $A \subset N$ . First observe that the choice of  $X(N)$  affects only the total surplus in the event that  $A = N$  (all agents can respond), and does not affect any agent's reservation utility. Therefore, in any profit-maximizing mechanism the principal chooses  $X(N)$  to maximize total surplus. But what is the probability of  $A = N$ ? If  $N$  is large and  $\epsilon_N$  is small, this probability can be approximated by

$$(1 - \epsilon_N)^N = [(1 - \epsilon_N)^{-1/\epsilon_N}]^{-N\epsilon_N} \approx e^{-N\epsilon_N}.$$

Therefore, if  $N\epsilon_N$  could be bounded regardless of  $N$ , then with a positive probability, all agents would accept, and an efficient outcome would obtain. On the other hand, if, as we assume,  $N\epsilon_N \rightarrow \infty$ , the probability that all agents accept goes to zero as  $N \rightarrow \infty$ . The principal's optimal choice of  $X(A)$  for acceptance sets  $A \neq N$  is determined by two considerations: first, it affects the equilibrium surplus when the set of agents who are able to respond is  $A$ ; and second, it affects the reservation utility of each agent  $i \in N \setminus A$  when the set of agents who are able to respond is  $A \cup i$ . Thus, unlike in the case of perfect certainty, the principal can no longer distinguish perfectly between the in- and out-of-equilibrium situations, and surplus maximization and rent extraction cannot be separated. As a result of the optimal trade-off between the two

43. Which is satisfied in all applications described in Section II.

motives, the principal asymptotically implements a nonpivotal outcome.

Observe that in contrast to Proposition 11, parts (i) and (ii) of Proposition 12 do not require  $\beta(\cdot)$  to be continuous. Intuitively, noise smooths all discontinuities, be they in mechanisms or in payoffs. Thus, Proposition 11 supports and extends the Grossman-Hart conjecture on takeovers: regardless of the mechanisms used by the raider (e.g., conditional bids, any-and-all bids, restricted bids, etc.), and regardless of how the firm's value depends on the raider's stake, with a large number of shareholders and noise, an inefficient nonpivotal outcome obtains.

#### *D. Relation to Other Noisy Asymptotics Results*

This subsection relates Proposition 12 to the asymptotic results of Rob [1989]; Mailath and Postlewaite [1990]; Levine and Pesendorfer [1995]; Pauzner [1997]; Fudenberg, Levine, and Pesendorfer [1998]; and Al-Najjar and Smorodinsky [1997].<sup>44</sup> The last two papers, in particular, suggest that many existing noisy asymptotic results can be understood in terms of a simple mathematical fact, here labeled the "Nonpivotalness (NP) Principle." The Principle says that when the actions of  $N$  agents stochastically affect a bounded real random variable, the agents' average expected ex ante influence on this variable is bounded by a number that goes to zero as  $N \rightarrow \infty$ , provided that noise does not vanish too quickly.

Fudenberg, Levine, and Pesendorfer [1998] apply the NP Principle to games in which the principal controls a bounded number of real variables, and show that small agents in such games asymptotically take the principal's actions as given. This result extends the analysis of Levine and Pesendorfer [1995], who study a Takeover game in which the raider, following shareholders' tenders, makes a one-dimensional decision—whether to take over (accepting all tenders) or not (accepting none)—and find that small anonymous shareholders asymptotically take the raider's decision as given.

Al-Najjar and Smorodinsky [1997] show that the NP Principle can also be applied to the setting of Pure Public Good (A10), where the principal controls unboundedly many variables—the level of the public good and the monetary transfers to  $N$  agents.

44. The last three papers were discovered by me, in unpublished form, after completion of this paper's first draft.

Their analysis relies on the additive separability of agents' utilities in the level of the public good and private transfers. The fact that agents are asymptotically nonpivotal for the level of the public good implies that they will not contribute anything to its provision.

Neither approach, however, can be readily extended to the general mechanism design setting of this paper, in which the principal chooses the  $N$ -dimensional trade and transfer vectors  $x(A)$  and  $t(A)$  for each acceptance set  $A$ . In particular, under Condition L the utility of each agent  $i$  depends not only on the aggregate trade  $X$ , but also on his own trade  $x_i$  and payment  $t_i$ . Since the number of variables controlled by the principal is unbounded, the result of Fudenberg, Levine, and Pesendorfer [1998] cannot be applied: while the NP principle implies that the average agent is asymptotically nonpivotal for the expected value of  $X(A)$ , he may still be pivotal for  $x_i(A)$  and  $t_i(A)$ . The approach of Al-Najjar and Smorodinsky [1997] also cannot be used here, since agent  $i$ 's private trade  $x_i$  and the aggregate trade  $X$  are not in general separable in the agent's utility function.

It turns out, however, that due to the fact that the principal optimally makes agents' participation constraints bind, we can focus on their *reservation utility*  $\beta(X(A))$ . The NP principle implies that in any mechanism, the average agent asymptotically takes the expected reservation utility  $E_A\beta(X(A))$  as given. This implies that the third term in the expectation in (9) vanishes in the limit, and therefore the principal's profit converges to its nonpivotal level. Using Condition L, we also establish that both the aggregate trade and total surplus converge in probability to their nonpivotal levels (Proposition 12 (ii) and (iii)). Without Condition L, total surplus would depend on the allocation of the aggregate trade among agents, and these results could not be obtained. (Indeed, welfare results are absent from Fudenberg, Levine, and Pesendorfer [1998].)

Finally, note that Proposition 12 is substantially more general than previous asymptotic results. Unlike in the setting of Pure Public Good [Rob 1989; Mailath and Postlewaite 1990], this proposition allows contracting to have a private, as well as public, component. While agents asymptotically take the level  $X$  of the "public good" as given, they may still trade their private good with the principal. Also, while Levine and Pesendorfer's [1995] result was obtained for specific conditional Takeover bids, Proposition 12 demonstrates that the same nonpivotal outcome predicted by

Grossman and Hart [1980] obtains asymptotically when the principal can use arbitrary voluntary mechanisms.

VIII. CONCLUSION

This paper identifies and studies “transaction costs” that arise in environments with multilateral externalities even when all agents can participate in contracting. When the principal lacks commitment power (Section IV), when she uses bilateral contracts or other restricted mechanisms (Sections III and VI), or when the number of agents is large and there is some noise in the execution of the mechanism (Section VII), inefficient contracting outcomes arise.

The identified inefficiencies are distinct from, although related to, those caused by asymmetric information. To be sure, the noise postulated for the noisy asymptotic result in Section VII is due to agents’ private information. However, as (i) only a small amount of private information is needed to explain large contracting inefficiencies, (ii) the same inefficiencies arise when the principal is restricted to “reasonable” mechanisms (such as bilateral contracts), and (iii) these inefficiencies are important in many economic applications, they may deserve a separate place in the arsenal of economic theory.

This paper has made only a first step toward understanding “transaction costs” arising in contracting with externalities. Many questions remain open, in particular those concerning the role of bargaining procedures and property rights in aggravating or alleviating contracting inefficiencies.

APPENDIX 1: PROOFS OF ASYMPTOTIC RESULTS

The following three lemmas will be used in the proofs.

LEMMA A.1 (**Triangle Inequality**). For any three sets  $A, B, C \subset \mathfrak{R}$ ,  $d_{[+]}(A, C) \leq d_{[+]}(A, B) + d_{[+]}(B, C)$ .

*Proof.* Follows from the triangle inequality:  $|(a - c)_{[+]}| \leq |(a - b)_{[+]}| + |(b - c)_{[+]}|$ . ■

LEMMA A.2. If  $d(M_N, M_\infty) \rightarrow 0$ ,  $M_\infty$  is compact, and  $\beta(\cdot)$  is upper semi-continuous, then  $d_+(\beta(M_N), \beta(M_\infty)) \rightarrow 0$ . Moreover, if  $\beta(\cdot)$  is continuous, then  $d(\beta(M_N), \beta(M_\infty)) \rightarrow 0$ .

*Proof.* Suppose the first statement does not hold. Then there exists a sequence  $\{N_k\}_{k=1}^\infty$  such that  $d_+(\beta(X_{N_k}), \beta(M_\infty)) \geq \delta > 0$  and

$X_{N_k} \in M_{N_k}$  for all  $k$ . Since  $\bar{X}$  is compact, we can choose this sequence so that  $X_{N_k} \rightarrow \hat{X} \in \bar{X}$ . Since  $\beta(\cdot)$  is u.s.c.,  $d_+(\beta(\hat{X}), \beta(M_\infty)) \geq \delta$ . On the other hand,  $d(\hat{X}, M_\infty) = \lim_{k \rightarrow \infty} d(X_{N_k}, M_\infty) = 0$ , which together with compactness of  $M_\infty$  implies that  $\hat{X} \in M_\infty$ , and consequently  $d_+(\beta(\hat{X}), \beta(M_\infty)) = 0$ —a contradiction. The proof of the second statement is obtained by replacing  $d_+(\cdot)$  with  $d(\cdot)$  and u.s.c. with continuity. ■

LEMMA A.3. If  $\pi_\infty$  is upper semi-continuous,  $M_\infty = \arg \max_{X \in \bar{X}} \pi_\infty(X)$ , and  $d(\pi_\infty(M_N), \pi_\infty(M_\infty)) \rightarrow 0$ , then  $d(M_N, M_\infty) \rightarrow 0$ .

*Proof.* Suppose not. Then there exists a sequence  $\{N_k\}_{k=1}^\infty$  such that  $d(X_{N_k}, M_\infty) \geq \delta > 0$  for all  $k$ . Since  $\bar{X}$  is compact, we can choose this sequence so that  $X_{N_k} \rightarrow \hat{X}$ , with  $d(\hat{X}, M_\infty) \geq \delta > 0$ . On the other hand, by assumption,  $\pi_\infty(X_{N_k}) \rightarrow \pi_\infty(M_\infty)$ , and u.s.c. of  $\pi_\infty(\cdot)$  implies that  $\pi_\infty(\hat{X}) \geq \pi_\infty(M_\infty)$ , and therefore  $\hat{X} \in M_\infty$ —a contradiction. ■

*Proof of Proposition 11.* By the asymptotic continuity of mechanisms and the continuity of  $\beta(\cdot)$  (which is equivalent to uniform continuity),

$$\begin{aligned} (10) \quad \sup_{X \in \Sigma_{\delta^N}^N(N)} |\pi_N(X) - \pi_\infty(X)| &= \sup_{X \in \Sigma_{\delta^N}^N(N)} |R_N(X) - \beta(X)| \\ &\leq \sup_{X \in \Sigma_{\delta^N}^N, i \in N} |\beta(X(N \setminus i)) - \beta(X)| \\ &\rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} &|\sup \pi_N(\Sigma_{\delta^N}^N(N)) - \sup \pi_\infty(\bar{X})| \\ &\leq |\sup \pi_N(\Sigma_{\delta^N}^N(N)) - \sup \pi_\infty(\Sigma_{\delta^N}^N(N))| + |\sup \pi_\infty(\Sigma_{\delta^N}^N(N)) \\ &\quad - \sup \pi_\infty(\bar{X})| \leq \sup_{X \in \Sigma_{\delta^N}^N(N)} |\pi_N(X) - \pi_\infty(X)| \\ &\quad + |\sup \pi_\infty(\Sigma_{\delta^N}^N(N)) - \sup \pi_\infty(\bar{X})|, \end{aligned}$$

and (10) and the asymptotic adequacy of mechanisms imply (i).

Now, the triangle inequality implies that

$$\begin{aligned} d(\pi_\infty(M_N), \pi_\infty(M_\infty)) &\leq d(\pi_\infty(M_N), \pi_N(M_N)) \\ &\quad + d(\pi_N(M_N), \pi_\infty(M_\infty)) \leq \sup_{X \in \Sigma_{\delta^N}^N(N)} |\pi_N(X) - \pi_\infty(X)| \\ &\quad + |\sup \pi_N(\Sigma_{\delta^N}^N(N)) - \sup \pi_\infty(\bar{X})|. \end{aligned}$$

Using (10) and (i), we see that  $d(\pi_\infty(M_N), \pi_\infty(M_\infty)) \rightarrow 0$ , which together with Lemma A.3 implies (ii).

Finally, note that  $W(X) = \pi_\infty(X) + \beta(X)$ , and therefore

$$d(W(M_N), W(M_\infty)) \leq d(\pi_\infty(M_N), \pi_\infty(M_\infty)) + d(\beta(M_N), \beta(M_\infty)).$$

The first term has been shown to go to zero. The second term goes to zero by Lemma A.2, since  $\beta(\cdot)$  is continuous and (ii) holds. Thus, we obtain (iii). ■

*Proof of Proposition 12.* The principal's expected profit can be rewritten as

$$\pi_N(\mathbf{X}) = E_A \pi_\infty(X(A)) + E_A \left[ \sum_{j \in A} \frac{1}{N} [\beta(X(A)) - \beta(X(A \setminus j))] \right].$$

The first term is the profit in a nonpivotal mechanism. The second term can be rearranged as

$$\begin{aligned} & \sum_{A \subset N} \sum_{j \in A} p_N^A \frac{1}{N} [\beta(X(A)) - \beta(X(A \setminus j))] \\ &= \sum_{A \subset N} A p_N^A \frac{1}{N} \beta(X(A)) - \sum_{A \subset N} (N - A) p_N^{A+1} \frac{1}{N} \beta(X(A)) \\ &= \sum_{A \subset N} p_N^A \frac{1}{N} \left[ A - (N - A) \frac{1 - \epsilon}{\epsilon} \right] \beta(X(A)) \\ &= E_A \left[ \frac{A - N(1 - \epsilon)}{N\epsilon} \beta(X(A)) \right], \end{aligned}$$

where  $p_N^A = \epsilon^{N-A}(1 - \epsilon)^A$ —probability of a given acceptance set  $A \subset N$ .

This term can be bounded using Jensen's inequality:<sup>45</sup>

$$\begin{aligned} \left| E_A \left[ \frac{A - N(1 - \epsilon)}{N\epsilon} \beta(X(A)) \right] \right| &\leq \bar{\beta} E_A \left| \frac{A - N(1 - \epsilon)}{N\epsilon} \right| \\ &\leq \bar{\beta} \sqrt{E_A \left( \frac{A - N(1 - \epsilon)}{N\epsilon} \right)^2} = \bar{\beta} \sqrt{\text{var} \left( \frac{A}{N\epsilon} \right)} \\ &= \bar{\beta} \sqrt{\frac{N\epsilon(1 - \epsilon)}{N^2\epsilon^2}} = \bar{\beta} \sqrt{\frac{1 - \epsilon}{N\epsilon}}, \end{aligned}$$

45. I am grateful to Jim Powell for suggesting this bound.



where  $\bar{\beta} = \sup |\beta(\mathfrak{X})|$ . Therefore,

$$(11) \quad \left| \sup_{\mathbf{x} \in \Sigma \mathfrak{D}_N} \pi_N(\Sigma \mathfrak{D}_N) - \sup_{\mathbf{x} \in \Sigma \mathfrak{D}_N} E_A \pi_\infty(X(A)) \right| \\ \leq \sup_{\mathbf{x} \in \Sigma \mathfrak{D}_N} |\pi_N(\mathbf{X}) - E_A \pi_\infty(X(A))| \leq \bar{\beta} \sqrt{\frac{1 - \epsilon_N}{N \epsilon_N}} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Now write

$$(12) \quad \left| \sup_{\mathbf{x} \in \Sigma \mathfrak{D}_N} \pi_N(\Sigma \mathfrak{D}_N) - \sup \pi_\infty(\bar{\mathfrak{X}}) \right| \\ \leq \left| \sup_{\mathbf{x} \in \Sigma \mathfrak{D}_N} \pi_N(\Sigma \mathfrak{D}_N) - \sup_{\mathbf{x} \in \Sigma \mathfrak{D}_N} E_A \pi_\infty(X(A)) \right| \\ + \left| \sup_{\mathbf{x} \in \Sigma \mathfrak{D}_N} E_A \pi_\infty(X(A)) - \sup \pi_\infty(\bar{\mathfrak{X}}) \right|.$$

The first term has been shown to go to zero, so it remains to show that the second term goes to zero as well. For this purpose, observe that  ${}^{\gamma M} \mathfrak{X}/N \subset A \mathfrak{X}/N \subset \bar{\mathfrak{X}}$  when  $A > \gamma N$ , and therefore,

$$(13) \quad \sup \pi_\infty(\bar{\mathfrak{X}}) \geq \sup_{\mathbf{x} \in \Sigma \mathfrak{D}_N} E_A \pi_\infty(X(A)) = E_A \sup \pi_\infty(A \mathfrak{X}/N) \\ \geq \sup \pi_\infty({}^{\gamma M} \mathfrak{X}/N) - \Pr \{A \leq \gamma N\} \cdot 2 \sup |\pi_\infty(\bar{\mathfrak{X}})|.$$

As  $N \rightarrow \infty$ , the first term in the last expression goes to  $\sup \pi_\infty(\bar{\mathfrak{X}})$  by asymptotic adequacy of  $\bar{\mathfrak{X}}$ . To bound the second term, we use Chebyshev's inequality, which says that

$$(14) \quad \Pr \left\{ \left| \frac{A - N(1 - \epsilon)}{N \epsilon} \right| \geq \delta \right\} \leq \frac{\text{var}(A)}{\delta^2 (N \epsilon)^2} = \frac{1 - \epsilon}{\delta^2 N \epsilon} \text{ for any } \delta > 0.$$

The inequality implies that

$$(15) \quad \Pr \{A \leq \gamma N\} \leq \Pr \left\{ \left| \frac{A - N(1 - \epsilon)}{N \epsilon} \right| \geq \frac{1 - \epsilon - \gamma}{\epsilon} \right\} \\ \leq \frac{(1 - \epsilon) \epsilon_N^2}{(1 - \epsilon - \gamma)^2 N \epsilon_N} \rightarrow 0 \text{ as } N \rightarrow \infty,$$

since by assumption  $\epsilon_N \rightarrow 0$  and  $N \epsilon_N \rightarrow \infty$ . Therefore, the double inequality (13) implies that  $\sup_{\mathbf{x} \in \Sigma \mathfrak{D}_N} E_A \pi_\infty(X(A)) \rightarrow \sup \pi_\infty(\bar{\mathfrak{X}})$  as  $N \rightarrow \infty$ . Now (12) and (11) imply (i).

To show (ii), rewrite the principal's problem as

$$\max_{\mathbf{x} \in \Sigma \mathfrak{D}} E_A \left[ \pi_\infty(X(A)) + \frac{A - N(1 - \epsilon)}{N \epsilon} \beta(X(A)) \right].$$

The expectation can be maximized statewise, i.e., for every  $A$ ,

$$\begin{aligned} M_N(A) &= \arg \max_{X \in A\mathfrak{X}/N} \pi_N^A(X), \text{ where } \pi_N^A(X) \\ &= \pi_\infty(X) + \frac{A - N(1 - \epsilon)}{N\epsilon} \beta(X). \end{aligned}$$

Using (14) and the assumption that  $N\epsilon_N \rightarrow \infty$ , we see that  $\sup_{X \in \bar{\mathfrak{X}}} |\pi_N^A(X) - \pi_\infty(X)| \xrightarrow{P} 0$ .

Now, by the triangle inequality, when  $A > \gamma N$ , we can write

$$\begin{aligned} d(\pi_\infty(M_N(A)), \pi_\infty(M_\infty)) &\leq d(\pi_\infty(M_N(A)), \pi_N^A(M_N(A))) \\ &\quad + d(\pi_N^A(M_N(A)), \pi_\infty(M_\infty)) \leq \sup_{X \in \bar{\mathfrak{X}}} |\pi_N^A(X) - \pi_\infty(X)| \\ &\quad + |\sup \pi_N^A(A\mathfrak{X}/N) - \sup \pi_\infty(\bar{\mathfrak{X}})| \leq 2 \sup_{X \in \bar{\mathfrak{X}}} |\pi_N^A(X) - \pi_\infty(X)| \\ &\quad + |\sup \pi_\infty(A\mathfrak{X}/N) - \sup \pi_\infty(\bar{\mathfrak{X}})| \leq 2 \sup_{X \in \bar{\mathfrak{X}}} |\pi_N^A(X) - \pi_\infty(X)| \\ &\quad + |\sup \pi_\infty(\lceil \gamma N \rceil \mathfrak{X}/N) - \sup \pi_\infty(\bar{\mathfrak{X}})|. \end{aligned}$$

The first term has been shown to go to zero in probability, and the second term goes to zero by the asymptotic adequacy of  $\bar{\mathfrak{X}}$ . Since  $\Pr\{A > \gamma N\} \rightarrow 1$  by (15), we have

$$(16) \quad d(\pi_\infty(\mathbf{M}_N), \pi_\infty(M_\infty)) \xrightarrow{P} 0.$$

Since  $\pi_\infty(\cdot)$  is u.s.c., Lemma A.3 now implies (ii).

For the welfare results, write  $W(X(A)) = \pi_\infty(X(A)) + \beta(X(A))$ , and therefore,

$$d_+(W(\mathbf{M}_N), W(M_\infty)) \leq d_+(\pi_\infty(\mathbf{M}_N), \pi_\infty(M_\infty)) + d_+(\beta(\mathbf{M}_N), \beta(M_\infty)).$$

The first term is zero. When  $\beta(\cdot)$  is u.s.c. and (ii), the second term goes to zero in probability by Lemma A.2. Thus, we obtain (iii).

Finally, write

$$d(W(\mathbf{M}_N), W(M_\infty)) \leq d(\pi_\infty(\mathbf{M}_N), \pi_\infty(M_\infty)) + d(\beta(\mathbf{M}_N), \beta(M_\infty)).$$

When  $\beta(\cdot)$  is continuous and (ii), Lemma A.2 implies that the second term goes to zero in probability. Using in addition (16), we obtain (iv). ■

## APPENDIX 2: EXISTENCE OF EQUILIBRIA WITH PRIVATE OFFERS

A set of sufficient conditions for equilibrium existence is provided by the following proposition.

PROPOSITION A.1. When  $\bar{x}_i$  is an interval for all  $i$ , and the function

$$g(x, \bar{x}) = f(x) + \sum_i u_i(x_i, \bar{x}_{-i})$$

is continuous in  $(x, \bar{x})$  and quasi-concave in  $x$ , we have  $\mathcal{C} \neq \emptyset$ .

*Proof.* A trade profile  $\bar{x}$  satisfies the equilibrium condition (5) if and only if  $\hat{x} \in B(\bar{x}) = \arg \max_{x \in \bar{x}_1 \times \dots \times \bar{x}_N} g(x, \bar{x})$ . Under the assumptions, the correspondence  $B(\bar{x})$  satisfies the conditions of Kakutani's fixed point theorem. Therefore, a solution  $\hat{x}$  exists. ■

Quasi-concavity of  $g(x, \bar{x})$  in  $x$  can be ensured by concavity of  $f(x)$  and concavity of  $u_i(x_i, \bar{x}_{-i})$  in  $x_i$  for all  $i$  and all  $\bar{x}_{-i}$ . In the context of Vertical Contracting (A1) with homogeneous final goods and Bertrand-Edgeworth downstream competition, this means that the supplier's cost function is convex and the revenue functions are quasi concave, which generalizes the setting of Hart and Tirole [1990] and Rey and Tirole [1996].

In an asymptotic setting with a large  $N$ , assuming that the second derivatives of  $F(\cdot)$ ,  $\alpha(\cdot)$ ,  $\beta(\cdot)$  exist and are bounded on  $\bar{\mathcal{X}}$ , we can write

$$g_N(x, \bar{x}) = F\left(\sum_i x_i\right) + \sum_i \left[ x_i \alpha\left(x_i + \sum_{j \neq i} \bar{x}_j\right) + \frac{1}{N} \beta\left(x_i + \sum_{j \neq i} \bar{x}_j\right) \right],$$

$$\frac{\partial^2 g_N(x, \bar{x})}{\partial x_k \partial x_l} = F''\left(\sum_i x_i\right) + 2\delta_{kl} \alpha'\left(x_k + \sum_{j \neq k} \bar{x}_j\right) + O\left(\frac{1}{N}\right),$$

where

$$\delta_{kl} = \begin{cases} 1 & \text{when } k = l, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, having  $F(\cdot)$  concave and  $\alpha'(X) < 0$  for all  $X$  is sufficient to satisfy the conditions of Proposition A.1 for  $N$  large enough. On the other hand, when  $\alpha'(X) > 0$  for all  $X$ , the conditions of Proposition A.1 are violated for a large  $N$ . This can be seen by

checking that the  $3 \times 3$  leading principal minor of the bordered Hessian of  $g_N(\cdot, \hat{x})$  at the point  $x = \hat{x} = (\bar{X}/N, \dots, \bar{X}/N)$  is positive for  $N$  large enough. Therefore, in this case the function  $g_N(\cdot, \hat{x})$  is not quasi concave.

Since  $\alpha'(\cdot) < 0$  asymptotically corresponds to the property of decreasing externalities, this suggests that the existence of equilibrium is intimately tied to this property. While I do not have a general result to this effect, I have a result under Condition S:

PROPOSITION A.2. Suppose that Condition S holds. Then

- (i) When externalities are decreasing and  $F(\cdot)$  is concave, we have  $\mathfrak{C} \neq \mathfrak{Q}$ .
- (ii) When  $u_i(1, x_{-i}) - u_i(0, x_{-i})$  is strictly increasing in  $x_{-i}$  (i.e., externalities are strictly increasing), any equilibrium  $\hat{x} \in \mathfrak{C}$  must have  $\hat{x}_1 = \dots = \hat{x}_N$ .

*Proof.* Under Condition S, we can write  $u_i(x) = U(x_i, \sum_j x_j)$ . Define

$$\Delta(X) = \begin{cases} [F(X+1) + U(1, X+1)] - [F(X) + U(0, X)] & \text{when } X \in \{0, \dots, N-1\}, \\ 0 & \text{when } X < 0 \text{ or } X > N-1. \end{cases}$$

Note that  $\Delta(X)$  can be interpreted in two ways (when agents hold passive beliefs): (i) as the principal's gain from trading with one more agent when she is expected to trade with  $X$  agents, and (ii) as the principal's loss from trading with one fewer agent when she is expected to trade with  $X+1$  agents. It is then clear that  $X \in \{0, \dots, N\}$  is a "pairwise equilibrium" (as defined in footnote 18) if and only if  $\Delta(X) \leq 0$  and  $\Delta(X-1) \geq 0$ . The point  $\hat{X} = \min \{X \in \{0, \dots, N\} : \Delta(X) \leq 0\}$  satisfies this condition, and must exist (the set is nonempty by virtue of including  $N$ , and it is finite). Therefore, a pairwise equilibrium  $\hat{X}$  must exist.

To see that  $\hat{X}$  must be a true equilibrium, consider the general multiagent deviation, in which the principal trades with  $k$  new agents and gives up trade with  $l$  old agents. The principal's gain from this deviation can be written as

$$F(\hat{X} + k - l) - F(\hat{X}) + k[U(1, \hat{X} + 1) - U(0, \hat{X})] - l[U(1, \hat{X}) - U(0, \hat{X} - 1)].$$

When  $k \geq l$ , this gain can be rewritten as

$$F(\hat{X} + k - l) - F(\hat{X}) + (k - l)[U(1, \hat{X} + 1) - U(0, \hat{X})] \\ + l[(U(1, \hat{X} + 1) - U(0, \hat{X})) - (U(1, \hat{X}) - U(0, \hat{X} - 1))].$$

When  $F(\cdot)$  is concave, the first term can be bounded from above by  $(k - l)\Delta(\hat{X}) \leq 0$ . With decreasing externalities, the second term must also be nonpositive. Thus, the deviation is unprofitable. Since a similar argument works when  $k < l$ ,  $\hat{X}$  must be a true equilibrium, which establishes (i).

To see (ii), suppose in negation that we have  $\hat{x} \in \mathcal{C}$  with  $\hat{x}_k = 1$  and  $\hat{x}_l = 0$ . Consider the principal's deviation in which she trades with one new agent and gives up trade with one old agent. Letting  $\hat{X} = \sum_i \hat{x}_i$ , the principal's gain can be written as

$$[U(1, \hat{X} + 1) - U(0, \hat{X})] - [U(1, \hat{X}) - U(0, \hat{X} - 1)].$$

With strictly increasing externalities, the principal's gain must be strictly positive, which contradicts the hypothesis that  $\hat{x} \in \mathcal{C}$ . ■

While this result does not rule out the possibility that private-offers equilibria exist with increasing externalities, it does reinforce our intuition that such existence is, in some sense, less likely than with decreasing externalities. One way to ensure existence is to allow the agents to hold arbitrary, and not just passive, beliefs. Segal and Whinston [forthcoming] examine such equilibria in the context of Exclusive Dealing (A2), which exhibits increasing externalities.

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