Midterm exam

1. Let $f$ be the function $\mathbb{R} \to \mathbb{R}$ given by

$$f(x) = 1 - (x - 1)^2.$$ 

Consider a sequence $(\alpha_n)_n$ in $\mathbb{R}$ constructed as follows: The value of $\alpha_0 \in \mathbb{R}$ is chosen and subsequent terms are obtained by iterating $f$, so $\alpha_{n+1} = f(\alpha_n)$ for all $n \geq 0$.

a. Suppose that $(\alpha_n)_n$ converges. What value(s) $L \in \mathbb{R}$ can $\lim(\alpha_n)_n$ take? Carefully justify your answer.

Either $L = 0$ or $L = 1$. If $(\alpha_n)_n$ converges then so does $(\alpha_{n+1})_n$ (it is a subsequence of a convergent sequence), and they must have the same limit. Now,

$$\alpha_{n+1} = f(\alpha_n) = 1 - (\alpha_n - 1)^2$$

On the right-hand side, $(1 - (\alpha_n - 1)^2)_n$ converges to $1 - (L - 1)^2$ (by standard “arithmetic of limits”-type results) and the left-hand side converges to $L$. Thus, we must have

$$L = 1 - (L - 1)^2 = 2L - L^2$$

and thus we obtain $L - L^2 = L(1 - L) = 0$. It follows that $L = 0$ or $L = 1$, as claimed. □

b. We have (i.) $f(x) \leq 1$ for all $x$, (ii.) $f(x) > x$ when $0 < x < 1$, and (iii.) $f(x) < x$ when $x < 0$ or $x > 1$. Prove (ii).

We have $f(x) = x(1 - x)$ and this is positive only if $x > 0$ and $1 - x > 0$ (which is the same as $0 < x < 1$) or $x < 0$ and $1 - x < 0$ (which is the same as $x < 0$ and $1 < x$, which is not possible). □

c. For each $L$ you listed in part (a), describe all values $\alpha_0 \in \mathbb{R}$ for which

$$(\alpha_n)_n = (\alpha_0, f(\alpha_0), f(f(\alpha_0)), \ldots)$$

converges to $L$. Justify your answer.

We have the following: $\lim(\alpha_n)_n = 0$ if $\alpha_0 = 0$ or $\alpha_0 = 2$, and $\lim(\alpha_n)_n = 1$ if $0 < \alpha_0 \leq 2$. Otherwise, the sequence diverges.

Proof. If $\alpha_m = 0$ for some index $m$, then $\lim(\alpha_n)_n = 0$ because $f(0) = 0$. Similarly, if $\alpha_m = 1$ for some index $m$, then $\lim(\alpha_n)_n = 1$ because $f(1) = 1$.

If $0 < \alpha_m < 1$ for some index $m$, then by (ii) we have $0 < \alpha_m < f(\alpha_m)$, and by induction we see that $(\alpha_n)_{n=m}^\infty$ is an increasing sequence. By (i), $(\alpha_n)_n$ is bounded above, and it follows that $(\alpha_n)_n$ converges. Since all terms $\alpha_n$ with $n \geq m$ are $\geq \alpha_m > 0$, the same is true of the limit $L$, so $\lim(\alpha_n)_n = 1$ in this case.
If $1 < \alpha_m < 2$ for some index $m$, then by a similar argument to that in (b), if $0 < f(\alpha_m) = \alpha_{m+1} < 1$. It follows from the previous paragraph that $\lim(\alpha_n)_n = 1$ in this case.

If $\alpha_m < 0$ for any index $m$, then the series diverges, since $f(\alpha_m) < \alpha_m$ and so $(\alpha_n)_{n=m}^{\infty}$ is decreasing and bounded away from 0, the least possible limit of the sequence. Thus, the sequence diverges if $\alpha_0 < 0$. If $\alpha_m > 2$ for any index $m$, then $f(\alpha_m) < 0$ (by a simple argument with inequalities), so $(\alpha_n)_n$ diverges in this case. $\square$

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions (in the usual topology on $\mathbb{R}$) and suppose that for all $q \in \mathbb{Q}$ we have $f(q) = g(q)$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$. We can build a sequence of rational numbers with limit $x$ as follows: For all $n \geq 1$, let $q_n$ be a rational number such that $x < q_n < x + \frac{1}{n}$ (this exists by density of rationals). It is clear that $(q_n)_n \to x$.

Now, by our assumption on $f$ and $g$, we have $f(q_n) = g(q_n)$ for all $n$. By sequential continuity of $f$ and $g$, since $(q_n) \to x$, we have $(f(q_n))_n \to f(x)$ and $(g(q_n))_n \to g(x)$. However, $(f(q_n))_n = (g(q_n))_n$ as sequences, so they have the same limit, and therefore $f(x) = g(x)$. $\square$

3. Let $X$ be a nonempty metric space, let $S$ be a dense subset of $X$, and let $T$ be a countable subset of $X$. Prove that there exists a sequence $(s_n)_n$ in $S$ with the following property: For all $t \in T$, there is a subsequence $(s_{n_i})_i$ of $(s_n)_n$ that converges to $t$.

If $T$ is empty the statement is vacuous. Suppose that $T$ is nonempty and let $(t_n)_{n=1}^{\infty}$ be an enumeration of $T$ (a sequence corresponding to a surjection $\mathbb{N} \to T$).

Construct a two-indexed sequence $a_{ij}$ as follows: For $i, j \geq 1$, $a_{ij}$ is an element of $S$ such that $d(a_{ij}, t_j) < \frac{1}{7}$. Such an element exists by the density of $S$ in $X$. Now, let $(s_n)_n$ be the sequence

$$(s_n)_n = (a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \ldots)$$

obtained by reading the matrix $A = [a_{ij}]_{ij}$ along its northeast-to-southwest diagonals. Note that every column of $A$ is a subsequence of $(s_n)_n$, and by construction, the limit of the subsequence in the $j$th column, namely $(a_{ij})_i$ (with $j$ fixed) is $t_j$. $\square$

4. The Koff topology is a topology $\kappa$ on $\mathbb{R}$ whose open sets are as follows:

i. $\emptyset \in \kappa$, and

ii. If $U$ is a nonempty subset of $\mathbb{R}$, then $U \in \kappa$ (that is, $U$ is open in the Koff topology) if and only if $U^c$ is a finite set.

You should take a moment to

- Convince yourself that $\kappa$ is indeed a topology on $\mathbb{R}$;
- To think about what the closed sets are in this topology; and
To think about the relationship between openness in the Koff topology and openness in the usual topology on $\mathbb{R}$.

By the usual topology on $\mathbb{R}$ we of course mean the topology induced by the Euclidean metric $d(x, y) = |x - y|$.

a. Let $K \subseteq \mathbb{R}$. Prove that if $K$ is compact in the usual topology, then $K$ is compact with respect to the Koff topology.

Follows immediately from the fact that every set that is open in the Koff topology is open in the usual topology (so every open cover of $K$ in the Koff topology is an open cover in the usual topology, and by compactness it can be refined to a finite subcover). □

b. Prove that $\mathbb{R}$ is compact with respect to the Koff topology.

Let $G$ be an open cover of $\mathbb{R}$ in the Koff topology. Then there is $U \in G$ which is not the empty set. Let $U^c = \{x_1, \ldots, x_n\}$. Since $G$ is a cover, for each $i$ we can find $U_i \in G$ such that $x_i \in U_i$. $\{U, U_1, \ldots, U_n\}$ is a finite subcover. □

c. Write down a subset $K$ of $\mathbb{R}$ which is is compact but not closed in the Koff topology.

All closed sets in $K$ are either finite or $\mathbb{R}$ itself. $[0, 1]$ is therefore not closed, but it is compact by (a). □

d. Write down a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous in the usual topology (on both domain and codomain) but not continuous with respect to the Koff topology (on both domain and codomain). To justify your answer, just write down a Koff-open (or Koff-closed) set $S$ such that $f^{-1}(S)$ is not Koff-open (or Koff-closed).

You may use any functions you know from calculus.

The sine function is for example not continuous as a function $\mathbb{R} \to \mathbb{R}$ under the Koff topology. This is because (by checking the definition of topological continuity in terms of closed sets) a Koff-continuous function must satisfy the following: the preimage of a finite set must be finite (or all of $\mathbb{R}$), but $(\sin)^{-1}(\{0\}) = \pi\mathbb{Z}$ is infinite. □

5. An base-$n$ expansive number is a real number $\gamma$ such that expressing $\gamma$ in base $n$ requires every digit. That is, when $\gamma$ is expressed as a string in base $n$, every $d \in \{0, \ldots, n-1\}$ must appear. For example,

$$\frac{1}{7} = 0.001\overline{1}_{2} = 0.01\overline{0}_{2} = 0.0\overline{2}_{3} = 0.02\overline{1}_{4} = \cdots$$

so $\frac{1}{7}$ is expansive in bases 2 and 3, but not in base 4, since it is missing the digit 3. Irrational numbers that “arise in nature” like $\pi$ or the Reciprocal Fibonacci constant

$$\psi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \cdots = 3.3598856662431775317201130291241845\ldots$$

are expected to be expansive in every base, though this is still not proven.

In class, I explained why the Cantor set is $[0, 1] \setminus S(3, 1)$, the set of all numbers in $[0, 1]$ that have at least one ternary expansion in which the 1 digit does not appear.
A *universally expansive number* is a real number that is expansive in *every base* \( n \geq 2 \). While it is challenging to write down an explicit example of a provably universally expansive number, we can still show such numbers exist!

\((\ast)\). (You don’t need to write anything down for this part.) Let \( n \geq 2 \) be an integer and let \( d \in \{0, \ldots, n - 1\} \). Define \( S(n, d) \) to be the set of all real numbers \( x \) whose base-\( n \) expansions *must* include the digit \( d \). Convince yourself that \( S(n, d) \) is open (you can wiggle around in \( S(n, d) \) without leaving!) and dense (every open interval overlaps with \( S(n, d)! \)) in \( \mathbb{R} \).

(You need to write something down for this part.) Prove, using the conclusion of \((\ast)\) and any theorem we’ve covered in class, that there is a universally expansive number.

The set of universally expansive numbers is \( S^* = \bigcap_{n=2}^{\infty} \bigcap_{d=0}^{n-1} S(n, d) \). This is a countable intersection, since \((n, d)\) ranges over a subset of \( \mathbb{N} \times \mathbb{N} \). Every term is an open dense subset of \( \mathbb{R} \), by \((\ast)\). By the Baire category theorem, \( S^* \) is dense, and therefore nonempty. It follows that there is a universally expansive number. \( \square \)