Logic and Probability

Probabilities on rich languages, random structures and 0-1 laws

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Measure theory vs. probabilities over a language

Probability spaces in the measure theoretic sense are structures $(\Omega, \mathcal{E}, \mu)$ with

- $(\Omega, \mathcal{E})$ a measurable space, i.e. we have
  - Ω is an arbitrary set
  - $\mathcal{E}$ is a σ-algebra over Ω, i.e., a subset of $\wp(\Omega)$ closed under complement and countable unions.
- $\mu : \mathfrak{E} \to [0, 1]$ a countably additive measure, i.e.
  - $\mu(\Omega) = 1$;
  - $\mu(\bigcup_{i \in \mathbb{N}}) = \sum_{n=0}^{\infty} \mu(E_i)$, when $E_i \cap E_j = \emptyset$ for $i \neq j$.

How do these relate to probabilities defined directly on logical languages?

Everyone says "consider the probability that $X \geq 0$," where X is a random variable, and only the pedant insists on replacing this phrase by "consider the measure of the set $\{\omega \in \Omega : X(\omega) \geq 0\}$." Indeed, when a process is specified, only the distribution is of interest, not a particular underlying sample space. In other words, practice shows that it is more natural in many situations to assign probabilities to *statements* rather than *sets*.

—Scott & Krauss 1966

Suppose we have a countable propositional language $\mathcal{L}$:

$$\varphi \quad ::= \quad A_1 \mid A_2 \mid \ldots \mid \varphi \wedge \varphi \mid \neg\varphi$$

We can define a probability $\mathbb{P} : \mathcal{L} \to [0, 1]$ directly on $\mathcal{L}$:

- $\mathbb{P}(\varphi) = 1$, for any tautology φ ;
- $\mathbb{P}(\varphi \vee \psi) = \mathbb{P}(\varphi) + \mathbb{P}(\psi)$, whenever $\vDash \neg(\varphi \wedge \psi)$.

Equivalent set of requirements:

- $\mathbb{P}(\varphi) = 1$, for any tautology ;
- $\mathbb{P}(\varphi) \leq \mathbb{P}(\psi)$ whenever $\vDash \varphi \to \psi$;
- $\mathbb{P}(\varphi) = \mathbb{P}(\varphi \wedge \psi) + \mathbb{P}(\varphi \wedge \neg\psi)$.

Some measure-theoretic notions

A family of subsets $\mathcal{R} \subseteq \wp(\Omega)$ forms a ring if

- $\emptyset \in \mathcal{R}$
- If $A, B \in \mathcal{R}$ then $A \cup B \in \mathcal{R}$ and $A \setminus B \in \mathcal{R}$

A measure μ is finite if $\mu(\Omega)$ is finite.

Given a family of subsets $\mathcal{F} \subseteq \wp(\Omega)$, let $\sigma(\mathcal{F})$ the smallest σ-algebra containing $\mathcal{F}$.

Theorem (Carathéodory's Extension Theorem)

Let μ be a measure on a ring $(\Omega, \mathcal{R})$. If μ is a finite measure that is σ-additive on $\mathcal{R}$, then there is a unique σ-additive measure μ' on $\sigma(\mathcal{R})$ that extends μ.

From Probabilities on Languages to Spaces

- Let $\mathcal{V}$ be the set of all valuations in language $\mathcal{L}$.
- Let $\mathcal{O} \triangleq \{[\![\varphi]\!] : \varphi \in \mathcal{L}\}$, where $[\![\varphi]\!] = \{v : v \vDash \varphi\}$. Then $\mathcal{O}$ forms a Boolean algebra, hence also a ring. Moreover, any probability measure $\mathbb{P}$ generates a measure that is σ-additive on $\mathcal{O}$. By the Carathéodory Extension Theorem, it uniquely extends to a σ-additive measure on the smallest σ-algebra extending $\mathcal{O}$ [this uses Compactness!].
- In fact, $\mathcal{O}$ forms a clopen basis of a topology on $\mathcal{V}$, which is homeomorphic to standard Cantor space (coin-tossing space: space of infinite binary sequences with clopen basis of cylinder sets). The σ-algebra generated by $\mathcal{O}$ is the standard Borel σ-algebra on Cantor space.
- In this way we can show that all functions $\mathbb{P} : \mathcal{L} \to [0, 1]$ can define all the usual probability measures (Borel measures).

Probabilities on propositional calculi are general, but not particularly expressive.

Let $\mathcal{L}$ be a first-order logical language, given by:

- a set $\mathcal{V}$ of individual variables ;
- a set $\mathcal{C}$ of individual constants ;
- a set $\mathcal{P}$ of predicate variables .

Terms and formulas of $\mathcal{L}$ are defined as usual:

$$\varphi \quad ::= \quad R(t_1, \ldots, t_n) \mid \varphi \wedge \varphi \mid \neg\varphi \mid \exists x\varphi \mid \forall x\varphi$$

Define $\mathcal{S}_{\mathcal{L}}$ to be the set of sentences of $\mathcal{L}$, i.e., formulas with no free variables, and $\mathcal{S}^0_{\mathcal{L}}$ to be the set of quantifier-free sentences of $\mathcal{L}$.

A probability on $\mathcal{L}' \subseteq \mathcal{S}_{\mathcal{L}}$ is a function $\mathbb{P} : \mathcal{L}' \to [0, 1]$, with

- $\mathbb{P}(\varphi) = 1$, for any first-order validity φ ;
- $\mathbb{P}(\varphi \vee \psi) = \mathbb{P}(\varphi) + \mathbb{P}(\psi)$, whenever $\vDash \neg(\varphi \wedge \psi)$.

Question: Given a probability $\mathbb{P} : \mathcal{S}^0_{\mathcal{L}} \to [0, 1]$, is there a natural extension of $\mathbb{P}$ to all of $\mathcal{S}_{\mathcal{L}}$?

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If there are only finitely many constants c such that $\mathbb{P}\big(R(c)\big) > 0$, then:

$$\mathbb{P}\big(\exists x R(x)\big) = \mathbb{P}\Big(\bigvee_{c \in \mathcal{C}} R(c)\Big)$$

What about in the case where the size of $\mathcal{C}$ is infinite?

Example

Consider a simple first-order arithmetical language $\mathcal{L}$, with a constant $\mathbf{n}$ for each $n \in \mathbb{Z}^{+} = \{1, 2, 3 \ldots\}$. Let $R(x)$ be a one-place predicate. Define a probability function $\mathbb{P} : \mathcal{S}^0_{\mathcal{L}} \to [0, 1]$ on the quantifier-free sentences so that:

- $\mathbb{P}\big(R(\mathbf{n})\big) \; = \; 2^{-(n+1)}$, for all $n \in \mathbb{N}$;
- $\mathbb{P}\big(\bigwedge_{i \leq k} R(\mathbf{n}_i)\big) \; = \; \prod_{i \leq k} \mathbb{P}\big(R(\mathbf{n}_i)\big)$.

In this case we should expect:

$$\mathbb{P}\big(\exists x R(x)\big) \; = \; \sum_{n=2}^{\infty} \frac{1}{2^n} \; = \; \frac{1}{2} \, .$$

Let us assume in what follows that we have a countably infinite set of constant symbols.

Definition (Gaifman's Condition)

A probability $\mathbb{P} : \mathcal{S}_{\mathcal{L}} \to [0,1]$ satisfies the Gaifman condition if for all formulas with one free variable $\varphi(x)$:

$$\mathbb{P}\big(\exists x \varphi(x)\big) \;=\; \sup \{\mathbb{P}\big(\bigvee_{i=1}^{n} \varphi(c_i)\big) \mid c_1, \ldots, c_n \in \mathcal{C}\} ,$$

or equivalently,

$$\mathbb{P}\big(\forall x \varphi(x)\big) \;=\; \inf \{\mathbb{P}\big(\bigwedge_{i=1}^{n} \varphi(c_i)\big) \mid c_1, \ldots, c_n \in \mathcal{C}\} .$$

Theorem (Gaifman 1964)

Given $\mathbb{P}' : \mathcal{S}^0_{\mathcal{L}} \to [0,1]$, there is exactly one extension $\mathbb{P}$ of $\mathbb{P}'$ to all of $\mathcal{S}_{\mathcal{L}}$ that satisfies the Gaifman condition.

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Proof of Uniqueness.

Suppose we have $\mathbb{P}_1$ and $\mathbb{P}_2$ that agree on all of $\mathcal{S}^0_{\mathcal{L}}$. We show by induction on quantifier complexity that they agree on all $\varphi \in \mathcal{S}_{\mathcal{L}}$. Suppose the $\mathbb{P}_i$'s agree all Π_n sentences. Let φ a Σ_{n+1} sentence. We have $\varphi = \exists\vec{x}\psi(\vec{x})$ where $\psi(\vec{x})$ is Π_n. Now, since both satisfy the Gaifman condition, we have

$$\mathbb{P}_i(\varphi) = \lim_{n\to\infty} \mathbb{P}_i\Big(\bigvee_{k_1,\dots,k_m<n} \psi(\mathbf{c}_{\mathbf{k_1}},\dots,\mathbf{c}_{\mathbf{k_m}})\Big).$$

Each $\psi(\mathbf{c}_{\mathbf{k_1}},\dots,\mathbf{c}_{\mathbf{k_m}})$ is a Π_n *sentence*. Since Π_n sentences are closed under disjunctions, each such $\bigvee_{k_1,\dots,k_m<n} \psi(\mathbf{c}_{\mathbf{k_1}},\dots,\mathbf{c}_{\mathbf{k_m}})$ is also a Π_n sentence, and by inductive hypothesis $\mathbb{P}_1$ and $\mathbb{P}_2$ must agree on it. This uniquely determines the limit above, and so the $\mathbb{P}_i$'s must agree on φ. The same argument works for Π_{n+1} sentences, using the closure of Σ_n sentences under conjunctions. □

Theorem (Gaifman 1964)

Given $\mathbb{P}' : \mathcal{S}^0_{\mathcal{L}} \to [0,1]$, there is exactly one extension $\mathbb{P}$ of $\mathbb{P}'$ to all of $\mathcal{S}_{\mathcal{L}}$ that satisfies the Gaifman condition.

Proof Sketch of Existence.

Consider the space Mod_ω of all countable models with a fixed countable domain (take as domain set of constants $\mathcal{C}$). As in propositional case, let $[\![\varphi]\!] \triangleq \{\mathcal{M} = (\mathcal{C}, \mathcal{I}) : \mathcal{M} \vDash \varphi\}$ for each $\varphi \in \mathcal{S}^0_{\mathcal{L}}$. This defines a Boolean algebra $\mathcal{B}_0$ (hence a ring) in the obvious way, and we can define a measure $\mu([\![\varphi]\!]) = \mathbb{P}(\varphi)$, which can be canonically uniquely extended (by Carathéodory again) to a (countably additive) measure μ^* on the full σ-algebra $\sigma(\mathcal{B}_0)$ (NB. we use compactness!). Lastly, $[\![\exists x \varphi(x)]\!] = \bigcup_{c \in \mathcal{C}} [\![\varphi(c)]\!]$, so all sets of this form are in the σ-algebra. If we define $\mathbb{P}^*\big(\exists x \varphi(x)\big) \triangleq \mu^*\big([\![\exists x \varphi(x)]\!]\big)$, then countable additivity guarantees the Gaifman condition. □

The space of models

We have built a measure μ on the space of countable models.

Mod_ω is the space of countable structures $\{\mathfrak{M} \text{ an } \mathcal{L}\text{-model} \mid dom(\mathfrak{M}) = \omega\}$ with the topology generated by opens

$$[\![\pm R(\bar{a})]\!] := \{\mathcal{M} \in \text{Mod}_\omega \mid \mathcal{M} \models \pm R(\bar{a})\} \text{ with } \bar{a} \in \omega^{<\omega}$$

This is a Polish space: it is homeomorphic to the Cantor space $(2^\omega, \mathcal{O})$ with $\mathcal{O}$ generated by cylinder sets.

The same is true in the propositional case. if we take the space $(\mathcal{V}, \mathcal{O})$ with $\mathcal{O}$ the topology generated by $[\![\bigwedge_{i\leq n} \pm p_i]\!] = \{v \in \mathcal{V} \mid v \vDash \bigwedge_{i\leq n} \pm p_i\}$.

In both cases, we can treat probability functions on our language $\mathcal{L}$ as probability measures on the standard Borel space Mod_ω.

In this sense we can get all the standard Borel measures: and we already have this with measures on propositional languages with countably many atomic propositions.

From Probabilities on Languages to Spaces

Given a probability measure $\mathbb{P}$ on $\mathcal{L}$, we can see it as

- A measure on the Lindenbaum-Tarski algebra $\mathcal{L}/\equiv$ (the algebra of equivalence classes of formulas modulo logical equivalence), where we let

$$\mathbb{P}^*([\varphi]) := \mathbb{P}(\varphi)$$

- The induced countably additive measure μ on the space of models (/valuations), which satisfies:

$$\mu(\{v \in \mathcal{V} \mid v \models \varphi\}) = \mathbb{P}(\varphi)$$

One should be careful about treating these as the same thing!

From Probabilities on Languages to Spaces

One important difference:

- Consider a probability measure $\mathbb{P}$ on an infinite (countable) propositional language. The measure μ induced by $\mathbb{P}$ on $\mathrm{Mod}(\mathcal{L})$ is countably additive.
- ...but he measure $\mathbb{P}^*$ on the Lindenbaum-Tarski algebra always *fails to be countably additive* [Amer, 1985] and even badly so [Seidenfeld].

Takeway:
We can translate between the logical and measure-theoretic perspective without losing anything essential. (There are however some subtle points to take into consideration, such as the issue of σ-additivity.)

Now: when can logic and probability genuinely illuminate one another?

From logic to probability and back: the case of random structures.

Asymptotic probability of graph properties

What is a typical property of a graph?

- Let $\mathbb{G}_n$ the set of all (labelled) graphs on n vertices.
- For a well-defined graph property F, define

$$p_n(F) := \frac{|\{G \in \mathbb{G}_n \mid G \text{ has } F\}|}{|\mathbb{G}_n|}$$

- When does $\mathrm{P}(F) = \lim_{n\to\infty} p_n(F)$ exist? What proportion of finite graphs has property P (asymptotically)?

Asymptotic probability

Consider various properties for F:

- G has a complete subgraph of size m: $\lim_{n\to\infty} p_n(F) = 1$.
- G is planar: $\lim_{n\to\infty} p_n(F) = 0$.
- G has an odd number of vertices: *no asymptotic probability*.

Which properties have a limiting probability? Which ones are *typical*, in the sense of occurring almost surely?

0-1 law.

Let φ a FOL sentence. Define

$$p_n(\varphi) := \frac{|\{G \in \mathbb{G}_n \mid G \vDash \varphi\}|}{|\mathbb{G}_n|}$$

0-1 law. Let φ a FOL sentence. Then $\lim_{n\to\infty} p_n(\varphi)$ always exists, and takes a value in $\{0, 1\}$.

All first-order properties (1) have a limiting probability and (2) are either typical or atypical!

Alice's Restaurant Property

You can get anything you want at Alice's Restaurant.

$$\forall x_1, \ldots x_k, y_1, \ldots, y_m$$
$$\Big(\bigwedge_{i \le k, j \le m} x_i \neq y_j \rightarrow \exists z \big(\bigwedge_{i \le k} z \neq x_i \wedge R(z, x_i) \wedge \bigwedge_{i \le m} z \neq y_i \wedge \neg R(z, y_i) \big) \Big)$$

Given $X = \{x_1, \ldots x_k\}$ and $Y = \{y_1, \ldots, y_m\}$ we say z as above is a *witness* for X and Y: we write $\mathrm{W}(z, X, Y)$.

There is a unique (up to isomorphism) countably infinite graph with the ARP.

Uniqueness: the AFP gives a winning strategy for Duplicator in EF_ω.
Existence?

Probabilistic construction

Take $\mathbb{N}$ as vertex set, and for each $(n, m) \in \mathbb{N}^2$ with $n \neq m$, toss a fair coin to decide if $R(n, m)$. This random process generates a countable random structure $(\mathbb{N}, R)$. Now:

The Random graph. The procedure above almost surely generates a graph satisfying the Alice's Restaurant Property.

So by drawing edges independently at random with probability 1/2, we almost-surely generate the *unique* countable graph satisfying ARP. This is the **Random/Rado graph** $\mathfrak{R}$.

Probabilistic construction

Proof.

Fix $A = \{a_1, ..., a_k\}$ and $B = \{b_1, ..., b_m\}$ two disjoint sets of vertices. List all vertices $\langle v_n \rangle_{n \in \omega}$ not belonging to either set. For any such v_n, $\mathrm{P}(\mathrm{W}(A, B, v)) = 1/2^{k+m}$. The probability that *no other vertex* is a witness is

$$\mathrm{P}\Big(\bigcap_n \neg \mathrm{W}(v_n, A, B)\Big) = \lim_{n \to \infty} \mathrm{P}\Big(\bigcap_{i \le n} \neg \mathrm{W}(v_n, A, B)\Big)$$
$$= \lim_{n \to \infty} (1 - 1/2^{k+m})^n = 0$$

(edges are drawn independently). Now $\mathrm{P}(\neg ARP)$ is at most

$$\mathrm{P}\Big(\bigcup_{A, B \in S} \Big(\bigcap_n \neg \mathrm{W}(v_n, A, B)\Big)\Big)$$

where S ranges over disjoints pairs of finite sets of vertices. This is a *countable* union of probability 0 events, so it has probability 0.

□

Asymptotic probabilities and random structures

Now for the 0-1 law. Let $\alpha_{k,m}$ denote the sentence

$$\forall x_1, \dots x_k, y_1, \dots, y_m \Big(\bigwedge_{i \leq k, j \leq m} x_i \neq y_j \rightarrow \exists z \, W(z, x_1, \dots x_k, y_1, \dots, y_m) \Big)$$

Let $T_R := \{\alpha_{n,m} \mid n, m < \omega\}$.

Thm (Glebskii et al. [1969], Fagin [1976]). Let φ a first-order sentence. The following are equivalent:

- $\lim_{n \to \infty} p_n(\varphi) = 1$
- φ holds on the random graph;
- $T_R \vdash \varphi$.

A sentence φ holds almost surely—in almost all finite graphs—if and only if it holds on the random graph.

$$Cn(T_R) = Th(\mathfrak{R}) = \{\varphi \in Sent \mid \lim_{n \to \infty} p_n(\varphi) = 1\}$$

The 0-1 law

Proof.

By our back-and-forth argument, T_R is ω-categorical, and has no finite models: so it is complete. It has $\mathfrak{R}$ as a model, and so $T_R \vdash \varphi$ is equivalent to φ holding on the random graph. Next, we show that $T_R \vdash \varphi$ entails $\lim_{n\to\infty} p_n(\varphi) = 1$. $T_R \vdash \varphi$ means that there is a finite set Γ of extension axioms $\alpha_{k,m}$ such that $\Gamma \vdash \varphi$. It is enough to show that each $\alpha_{k,m}$ holds (asymptotically) almost surely.

As before, for a finite graph G of size n and two disjoint subsets $A, B \subseteq G$ of respective sizes k and m, the probability that *no* $v \in G \setminus (A \cup B)$ is a witness is $(1 - 1/2^{k+m})^{n-k-m}$.

□

The 0-1 law

Proof.

For sufficiently large n, $\alpha_{k,m}$ fails with probability at most

$$\binom{n}{k}\binom{n-k}{m}(1-1/2^{k+m})^{n-k-m}$$

and indeed an cruder upper bound for $\lim_{n\to\infty} p_n(\neg\alpha_{k,m})$ is

$$\lim_{n\to\infty} n^{k+m}(1-1/2^{k+m})^{n-k-m} = 0,$$

Now the expression is of the form $n^\alpha \times \beta^{n-\alpha}$ with α, β constants and $0 < \beta < 1$: the term $\beta^{n-\alpha}$ going to 0 exponentially, while n^α has only polynomial growth. So it goes to 0, and so we conclude $\lim_{n\to\infty} p_n(\neg\alpha_{k,m}) = 0$. □

The 0-1 law

Proof.

Lastly, we show that $\lim_{n\to\infty} p_n(\varphi) = 1$ entails $T_R \vdash \varphi$. Suppose $T_R \nvdash \varphi$. By completeness of T, we have $T_R \vdash \neg\varphi$. By the previous argument, this means that $\lim_{n\to\infty} p_n(\neg\varphi) = 1$, and so φ cannot hold in almost all finite graphs. □

Consequences

Thm Let φ a first-order sentence. The following are equivalent

- $\lim_{n\to\infty} p_n(\varphi) = 1$
- φ holds on the random graph;
- $T_R \vdash \varphi$.

Trakhtenbrot:

sure properties over finite structures are undecidable

The theory $T_R := \{\alpha_{n,m} \mid n, m < \omega\}$ is ω-categorical and so it is complete. The axiomatisation is also recursive. Consequence:

almost sure properties over finite graphs are decidable! (in fact, PSPACE)

Bonus: constructing the Rado graph

We built the random graph by randomly (i.i.d) deciding on each potential edge $(a, b) \in \mathbb{N}^2$. But the infinite random graph is *easy to get*.

The brute-force construction: starting from the empty graph, build an infinite increasing sequence of graphs $G_0 \subsetneq \ldots G_n \subsetneq G_{n+1} \subsetneq \ldots$ as follows:

Given $G_n = (V_n, E_n)$, let $G_{n+1} = (V_{n+1}, E_{n+1})$ where

- $V_{n+1} := V_n \cup \{v_A \mid A \subseteq V_n\}$,
- $E_{n+1} \cap V_n^2 = E_n$,
- for all $A \subseteq V_n$, we let $E_{n+1}(v_A, x) \Leftrightarrow x \in A$

At each stage, for each subset of vertices, we add a vertex that has precisely this subset as neighbours.

By design, $\boxed{G_\omega := \bigcup_{n \in \mathbb{N}} G_n}$ is an infinite countable graph satisfying ARP.

Set-theoretic construction

- Take $(M, \in)$, a countable model of ZFC.
- For $a, b \in M$, define $R(a, b)$ if and only if $a \in b$ or $b \in a$.
- Then (M, R) is isomorphic to the Rado graph.

Why? Foundation! Let $a_1, \dots a_n, b_1, \dots, b_m \in M$ with the a's and b's pairwise distinct. Consider the set

$$z := \{a_1, ..., a_n, \{b_1, ..., b_m\}\}$$

Note that $R(z, b_i)$ would mean that there are $\in$-cycles in M.
(M, R) is thus a countable graph satisfying ARP, and so $(M, \in) \cong \mathfrak{R}$.
(what if we take non well-founded set theory, e.g. ZFA?)

Number-theoretic construction (Payley)

Let $V := \{p \in \mathbb{P} \mid p \equiv 1 \pmod 4\}$, and let $R(p, q)$ if and only if $\exists x \in \{0, ..., q\}, p \equiv x^2 \pmod p$. Then $(V, R) \cong \mathfrak{R}$.

Let $\{u_1, ..., u_k\}$ and $\{v_1, .., v_m\}$ disjoint sets in V. Pick some b_i's st. $\neg\exists x, x^2 \equiv b_i \pmod{v_i}$.
By the Chinese Remainder Theorem, there is an $x \in \mathbb{N}$ such that

$$x \equiv 1 \pmod 4$$
$$x \equiv 1 \pmod{u_i} \quad \text{for } i \leq k$$
$$x \equiv b_i \pmod{v_i} \quad \text{for } i \leq m$$

and any number in the progression $\langle x + nd \rangle_n$ ($d = 4u_1 ... u_k v_1 ... v_n$) is also a solution to the above congruences. By Dirichlet's Theorem on arithmetic progressions, there exists a *prime* p' of this form, so that $p' = x$ satisfies the above. Then p' is a witness for $\{u_1, ..., u_k\}$ and $\{v_1, .., v_m\}$, as desired.

Properties of the random graph

What is special about *the random graph*?

- Uniqueness (back-and-forth)
- Almost-sure theory
- Universality
- Symmetry (ultra-homogeneous)
- Its relation to the class of finite graphs: a kind of limit, encoding probabilistic information.

This construction (and the 0-1 law) generalises to finite relational signatures: we can carry over the same general model-theoretic construction for the class of all finite models (via Fraïssé limits)

Perspectives on typicality

Random structures offer fertile ground for exploring different notions of typicality:

- Asymptotic over finite structures
- Measure theoretic
    - Probability space $(\mathrm{Mod}_\omega, \mathcal{F}, \mu)$ with $\mathcal{F}$ the Borel algebra of the underlying topology. The Lebesgue measure concentrates on the isomorphism class of the random graph (assigns it measure one). [*Symmetric probabilistic constructions*: μ a S_∞-invariant measure, i.e for every Borel set A and permutation $g \in S_\infty$, $\mu(A) = \mu(gA)$].
- Topological:
    - Seeing Mod_ω as a topological space, the isomorphism class of the random graph forms a co-meagre set (*topologically large*).

But these notions of typicality need not always agree with one another. How to they relate? By virtue of which property of a theory or class of structures?

Conclusion

Random structures lie at the cusp of probability and logic, bridging together model theory and combinatorics. They can be put to use to:

- establish asymptotic 0-1 laws for logics over classes of finite structures
- display infinitary structures ‘approximating’ finite ones
- investigate the connection between symmetries of a structure and probabilistic models
- explore the relationship between topological and measure-theoretic notions of typicality.

Tomorrow:

Probabilistic grammars and probabilistic programs