1. Let $f : \mathbb{R} \to \mathbb{R}$ be any continuous function of period 2 (i.e., $f(t + 2) = f(t)$ for all $t \in \mathbb{R}$) such that $f(t) = 0$ for all $t \in [0, \frac{1}{3}]$ and $f(t) = 1$ for all $t \in [\frac{2}{3}, 1]$.

For example, $f$ could be the function whose graph on $[0, 2]$ is obtained by drawing the line segments $(0, 0) \to (\frac{1}{3}, 0) \to (\frac{2}{3}, 1) \to (\frac{4}{3}, 1) \to (\frac{5}{3}, 0) \to (2, 0)$ and then repeating this pattern to obtain a continuous function $\mathbb{R} \to \mathbb{R}$ which is periodic of period 2.

Having chosen such an $f$, let

$$x(t) = \sum_{n=1}^{\infty} f(3^{2n-1}t) \cdot 2^{-n} \quad \text{and} \quad y(t) = \sum_{n=1}^{\infty} f(3^{2n}t) \cdot 2^{-n}$$

and let $\Phi : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (x(t), y(t))$.

a. Prove that $x : \mathbb{R} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$ are continuous functions. (It follows that their Cartesian product, $\Phi : \mathbb{R} \to \mathbb{R}^2$, is continuous by something we proved long ago in this class.) Hint: You'll want to use the Weierstrass M-test.

b. Let $I = [0, 1]$ and let $x_0, y_0 \in I$. We can expand these two numbers in binary:

$$x_0 = \sum_{n=1}^{\infty} a_{2n-1} \cdot 2^{-n} \quad \text{and} \quad y_0 = \sum_{n=1}^{\infty} a_{2n} \cdot 2^{-n}$$

(the weird indexing of $(a_m)$ is on purpose) and then use these expansions to encode a third number

$$t_0 = \sum_{m=1}^{\infty} 2a_m \cdot 3^{-(m+1)}$$

Note that $t_0 \in I$—in fact, $t_0$ is in the Cantor set, since it is a number in $I$ with a base-3 expansion that does not use the 1 digit!

Prove that for any $k \geq 1$, $f(3^kt_0) = a_k$.

c. Prove that $\Phi$ is a continuous surjection from $I$ (the closed unit interval in $\mathbb{R}$) to $I^2$ (the closed unit square in $\mathbb{R}$).

2. A detail we more-or-less skipped in our proof of the Stone–Weierstrass theorem is the existence of a sequence of polynomials that converges uniformly to the absolute value function $| \cdot |$ on $[-1, 1]$ (this was “Lemma 0”). It turns out that we can prove this directly (i.e., without Bernstein’s proof of Weierstrass’ original theorem).

Let $P_0(x) = 0$ and for every $n \geq 0$ let

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n(x)^2}{2}$$
The first few such polynomials (after $P_0$) are

$$P_1(x) = \frac{1}{2} x^2 \quad P_2(x) = -\frac{1}{8} x^4 + x^2 \quad P_3(x) = -\frac{1}{128} x^8 + \frac{1}{8} x^6 - \frac{5}{8} x^4 + \frac{3}{2} x^2$$

(Note that these polynomials double in degree after each iteration, and the coefficients are not “numerically stable”: for example, the coefficient of $x^2$ in $P_n$ is $\frac{n}{2}$, which tends to $+\infty$).

a. Prove the identity

$$|x| - P_{n+1}(x) = (|x| - P_n(x)) \left(1 - \frac{|x| + P_n(x)}{2}\right)$$

b. Prove that $0 \leq P_n(x) \leq |x|$ for all $n \geq 0$ and all $x \in [-1, 1]$

c. Prove that $0 \leq |x| - P_n(x) \leq |x| \left(1 - \frac{|x|}{2}\right)^n$ for all $n \geq 0$ and all $x \in [-1, 1]$. Conclude that $(P_n) \to |\cdot|$ pointwise on $[-1, 1]$.

d. Prove that $r \left(1 - \frac{r}{2}\right)^n \leq \frac{2}{n+1}$ for all $n \geq 0$ and all $r \in [0, 1]$. Conclude that $(P_n) \to |\cdot|$ uniformly on $[-1, 1]$.