Logic and Probability

The Quantitative/Qualitative Interface

Thomas Icard & Krzysztof Mierzewski

August 9, 2022

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- Today we expand this to prominent kinds of probabilistic reasoning, including inference.
- Default reasoning and rules for acceptance of hypotheses in uncertain contexts have been studied in both logical and probabilistic settings. How do these approaches relate?

Some people think it is possible to try to save monotonicity by saying that what was in your mind was not a general rule [...] but a probabilistic rule. So far these people have not worked out any detailed epistemology for this approach, i.e. exactly what probabilistic sentences should be used. Instead AI has moved to directly formalizing nonmonotonic logical reasoning.

—John McCarthy, 1990

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$$\frac{\varphi \mathrel{|\!\sim} \psi \qquad \varphi \mathrel{|\!\sim} \chi}{\varphi \wedge \psi \mathrel{|\!\sim} \chi} \quad (\textsc{Cautious Monotonicity})$$

Semantics for System **P**: Preferential Models

A preferential model is a triple $(W, \preceq, V)$ of a set W of states, a preorder $\preceq$, and a propositional valuation $V : \mathrm{Prop} \to \wp(W)$.

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We can then extend a propositional logical language to include a conditional operator $\varphi \rightsquigarrow \psi$:

$$\mathcal{M} \models \varphi \rightsquigarrow \psi \qquad \text{iff} \qquad \min_{\preceq}([\![\varphi]\!]_{\mathcal{M}}) \subseteq [\![\psi]\!]_{\mathcal{M}} \; .$$

Theorem (Veltman 1986; Kraus et al. 1990)

For any consequence relation $\mid\!\sim$ closed under the rules of **P**, there is a model $\mathcal{M}$, such that:

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Conversely, given any model $\mathcal{M}$, there is a **P**-consequence relation $\mathrel{|\!\sim}$ satisfying the above equivalence.

Rational Consequence Relations: System **R**

Some have proposed a stronger rule than CAUTIOUS MONOTONICITY:

$$\frac{\varphi \mid\!\sim \psi_1 \qquad \varphi \not\mid\!\sim \neg\psi_2}{\varphi \wedge \psi_2 \mid\!\sim \psi_1} \qquad \text{(RATIONAL MONOTONICITY)}$$

The system that results from adding this rule to **P** is called **R**.

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The models for **R** are just like those for **P**, namely triples $(W, \preceq, V)$, except $\preceq$ is now assumed to be a total preorder:

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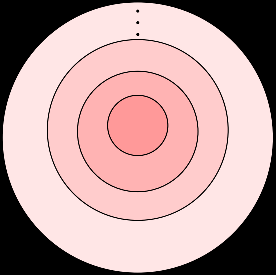

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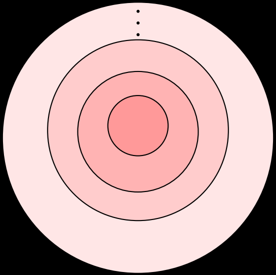

Theorem (Lehmann & Magidor)

R consequence relations correspond to total models.

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- Suppose this agent receives (veridical) information φ. How should $\mathcal{K}$ be revised in light of this fact?

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8. If $\neg\psi \notin \mathcal{K} * \varphi$, then $Cl(\mathcal{K} * \varphi \cup \{\psi\}) \subseteq \mathcal{K} * (\varphi \wedge \psi)$.

Theorem (Folklore)

1. Suppose we are given a knowledge base $\mathcal{K}$ and a revision operation $*$. Define a consequence relation $\mid\sim$ so that

$$\varphi \mid\sim \psi \qquad \text{iff} \qquad \psi \in \mathcal{K} * \varphi \; .$$

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Then $\mid\!\sim$ satisfies all the rules of **R**.

2. Suppose $\mid\!\sim$ is a consequence relation satisfying **R**, and such that $\varphi \mid\!\sim \bot$ only if $\vDash \neg\varphi$. If we define

$$\mathcal{K} \triangleq \{\psi \mid \top \mid\!\sim \psi\} \qquad \text{and} \qquad \mathcal{K} * \varphi \triangleq \{\psi : \varphi \mid\!\sim \psi\}\ ,$$

then this gives us an AGM belief revision operation.

How do these systems to relate to probability?

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if $\forall \epsilon > 0$, $\exists \delta > 0$, such that for all $\mathbb{P} : \mathcal{L} \to [0, 1]$:

$$\text{if } \mathbb{P}(\beta_i | \alpha_i) > 1 - \delta \text{ for all } i \leq n, \text{ then } \mathbb{P}(\psi | \varphi) > 1 - \epsilon.$$

Probabilistic Semantics for System **P**

Theorem (Adams 1966)

A default statement $\varphi \mid\!\sim \psi$ is derivable from a set Δ of default statements using the rules of **P**, if and only if $\Delta \vDash_A \varphi \mid\!\sim \psi$.

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Theorem (Harper 1975; Hawthorne 1998)

1. Given a conditional probability function $\mathbb{P}$, define a consequence relation $\varphi \mathrel{|\!\sim} \psi$, which holds just in case $\mathbb{P}(\psi \mid \varphi) = 1$. Then $\mathrel{|\!\sim}$ is an **R** relation.

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1. Given a conditional probability function $\mathbb{P}$, define a consequence relation $\varphi \mid\!\sim \psi$, which holds just in case $\mathbb{P}(\psi \,|\, \varphi) = 1$. Then $\mid\!\sim$ is an **R** relation.
2. For any **R** relation $\mid\!\sim$, there is a conditional probability function $\mathbb{P}$ such that:
   - $\mathbb{P}(\psi \,|\, \varphi) = 1$, iff $\varphi \mid\!\sim \psi$;
   - $\mathbb{P}(\psi \,|\, \varphi) = 0$, iff $\varphi \mid\!\sim \neg\psi$;
   - $0 < \mathbb{P}(\psi \,|\, \varphi) < 1$, iff $\varphi \not\mid\!\sim \psi$ and $\varphi \not\mid\!\sim \neg\psi$.

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and let $t = 0.5$.

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and let $t = 0.5$. Then $\top \mathrel{|\!\sim} A \vee B$ and $\top \mathrel{|\!\not\sim} \neg(A \vee C)$, but $A \vee C \mathrel{|\!\not\sim} A \vee B$.

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Theorem (Paris & Simmonds 2009)

Threshold semantics cannot be axiomatized by any finite set of rules of this form. In particular system **O** is not complete.

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From a more general perspective, these can be seen as encoding a logic of acceptance:

Given prior $\mathbb{P}$, after learning φ, accept ψ.

Implicit *acceptance rule*:

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Compare this to

$$\varphi \mid\!\sim \psi \qquad \text{iff} \qquad \psi \in \mathcal{K} * \varphi\ .$$

Question: how do we get $\mathcal{K}$ and $*$ from $\mathbb{P}$?

Acceptance rules aim at answering the following question:

Question

Suppose our information/degrees of belief are described by a probability space $(W, \mathcal{E}, \mu)$. Which propositions/hypotheses should we (tentatively) accept simpliciter?

- Bridge between probabilistic credences and qualitative beliefs (collection of accepted propositions).
- Relating probability spaces to nonmotononic consequence relations.
- Relating dynamics of probabilistic update to belief revision operators.

Acceptance rules and the logic of uncertain reasoning

Acceptance rule: translation between credences and all-or-nothing beliefs.

☞ What are well-behaved acceptance rules?

Desiderata:

- Non-trivial: do not rule out the acceptance of uncertain hypotheses.
- Consistent: never accept contradictions.
- Conjunctive: accepting A and accepting B entails accepting $A \cap B$.
- High-probability: only accept high-probability propositions (above some threshold $t > 0.5$)

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⚠ The Lockean rule does not meet these criteria! (*Lottery Paradox*)

Acceptance rules on finite spaces

Let $(W, \mathcal{E}, \mu)$ be a *finite* probability space, $X, Y, \ldots \in \mathcal{E}$ and $\Delta_{\mathcal{E}}$ the set of all probability measures on $\mathcal{E}$.

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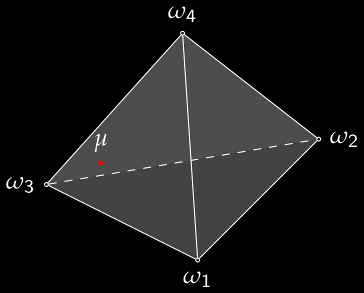
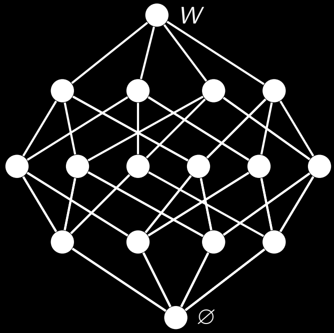
An *acceptance rule* is a map $\alpha : \Delta_{\mathcal{E}} \to \mathcal{E}$. The agent accepts $X \in \mathcal{E}$ if and only if $\alpha(\mu) \subseteq X$: i.e., $\alpha(\mu)$ is the strongest accepted proposition.





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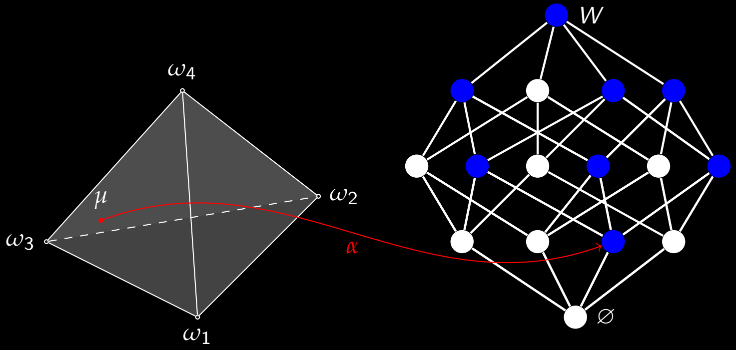

Revision & the Reverend: Bayes & Boole

μ

Revision & the Reverend: Bayes & Boole

$$\mu \xrightarrow[\text{Bayesian conditioning}]{|_X} \mu(\cdot \mid X)$$

Revision & the Reverend: Bayes & Boole

$$\mu \xrightarrow[\text{Bayesian conditioning}]{|_X} \mu(\cdot \mid X)$$

K

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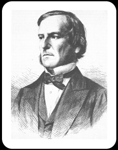
$$\mu \xrightarrow[\text{Bayesian conditioning}]{\;|_X\;} \mu(\cdot \mid X)$$

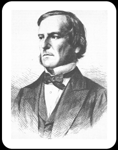
K



Revision & the Reverend: Bayes & Boole

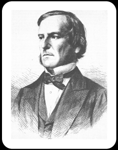
$$\mu \xrightarrow[\text{Bayesian conditioning}]{|_X} \mu(\cdot \mid X)$$

$$K \xrightarrow[\text{revision operator}]{*_X} K^*X$$



Revision & the Reverend: Bayes & Boole

$$
\begin{tikzcd}
\mu \arrow[rr, "|_X", "\text{Bayesian conditioning}"'] \arrow[d, "\text{acceptance}"] & & \mu(\cdot \,|\, X) \arrow[d, "\text{acceptance}"'] \\
K \arrow[rr, "*_X", "\text{revision operator}"'] & & K^{*}X
\end{tikzcd}
$$



Dynamic compatibility: tracking

Tracking. A qualitative revision policy maps each $\mu \in \Delta_{\mathcal{E}}$ to a belief set $\alpha(\mu)$ and a revision operator $*$ applicable to that belief set. It *tracks* Bayesian conditioning if, for any measure μ and proposition $X \in \mathcal{E}$ with $\mu(X) > 0$,

$$\alpha(\mu)^* X = \alpha(\mu(\cdot \mid X)).$$

$$\begin{tikzcd}
\mu \arrow[r, "|_X", "\text{Bayesian conditioning}"'] \arrow[d, "\alpha", "\text{acceptance rule}"'] & \mu(\cdot \mid X) \arrow[d, "\alpha"] \\
\alpha(\mu) \arrow[r, "*_X", "\text{revision operator}"'] & \alpha(\mu)^* X = \alpha(\mu(\cdot \mid X))
\end{tikzcd}$$

Stability (*Skyrms, Arló-Costa, Leitgeb*)

Let $(W, \mathcal{E}, \mu)$ a probability space and $t \in (0.5, 1]$.
A hypothesis $H \in \mathcal{E}$ is (μ, t)-*stable* if and only if $\forall\, Y \in \mathcal{E}$ such that $H \cap Y \neq \varnothing$ and $\mu(Y) > 0$, $\mu(H \mid Y) \geq t$.

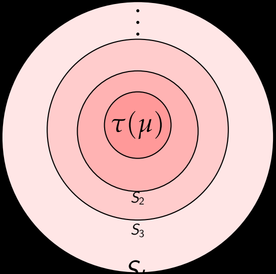
H is stable if it has *resiliently high probability* under new information.

☞ On finite spaces, the collection of stable sets, written $\mathfrak{S}^t(\mu)$, is **well-ordered by logical strength**. *(why?)*

Stability (*Skyrms, Arló-Costa, Leitgeb*)

Let $S = (W, \mathcal{E}, \mu)$ a probability space and $t \in (0.5, 1)$. We define the collection $\mathsf{L}(\mu)$ of accepted hypotheses as follows:

$$\begin{aligned}\mathsf{L}(\mu) &:= \{X \in \mathcal{E} \mid \exists A \in \mathfrak{S}^t(\mu),\, A \subseteq X\} \\ &= \{X \in \mathcal{E} \mid \tau(\mu) \subseteq X\} \text{ where } \tau(\mu) = \min_{\subseteq} \mathfrak{S}^t(\mu)\end{aligned}$$



Stability and tracking

The stability rule gives an AGM revision policy. Does this AGM policy commute with conditioning?

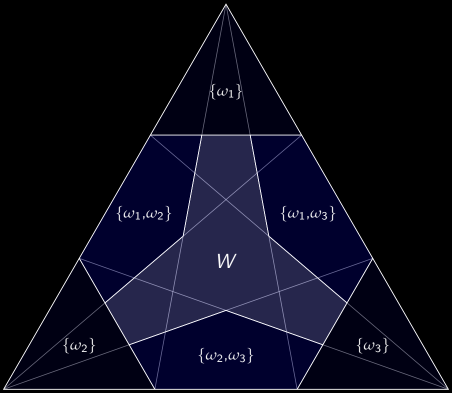
$$\begin{array}{ccc} \mu & \xrightarrow{\;|_X\;} & \mu(\cdot \mid X) \\ \big\downarrow \mathsf{L} & & \big\downarrow \mathsf{L} \\ \mathsf{L}(\mu) & \xrightarrow{\;*_X\;} & \mathsf{L}(\mu)^* X =_? \mathsf{L}\big(\mu(\cdot \mid X)\big) \end{array}$$

The revision $\mathsf{L}(\mu) \to \mathsf{L}\big(\mu(\cdot \mid X)\big)$ is not in general AGM, as it can fail Inclusion.

☞ *Tracking fails* for the stability rule.

Theorem (No-Go Theorem (Lin & Kelly, 2012))

Let $\alpha : \Delta_{\mathcal{E}} \to \mathcal{E}$ be any sensible acceptance rule. No AGM revision policy based on α tracks Bayesian conditioning.



Belief revision as 'lossy' Bayesian reasoning

In spite of the No-Go Theorem, is there still a way to bridge AGM revision and Bayesian conditioning? How can a Bayesian agent make sense of AGM revision?

Observation: AGM revision can be represented as Bayesian conditioning on a distinguished probabilistic representative of the agent's qualitative belief state.

Consider an agent with a belief set K. Credences compatible with K: all μ such that $\mathsf{L}(\mu) = K$.

Max Entropy Principle

$$\text{Entropy: } \mathcal{H}(\mu) = \sum_{\omega \in W} -\mu(\omega) \log \mu(\omega)$$

- Entropy as a measure of uncertainty.

Maximum Entropy Principle (MEP):
Select a distribution with maximal entropy consistent with the constraints imposed by the available information (if such distributions exist).

- Max Entropy distribution thought of as the least biased representation of the agent's belief state, given the constraints.

Max Entropy Principle

In making inferences on the basis of partial information we must use that probability distribution which has maximum entropy subject to to whatever is known. This is the only unbiased assignment we can make; to use any other would amount to arbitrary assumption of information which by hypothesis we do not have. [...] The maximum entropy distribution may be asserted for the positive reason that it is uniquely determined as the one which is maximally noncommittal with regard to missing information.

– E.T. Jaynes, *Information Theory and Statistical Mechanics*

AGM from Bayes via Max Entropy

For finite probability spaces:

Let K a consistent belief set on $\mathcal{E}$, and fix $t \in (0.5, 1)$. Then there is a unique maximal entropy distribution $\mu \in \Delta_{\mathcal{E}}$ such that $\mathsf{L}(\mu) = K$. Moreover, for any positive probability $X \in \mathcal{E}$, we have $\mathsf{L}(\mu(\,\cdot\,|X)) = K^{*}X$, where $*$ is the AGM revision operator generated by $\mathfrak{S}(\mu)$.

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$$
\begin{tikzcd}
\mu \\
K \arrow[u, "\substack{\text{Max}\\ \text{Entropy}\\ \text{in } \mathsf{L}^{-1}(K)}"]
\end{tikzcd}
$$

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$$\begin{tikzcd}
\mu \arrow[rrr, "|_X", "\text{Bayesian conditioning}"'] & & & \mu(\cdot \,|\, X) \\
K \arrow[u, "\substack{\text{Max} \\ \text{Entropy} \\ \text{in } \mathsf{L}^{-1}(K)}"] & & &
\end{tikzcd}$$

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$$
\begin{array}{ccc}
\mu & \xrightarrow[\text{Bayesian conditioning}]{|_X} & \mu(\cdot \mid X) \\
\text{Max Entropy in } \mathsf{L}^{-1}(K) \uparrow & & \downarrow \mathsf{L} \text{ acceptance} \\
K & & \mathsf{L}(\mu(\cdot \mid X))
\end{array}
$$

AGM from Bayes via Max Entropy

For finite probability spaces:

Let K a consistent belief set on $\mathcal{E}$, and fix $t \in (0.5, 1)$. Then there is a unique maximal entropy distribution $\mu \in \Delta_{\mathcal{E}}$ such that $\mathsf{L}(\mu) = K$. Moreover, for any positive probability $X \in \mathcal{E}$, we have $\mathsf{L}(\mu(\,\cdot\,|X)) = K^*X$, where $*$ is the AGM revision operator generated by $\mathfrak{S}(\mu)$.

$$
\begin{tikzcd}
\mu \arrow[rr, "|_X", "\text{Bayesian conditioning}"'] & & \mu(\cdot \mid X) \arrow[d, "\mathsf{L}\text{ acceptance}"] \\
K \arrow[u, "\substack{\text{Max}\\ \text{Entropy}\\ \text{in } \mathsf{L}^{-1}(K)}"] \arrow[rr, dashed, "*_X", "\text{AGM}"'] & & K^*X = \mathsf{L}(\mu(\cdot \mid X))
\end{tikzcd}
$$

Belief revision as 'lossy' Bayesian reasoning

The same obtains even for a finer representation of belief: a plausibility ranking.

$$\begin{tikzcd}
\mu \arrow[rr, "|_X"] \arrow[dd, bend right, "\mathsf{L}"'] & & \mu(\cdot \mid X) \arrow[d] \arrow[dd, bend left, "\mathsf{L}"] \\
\mathfrak{S} \arrow[u, "\text{Max Entropy}"'] \arrow[rr, "\text{Restriction to } X"'] \arrow[d, "\min"'] & & \mathfrak{S} \upharpoonright X \arrow[d, "\min"] \\
\mathsf{L}(\mu) \arrow[rr, dashed, "\text{AGM}"] & & \mathsf{L}(\mu(\cdot \mid X))
\end{tikzcd}$$

(For the revision to be autonomous—fully determined by the qualitative structure alone—the agent must retain the plausibility ranking $\mathfrak{S}$.)

AGM revision = stable belief + MaxEnt + Bayes

- AGM reasoners behave as if they were lossy Bayesian reasoners; they revise their beliefs *as if* they were relying on the maximum entropy representative of their belief state.
- Dynamic compatibility can be (approximately) recovered, and AGM represented as 'coarse-grained' Bayesian reasoning.
- A grain of salt: representation vs. rationalisation.
- A grain of truth? Resource-rational approaches to cognition [Lieder & Griffiths 2020].

KM (2020). Probabilistic Stability, AGM Operators and Maximum Entropy. *The Review of Symbolic Logic*, 1-34.

Bridges: a potpourri

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mu \arrow[r, "|_X", "\text{Bayes}"'] \arrow[d, "\alpha", "\text{acceptance rule}"'] & \mu(\cdot \mid X) \arrow[d, "\alpha"] \\
\alpha(\mu) \arrow[r, "*_X", "\text{revision operator}"'] & \alpha(\mu)^* X = \alpha(\mu(\cdot \mid X))
\end{tikzcd}$$

General models for revision policies: $(W, \mathcal{E}, \sigma)$ with $\sigma : \mathcal{E} \to \mathcal{E}$ a selection function: here, $\sigma(A)$ is the revised state after learning A.

A selection function specifies a belief set $K = \sigma(W)$ together with a qualitative revision policy $K^* A := \sigma(A)$.

$$\varphi \mid\!\sim \psi \text{ if and only if } \sigma([\![\varphi]\!]) \subseteq [\![\psi]\!]$$

Bridges: a potpourri

$$\begin{tikzcd}[column sep=huge] \mu \arrow[r, "|_X", "\boxed{\text{Bayes}}"'] \arrow[d, "\alpha", "\text{acceptance rule}"'] & \mu(\cdot \mid X) \arrow[d, "\alpha"] \\ \alpha(\mu) \arrow[r, "*_X", "\boxed{\text{AGM}}"'] & \alpha(\mu)^* X = \alpha(\mu(\cdot \mid X)) \end{tikzcd}$$

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$\varphi \mid\!\sim \psi$ if and only if $\sigma(\llbracket \varphi \rrbracket) \subseteq \llbracket \psi \rrbracket$

Bridges: a potpourri

$$
\begin{tikzcd}
\mu \arrow[r, "|_X", "\boxed{\text{Bayes}}"'] \arrow[d, dashed, no head, "\text{acceptance rule?}"', "\alpha"] & \mu(\cdot \mid X) \arrow[d, dashed, no head, "\alpha?"] \\
\alpha(\mu) \arrow[r, "*_X", "\boxed{\text{AGM}}"'] & \alpha(\mu)^* X = \alpha(\mu(\cdot \mid X))
\end{tikzcd}
$$

AGM = Stability + MaxEnt + Bayes

$$\begin{tikzcd}
\mu \arrow[rr, "|_X", "\boxed{\text{Bayes}}"'] & & \mu(\cdot \mid X) \arrow[d, "\mathsf{L}"] \\
\mathsf{L}(\mu) \arrow[u, "\text{Max Ent}", "\mathsf{L}"'] \arrow[rr, "*_X", "\boxed{\text{AGM}}"'] & & \mathsf{L}(\mu)^* X = \mathsf{L}(\mu(\cdot \mid X))
\end{tikzcd}$$

Nonmonotonic consequence relations closed under system **R**.

$\sigma(X) := \min_{\preceq}(X)$ for $\preceq$ a *total preorder* on W.

$\omega_i \preceq \omega_j$ if and only if for any (μ, t)-stable H, if $\omega_i \in H$ then $\omega_j \in H$

System **P** $=$ Odds Ratio $+$ Bayes (*Lin and Kelly, 2012*)

$$
\begin{tikzcd}
\mu \arrow[rr, "|_X"', "\boxed{\text{Bayes}}"] \arrow[d, "\kappa", "\text{odds-ratio rule}"'] & & \mu(\cdot \mid X) \arrow[d, "\kappa"] \\
\kappa(\mu) \arrow[rr, "*_X", "\boxed{\text{Shoham revision}}"'] & & \kappa(\mu)^* X = \kappa(\mu(\cdot \mid X))
\end{tikzcd}
$$

Nonmonotonic consequence relations closed under system **P**.

$\sigma(X) := \min_{\prec}(X)$ for $\prec$ a *partial order* on W.

$\omega_i \prec \omega_j$ if and only if $\mu(\omega_i) > q \cdot \mu(\omega_j)$ $(q \geq 1)$

Probabilistically stable revision

$$\begin{tikzcd} \mu \arrow[rr, "|_X", "\boxed{\text{Bayes}}"'] \arrow[d, "\boxed{\text{Stability}}"'] & & \mu(\cdot \mid X) \arrow[d, "\mathsf{L}"] \\ \mathsf{L}(\mu) & & \mathsf{L}(\mu)^* X := \mathsf{L}(\mu(\cdot \mid X)) \end{tikzcd}$$

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$$\begin{tikzcd} \mu \arrow[r, "|_X", "\boxed{\text{Bayes}}"'] \arrow[d, "\boxed{\text{Stability}}"'] & \mu(\cdot \mid X) \arrow[d, "\mathsf{L}"] \\ \mathsf{L}(\mu) \arrow[r, dashed, "*_X", "\text{revision operator?}"'] & \mathsf{L}(\mu)^* X := \mathsf{L}(\mu(\cdot \mid X)) \end{tikzcd}$$

Probabilistically stable revision

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mu \arrow[r, "|_X"] \arrow[r, phantom, shift right=4ex, "\boxed{\text{Bayes}}"] \arrow[d, swap, "\boxed{\text{Stability}}"] & \mu(\cdot \mid X) \arrow[d, "\mathsf{L}"] \\
\mathsf{L}(\mu) \arrow[r, dashed, "*_X", "\text{s.s.s operator } \sigma_\mu"'] & \mathsf{L}(\mu)^* X := \mathsf{L}(\mu(\cdot \mid X))
\end{tikzcd}$$

Characterise *strongest stable set* operators $\sigma_\mu : X \mapsto \tau\big(\mu(\cdot \mid X)\big)$.

Representation result: these operators σ encode the behaviour of a (special class of) comparative probability orders.

Probabilistically stable revision is not *trackable* using order minimisation.

Probabilistically stable revision

Theorem (M.)

Let $(W, \mathcal{E}, \sigma)$ be a selection structure. Then σ is a strongest stable set operator $\sigma_\mu : X \mapsto \tau(\mu(\cdot \mid X))$ for some (regular) measure μ on $\mathcal{E}$ if and only if the following hold:

(S1) $\sigma(X) = \varnothing$ *only if* $X = \varnothing$

(S2) $\sigma(X) \subseteq X$

(S3) *If* $\sigma(A) \cap B \neq \varnothing$*, then* $\sigma(A \cap B) \subseteq \sigma(A) \cap B$

(S4) *If* $X_i \setminus X_j \subseteq \sigma(X_i)$ *for all* $i \neq j \leq n$*, then* $\sigma(\bigcup_{i \leq n} X_i) \subseteq \bigcup_{i \leq n} \sigma(X_i)$

(SC) *If* $(A_i)_{i \leqslant n} =_0 (B_i)_{i \leqslant n}$ *and* $\forall i \leqslant n$, $A_i \succcurlyeq^*_\sigma B_i$*, then* $\forall i \leqslant n$, $A_i \preccurlyeq^*_\sigma B_i$.

Say $A \gg B$ iff $\sigma(A \cup B) \subseteq A \setminus B$. We define

$$A \succcurlyeq^*_\sigma B \Leftrightarrow \text{ either } \big[A \gg B\big] \text{ or } \big[B \text{ is a } \mathcal{E}\text{-atom } \& \ B \not\gg A\big]$$

$\rightsquigarrow$ (SC): Finite Cancellation-style axiom for comparative probability.

Probabilistically stable revision

$$\frac{}{\varphi \mathrel{|\!\sim} \varphi} \quad (\textsc{Refl.})$$

$$\frac{\models \varphi_1 \leftrightarrow \varphi_2 \qquad \varphi_1 \mathrel{|\!\sim} \psi}{\varphi_2 \mathrel{|\!\sim} \psi} \ (\text{L-E}) \qquad \frac{\models \psi_1 \to \psi_2 \qquad \varphi \mathrel{|\!\sim} \psi_1}{\varphi \mathrel{|\!\sim} \psi_2} \ (\text{R-W})$$

$$\frac{\varphi \mathrel{|\!\sim} \psi_1 \qquad \varphi \mathrel{|\!\sim} \psi_2}{\varphi \mathrel{|\!\sim} \psi_1 \wedge \psi_2} \quad (\textsc{And}) \qquad \frac{\varphi_1 \mathrel{|\!\sim} \psi \qquad \varphi_2 \mathrel{|\!\sim} \psi}{\varphi_1 \vee \varphi_2 \mathrel{|\!\sim} \psi} \quad (\textsc{Or})$$

$$\frac{\varphi \;\not\mathrel{|\!\sim} \neg\psi \qquad \varphi \mathrel{|\!\sim} \chi}{\varphi \wedge \psi \mathrel{|\!\sim} \chi} \quad (\textsc{Rational Monotonicity})$$

Probabilistically stable revision

$$\frac{}{\varphi \mid\!\sim \varphi} \quad (\textsc{Refl.})\checkmark$$

$$\frac{\vDash \varphi_1 \leftrightarrow \varphi_2 \qquad \varphi_1 \mid\!\sim \psi}{\varphi_2 \mid\!\sim \psi} \quad (\text{L-E})\checkmark \qquad \frac{\vDash \psi_1 \rightarrow \psi_2 \qquad \varphi \mid\!\sim \psi_1}{\varphi \mid\!\sim \psi_2} \quad (\text{R-W})\checkmark$$

$$\frac{\varphi \mid\!\sim \psi_1 \qquad \varphi \mid\!\sim \psi_2}{\varphi \mid\!\sim \psi_1 \wedge \psi_2} \quad (\textsc{And})\checkmark \qquad \frac{\varphi_1 \mid\!\sim \psi \qquad \varphi_2 \mid\!\sim \psi}{\varphi_1 \vee \varphi_2 \mid\!\sim \psi} \quad (\textsc{Or})$$

$$\frac{\varphi \;\not\mid\!\sim \neg\psi \qquad \varphi \mid\!\sim \chi}{\varphi \wedge \psi \mid\!\sim \chi} \quad (\textsc{Rational Monotonicity})\checkmark$$

Probabilistically stable revision

$$\frac{}{\varphi \mid\!\sim \varphi} \quad \text{(Refl.)}\checkmark$$

$$\frac{\vDash \varphi_1 \leftrightarrow \varphi_2 \qquad \varphi_1 \mid\!\sim \psi}{\varphi_2 \mid\!\sim \psi} \quad \text{(L-E)}\checkmark \qquad \frac{\vDash \psi_1 \rightarrow \psi_2 \qquad \varphi \mid\!\sim \psi_1}{\varphi \mid\!\sim \psi_2} \quad \text{(R-W)}\checkmark$$

$$\frac{\varphi \mid\!\sim \psi_1 \qquad \varphi \mid\!\sim \psi_2}{\varphi \mid\!\sim \psi_1 \wedge \psi_2} \quad \text{(And)}\checkmark \qquad \frac{\varphi_1 \mid\!\sim \psi \qquad \varphi_2 \mid\!\sim \psi}{\varphi_1 \vee \varphi_2 \mid\!\sim \psi} \quad \text{(Or)}✗$$

$$\frac{\varphi \not\mid\!\sim \neg\psi \qquad \varphi \mid\!\sim \chi}{\varphi \wedge \psi \mid\!\sim \chi} \quad \text{(Rational Monotonicity)}\checkmark$$

We saw:

- Common systems for non-monotonic reasoning, their standard 'qualitative' and probabilistic semantics, and belief revision;
- The tracking problem: in what sense can logical belief revision be compatible, or emerge from, probabilistic reasoning and Bayesian updating in particular?
- Acceptance rules as bridge principles between quantitative and all-or-nothing beliefs, and the non-monotonic/defeasible reasoning principles they validate.

Next time:

- What is the relationship between measure-theoretic probability and probability functions on formal languages?
- When logic and probability shed light on each other: random structures and the 0-1 law.