Homework 5 - Solutions

1. a. Consider the step function \( \chi = \phi - \psi \). For \( x \in [a, b] \), \( \chi(x) \neq 0 \) if and only if \( x \in S \), which is a finite set of points. We can then write \( \chi \) as some expression of the form

\[
\chi = \sum_{k=1}^{n} a_k 1_{I_k},
\]

where each \( I_k \) is a degenerate interval, i.e. \( I_k = \{x_k\} \) for some \( x_k \in [a, b] \). In this notation we have \( \chi(x) = a_k \) if \( x = x_k \), and \( \chi(x) = 0 \) otherwise. Then

\[
\int_{a}^{b} \chi = \sum_{k=1}^{n} a_k |I_k| = \sum_{k=1}^{n} a_k \cdot 0 = 0.
\]

Since \( \chi = \phi - \psi \) and the integral is linear on step functions, we have

\[
\int_{a}^{b} \chi = \int_{a}^{b} (\phi - \psi) + \int_{a}^{b} \psi = 0 + \int_{a}^{b} \psi = \int_{a}^{b} \psi,
\]

as desired.

b. Let \( \phi \leq f \) on \([a, b]\). Define the step function \( \psi \) via

\[
\psi(x) = \begin{cases} 
  g(x) & \text{if } x \in S \\
  \phi(x) & \text{otherwise}. 
\end{cases}
\]

If \( x \in S \), then \( \psi(x) = g(x) \), so \( \psi(x) \leq g(x) \). If \( x \notin S \), then \( \psi(x) = \phi(x) \leq f(x) = g(x) \). Thus \( \psi \leq g \). Also, \( \psi(x) = \phi(x) \) unless \( x \in S \), which is a finite set. By problem 1a, \( \int_{a}^{b} \phi = \int_{a}^{b} \psi \), so since \( \psi \leq g \) we have \( \int_{a}^{b} \phi \in \mathcal{L}(g, [a, b]) \). Thus

\[
\sup \mathcal{L}(f, [a, b]) \leq \sup \mathcal{L}(g, [a, b]),
\]

so \( \int_{a}^{b} f \leq \int_{a}^{b} g \). This argument is completely symmetric in \( f \) and \( g \), so we also have \( \int_{a}^{b} g \leq \int_{a}^{b} f \), and thus \( \int_{a}^{b} f = \int_{a}^{b} g \) as desired.

2. Let \( \phi \leq f \) be any step function. We’ll show that \( \int_{a}^{b} \phi \in \mathcal{L}(g, [a, b]) \), which (as in problem 1b) is enough to show that \( \int_{a}^{b} f \leq \int_{a}^{b} g \).
Since \( \phi \) is a step function, it is a finite linear combination of intervals. In this linear combination, we can separate the degenerate intervals from the nondegenerate intervals, giving us that

\[
\phi = \sum_{k=1}^{n} a_k 1_{I_k} + \sum_{l=1}^{m} b_l 1_{J_l},
\]

where each \( I_k \) is a nondegenerate interval (i.e. not a single point) and each \( J_l \) is a degenerate interval. We perform two simplifications. First, we assume without loss of generality that the \( I_k \)'s are disjoint intervals. Second, we consider instead the step function \( \psi \) given by

\[
\psi = \sum_{k=1}^{n} a_k 1_{I_k} + \sum_{l=1}^{m} \min\{b_l, 0\} 1_{J_l}.
\]

There are finitely many \( J_l \)'s, so \( \psi \) and \( \phi \) differ at finitely many points. Thus by problem 1a, \( \int_{a}^{b} \phi = \int_{a}^{b} \psi \). Note also that \( \psi \leq \phi \) at all \( x \).

Claim: \( \psi \leq g \).

**Proof of claim.** Assume not. Then there exists some \( I_k \) and some \( x \in I_k \) with \( g(x) < \psi(x) = a_k \). By continuity, there exists an open neighborhood \( U \) of \( x \) such that for all \( y \in U \), \( g(y) < \psi(y) \). Since \( D \) is dense, there exists some \( y \in D \cap U \). But for this \( y \) we have

\[
\psi(y) \leq \phi(y) \leq f(y) \leq g(y),
\]

which contradicts the fact that \( g(y) < \psi(y) \).

Thus \( \psi \leq g \).

Since \( \psi \leq g \) and \( \int_{a}^{b} \phi = \int_{a}^{b} \psi \), we must have \( \int_{a}^{b} \phi \in L(g, [a, b]) \), so \( \int_{a}^{b} f \leq \int_{a}^{b} g \).

3. Let \( g : [0, 1] \to \mathbb{R} \) be given by \( g(x) = 0 \) for all \( x \). Then \( g \) is continuous, and \( T(x) \leq g(x) \) whenever \( x \notin \mathbb{Q} \). Thus \( T(x) \leq g(x) \) for a dense subset of the interval, \( T \) is bounded, and \( g \) is continuous, so by problem (2) we have \( \int_{0}^{1} T \leq \int_{0}^{1} g = 0 \). Meanwhile \( T \geq 0 \), so \( \int_{0}^{1} T \) must in fact be equal to 0.

It remains to prove that \( \int_{0}^{1} T \leq 0 \) to show that \( \int_{0}^{1} T = \int_{0}^{1} T \). Define a step function \( \psi_n \) on \([0, 1]\) via

\[
\psi_n(x) = \max \left\{ T(x), \frac{1}{n} \right\}.
\]

For each \( n \), there are finitely many points \( x \) where \( T(x) > \frac{1}{n} \), so \( \psi_n \) is a finite linear combination of intervals (many of which are degenerate). Also, \( \psi_n(x) \neq \frac{1}{n} \) only on finitely many points, so by problem 1a, \( \int_{0}^{1} \psi_n = \int_{0}^{1} (1/n) = 1/n \). Thus as \( n \to \infty \), \( \int_{0}^{1} \psi_n \to 0 \). Thus

\[
\int_{0}^{1} T \leq \inf_n \int_{0}^{1} \psi_n = 0,
\]

so \( \int_{0}^{1} T = \int_{0}^{1} T \). Thus \( T \) is Riemann-Darboux integrable on \([0, 1]\).