1. Let $U$ be any open cover of $X$. We first show the following claim:

Claim: There exists $\varepsilon > 0$ such that for all $x \in X$, there exists $U \in U$ such that the open ball $B_\varepsilon(x)$ is contained in $U$.

Proof of claim. Assume not. Then for all $n \in \mathbb{N}$, there exists some $x_n \in X$ such that $B_{1/n}(x_n)$ is not contained in any open set $U$ in our cover $U$. Then the $(x_n)_n$ form a sequence in $X$, so by hypothesis it has a convergent subsequence $(x_{n_k})_k$ converging to some limit $L \in X$. Since $L \in X$, it is covered by $U$, so there exists an open set $U_L \in U$ with $L \in U_L$. Since $U_L$ is open, for some $\varepsilon_L$, $B_{\varepsilon_L}(L) \subseteq U$. Let $K$ be large enough that $\frac{1}{n K} < \varepsilon_L/2$ and that $x_{n_k} \in B_{\varepsilon/L/2}(L)$, which is possible because $(x_{n_k})_k$ converges to $L$. But then by the triangle inequality, we have

$$B_{1/n_K}(x_{n_K}) \subseteq B_{\varepsilon_L/2}(x_{n_K}) \subseteq B_{\varepsilon_L}(L) \subseteq U_L,$$

which contradicts the assumption that $B_{1/n_K}(x_{n_K})$ is not contained in any open set $U$ of $U$.

Thus such an $\varepsilon > 0$ exists. \qed

Let $\varepsilon > 0$ satisfy the conditions of the claim. Then we further claim the following:

Claim: There exist finitely many points $x_1, \ldots, x_n \in X$ such that $\{B_\varepsilon(x_1), \ldots, B_\varepsilon(x_n)\}$ covers $X$.

Proof of claim. Assume not. Define a sequence $(x_n)_n \subseteq X$ as follows:

- Let $x_1 \in X$ be any point.
- For $n \geq 2$, let $x_n$ be any point contained in $X \setminus \bigcup_{k<n} B_\varepsilon(x_k)$.

Our assumption implies that this sequence does not terminate after finitely many steps. Also note that for all $n \neq m$, without loss of generality with $n < m$, we have $x_m \notin B_\varepsilon(x_n)$, so $d(x_n, x_m) \geq \varepsilon$. By hypothesis, $(x_n)_n$ has a convergent subsequence $(x_{n_k})_k$ converging to a limit $L \in X$. Let $j$ and $k$ be such that $d(x_{n_k}, L) < \varepsilon/2$ and $d(x_{n_j}, L) < \varepsilon/2$. By the triangle inequality, $d(x_{n_k}, x_{n_j}) < \varepsilon$, a contradiction. This completes the proof. \qed
Let \( x_1, \ldots, x_n \in X \) be such that \( \{B_\varepsilon(x_1), \ldots, B_\varepsilon(x_n)\} \) covers \( X \). Then for \( 1 \leq k \leq n \), let \( U_k \in \mathcal{U} \) be an open set with \( B_\varepsilon(x_k) \subseteq U_k \). Since \( \{B_\varepsilon(x_1), \ldots, B_\varepsilon(x_n)\} \) covers \( X \), so does \( \{U_1, \ldots, U_n\} \). Thus \( \mathcal{U} \) has a finite subcover, so any open cover of \( X \) has a finite subcover, and thus \( X \) is compact.

2. a. The set \( \mathbb{R} \) has the same cardinality as the power set of \( \mathbb{N} \), i.e. \( |\mathbb{R}| = 2^{\aleph_0} \). Then the set of all functions \( \mathbb{R} \to \mathbb{R} \) has cardinality

\[
|\{f : \mathbb{R} \to \mathbb{R}\}| = |\mathbb{R}|^{|\mathbb{R}|} = (2^{\aleph_0})^{2^{\aleph_0}} = 2^{\aleph_0 \cdot 2^{\aleph_0}} = 2^{2^{\aleph_0}}.
\]

b. Let \( C \) be the set of continuous functions from \( \mathbb{R} \to \mathbb{R} \). By problem (2) on the midterm, any element \( f \in C \) is determined by its values on \( \mathbb{Q} \). Thus the cardinality of \( C \) is the same as the cardinality of the set of continuous functions from \( \mathbb{Q} \to \mathbb{R} \). Let \( D \) be the set of continuous functions from \( \mathbb{Q} \to \mathbb{R} \), and let \( E \) be the set of all functions from \( \mathbb{Q} \to \mathbb{R} \). Then

\[
|D| \leq |E| = |\mathbb{R}|^{|\mathbb{Q}|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}.
\]

On the other hand, \( D \) contains the set of constant functions \( \{f_r : \mathbb{Q} \to \mathbb{R}, f_r(q) = r : r \in \mathbb{R}\} \). This set is in bijection with \( \mathbb{R} \), so the cardinality of the constant functions is \( 2^{\aleph_0} \). Thus \( |D| \geq 2^{\aleph_0} \). Since we also have \( |D| \leq 2^{\aleph_0} \) above, it must be that \( |D| = 2^{\aleph_0} \).

3. Fix \( \varepsilon > 0 \). Since \( f : X \to Y \) is uniformly continuous, there exists \( \delta > 0 \) such that if \( |x - y| < \delta \), then \( |f(x) - f(y)| < \varepsilon \). Since \( (x_n)_n \) is Cauchy, there exists \( N \in \mathbb{N} \) such that if \( n, m \geq N \), \( |x_n - x_m| < \delta \). But then for all \( n, m \geq N \), \( |f(x_n) - f(x_m)| < \varepsilon \), by our choice of \( \delta \). Thus \( (f(x_n))_n \) is Cauchy.

4. a. Assume that \( f \) is uniformly continuous. For \( x \in S \), let \( (x_n)_n \) be any sequence of points in \( S \) with \( \lim_{n \to \infty} x_n = x \). Then define \( f^* \) via

\[
f^*(x) = \lim_{n \to \infty} f(x_n).
\]

For a given choice of \( (x_n)_n \), \( (x_n)_n \) is Cauchy, so by problem 3, \( (f(x_n))_n \) is Cauchy as well, so by completeness of \( Y \) it converges to some value, i.e. the limit is well-defined. Note that when \( x \in S \), by continuity of \( f \) on \( S \) this definition gives that \( f^*(x) = f(x) \).

Also, for any two convergent sequences \( (x_n)_n \) and \( (y_n)_n \) of points in \( S \), if \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n \), then the sequence \( (z_n)_n \) given by \( z_{2n+1} = x_n \) and \( z_{2n} = y_n \) also converges to the same limit. Thus \( (z_n)_n \) is Cauchy, so \( (f(z_n))_n \) is as well. Since \( (f(z_n))_n \) converges to a limit in \( Y \), \( (f(x_n))_n \) and \( (f(y_n))_n \) must converge to the same limit, i.e. we have

\[
\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(y_n).
\]

Thus the definition of \( f^* \) is well-defined, i.e. the value of \( f^*(x) \) is independent of the Cauchy sequence chosen.
Throughout, we will use the sequential compactness definition of compactness, which is equivalent by problem 1 and the sequential compactness theorem.

Finally, let \( x \in \partial S \). Let \((x_n)_n\) be a sequence in \( S \) with \((x_n)_n \to x \). Then for all \( n, f(x_n) = f^*(x_n), \) so

\[
\lim_{n \to \infty} f^*(x_n) = \lim_{n \to \infty} f(x_n) = f^*(x_n),
\]

as desired.

b. Let \( S = (0, \infty) \), and let \( f(x) = \frac{1}{x} \). Then this is continuous on \( S \), but not uniformly continuous. To see this, let \( \epsilon = 1 \) and fix any \( \delta > 0 \), which we can assume to be less than 1/2. Let \( x = \delta \) and let \( y = \delta/2 \). Then \( |x - y| = \delta/2 < \delta \), but \( |f(x) - f(y)| = |\frac{1}{\delta} - \frac{2}{\delta}| = \frac{1}{\delta} \), which is greater than 1 since \( \delta < 1/2 \). Thus \( f \) fails uniform continuity.

Meanwhile, \( \bar{S} = [0, \infty) \). Any function \( f^* : \bar{S} \to \mathbb{R} \) extending \( f \) must define a value \( f^*(0) \). But for all \( y > 0 \) with \( y < \frac{1}{f^*(0) + 1} \), we have

\[
f^*(y) = f(y) = \frac{1}{y} > f^*(0) + 1,
\]

so \( \lim_{y \to 0} f^*(y) \neq 0 \), and thus \( f^* \) cannot be continuous at 0.

5. Throughout, we will use the sequential compactness definition of compactness, which is equivalent by problem 1 and the sequential compactness theorem.

Assume first that \( f \) is continuous. Let \((x_n, f(x_n))_n \subseteq G_f \) be any sequence. Since \( E \) is compact, \((x_n)_n \subseteq E \) has a convergent subsequence \((x_{n_k})_k \), which approaches a limit \( x \). Since \( f \) is continuous, \( f(x_{n_k}) \) approaches \( f(x) \) as \( x_{n_k} \) approaches \( x \), so \((x_{n_k}, f(x_{n_k}))_k \) converges to \((x, f(x)) \). Thus any sequence of points in \( G_f \) has a convergent subsequence, so \( G_f \) is compact.

Now assume that \( f \) is discontinuous at a point \( x \in E \). Then there exists some \( \epsilon > 0 \) such that for any \( n \in \mathbb{N} \), for some \( x_n \in E \) with \( |x_n - x| < \frac{1}{n}, |f(x_n) - f(x)| \geq \epsilon \). Consider a sequence \((x_n)_n \) of such points, and the corresponding sequence of points \((x_n, f(x_n))_n \subseteq G_f \). For any subsequence \((x_{n_k}, f(x_{n_k}))_k \), the sequence of points \((x_{n_k})_k \) converges to \( x \), so any limit point would have to be the point \((x, f(x)) \in G_f \). But \((f(x_{n_k}))_k \) does not converge to \( f(x) \), since each \( f(x_{n_k}) \) is a distance more than \( \epsilon \) away from \( f(x) \). Thus the sequence \((x_n, f(x_n))_n \) has no convergent subsequence, so \( G_f \) is not sequentially compact, and thus not compact.

6. We first note that for any \( \frac{p}{q} \in \mathbb{Q} \) in lowest terms and for any \( n \in \mathbb{N}, \frac{p}{q} + n = \frac{p+nq}{q} \). The greatest common divisor of \( p + nq \) and \( q \) is the same as the greatest common divisor of \( p \) and \( q \), so \( \frac{p+nq}{q} \) is also in lowest terms. But then

\[
T\left( \frac{p}{q} \right) = \frac{1}{q} = T\left( \frac{p+nq}{q} \right).
\]
Thus $T$ is periodic, so it suffices to show that $T$ is continuous at $x$ for all $x \in [0, 1]$.  

**Claim:** For any $x \in [0, 1]$ and for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \neq x$ with $|y - x| < \delta$, $|f(y)| < \varepsilon$.

**Proof of claim.** Let $N \in \mathbb{N}$ be large enough that $\frac{1}{N} < \varepsilon$. There are finitely many rational numbers $\frac{p}{q} \in [0, 1]$ in lowest terms with $q \leq N$; let $S_N$ be the set of these rational numbers. Then let

$$
\delta = \min_{y \in S_N, y \neq x} |y - x|.
$$

If $y \neq x$ and $|y - x| < \delta$, then $y$ cannot be in $S_N$, so either $y$ is irrational and $f(y) = 0$, or $y$ is rational and $0 < f(y) < \frac{1}{N} < \varepsilon$. In either case, we have that $|f(y)| < \varepsilon$ whenever $|y - x| < \delta$, as desired.

For $x \in \{0, 1\}$, the same argument works, but instead considering rationals within a shifted interval (i.e. $(1/2, 3/2)$ when $x = 1$, or $(-1/2, 1/2)$ when $x = 0$). This completes the proof of the claim. \hfill \Box

When $x$ is irrational and $f(x) = 0$, the claim shows that $f$ is continuous at $x$. When $x$ is rational (say $x = \frac{p}{q}$), we have that

$$
f(x^-) = \lim_{y \to x^-} f(y) = 0,
$$

and similarly for $f(x^+)$. However, $f(x) = \frac{1}{q} \neq 0$, so $f$ has a discontinuity of the first kind at $x$.  