Homework 2 - Solutions

1. Let $P \subseteq \mathbb{C}$ be an arbitrary subset, and assume by contradiction that it satisfies Axiom 12. By part (ii), one of $i$ and $-i$ is in $P$. Thus $-1 = i^2 = (-i)^2$ must be in $P$, since it is the product of two elements of $P$. But then $1 = (-1)^2 \in P$, so we have both $1 \in P$ and $-1 \in P$, which contradicts Axiom 12, part (ii).

2. *. Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ be positive, with $a, b, c, d \in \mathbb{N}_{\geq 1}$. Let $n = bc + 1$. Then

$$ad \geq 1 \Rightarrow a \geq \frac{1}{d} \Rightarrow ac \geq \frac{c}{d} \Rightarrow \frac{abc}{b} \geq \frac{c}{d} \Rightarrow \frac{bc}{b} \geq \frac{c}{d} \Rightarrow n\frac{a}{b} > \frac{c}{d},$$

so there exists $n \in \mathbb{N}$ with $n\frac{a}{b} > \frac{c}{d}$.

*. Note that for $\frac{A(x)}{B(x)} \in \mathbb{R}(x)$, by multiplying by $-\frac{1}{-1}$ if necessary, we can always assume that $B(x)$ has a positive leading coefficient. Then $\frac{A(x)}{B(x)} \in P$ if and only if $A(x)$ has a positive leading coefficient. We will use this condition for the remainder of the problem.

If $\frac{A(x)}{B(x)} \neq 0$, then $A(x)$ and $-A(x)$ have leading coefficients of opposite sign, so exactly one of them has positive leading coefficient. Thus exactly one of $\frac{A(x)}{B(x)}$ and $-\frac{A(x)}{B(x)}$ is in $P$.

Also, if $\frac{A(x)}{B(x)}, \frac{C(x)}{D(x)} \in P$, then all four of $A(x), B(x), C(x)$, and $D(x)$ have positive leading coefficients. Thus $A(x)C(x), B(x)D(x), A(x)D(x)$, and $C(x)B(x)$ also have positive leading coefficients, so $\frac{A(x)C(x)}{B(x)D(x)} \in P$ and $\frac{A(x)D(x)+C(x)B(x)}{B(x)D(x)} \in P$. Thus $P$ is closed under addition and multiplication, so it satisfies Axiom 12.

a. Note that with this ordering $c < x$ for all constants $c$, since $x - c = \frac{x-c}{1}$ has positive leading coefficients in both the numerator and denominator. With this observation it is easy to see that the Archimedean property fails: Even though $1, x > 0$, we have $n \cdot 1 < x$ for all $n \in \mathbb{N}$. 

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b. Consider the set \( N \subseteq \mathbb{R} \). We showed in (a) that \( N \) is bounded above (by \( x \)). Suppose that \( M \) is an upper bound on \( N \). Then for all \( n \in N \) we have \( n + 1 \leq M \), so \( n \leq M - 1 < M \), so \( M \) is never the least upper bound.

3. * Assume BMC holds and that \( (s_n)_n \) is a decreasing sequence in \( F \) that is bounded below. Then \( (-s_n)_n \) is an increasing sequence in \( F \) that is bounded above, and so it has a limit \( L \) by BMC; the decreasing sequence \( (s_n)_n \) will converge to \(-L\), so it also has a limit in \( F \). Thus BMC implies the decreasing-and-bounded-below version; by a nearly identical argument, the decreasing-and-bounded-below version implies BMC.

Assume that BMC holds; by the optional portion of this problem, the decreasing-and-bounded-below version is also true. Let \( S \subseteq F \) be an arbitrary nonempty subset that is bounded above. We will construct two sequences, \( (a_n)_n \) and \( (b_n)_n \), with the following properties:

- For all \( n \), \( a_n \in S \) and \( b_n \) is a strict upper bound for \( S \), i.e. \( b_n \) is an upper bound with \( b_n \not\in S \).
- The sequence \((a_n)_n\) is increasing and the sequence \((b_n)_n\) is decreasing.
- The two sequences \((a_n)_n\) and \((b_n)_n\), which converge by the BMC, have the same limit.

Construct \((a_n)_n\) and \((b_n)_n\) as follows. Let \( a_1 \in S \) be chosen arbitrarily, and let \( b_1 \) be any upper bound for \( S \) not contained in \( S \). Then, for all larger \( n \), consider \( x_n = \frac{a_{n-1} + b_{n-1}}{2} \). We then have two options, based on whether or not \( x_n \) is an upper bound for \( S \):

- If \( x_n > s \) for all \( s \in S \) (so \( x_n \not\in S \)), then let \( b_n = x_n \) and let \( a_n = a_{n-1} \).
- If there exists \( s \in S \) with \( x_n \leq s \), then let \( a_n = s \) and let \( b_n = b_{n-1} \).

We now show that \((a_n)_n\) and \((b_n)_n\) have the desired properties. As we’ve constructed it, every \( a_n \) is in \( S \), and every \( b_n \) is a strict upper bound of \( S \). Thus \( a_n < b_m \) for all \( n, m \in N \), so for all \( n \in N \), \( a_n < \frac{a_n + b_n}{2} < b_n \). But \( a_{n+1} \) is either \( a_n \) or the average of \( a_n \) and \( b_n \), so \((a_n)_n\) must be increasing, and likewise \( b_{n+1} \) is either \( b_n \) or the average, so \((b_n)_n\) must be decreasing. By the BMC, \((a_n)_n\) converges to some limit \( L_a \in F \) and \((b_n)_n\) converges to some limit \( L_b \in F \).

Assume by contradiction that \( L_a \neq L_b \). Since \( a_n < b_m \) for all \( n, m \in N \), \( L_a < L_b \), and there are no elements of either \((a_n)_n\) or \((b_n)_n\) between \( L_a \) and \( L_b \). Let \( \varepsilon < \frac{L_b - L_a}{2} \). Since \( L_a \) and \( L_b \) are the respective limits of \((a_n)_n\) and \((b_n)_n\), there exists \( m_a \in N \) such that \( L_a - a_{m_a} < \varepsilon \), and \( m_b \in N \) with \( b_{m_b} - L_b < \varepsilon \). Taking \( m = \max\{m_a, m_b\} \), we have \( L_a - a_m < \varepsilon \) and \( b_m - L_b < \varepsilon \). But this gives

\[
L_a - a_m < \frac{L_b - L_a}{2} < \frac{b_m - a_m}{2},
\]

so \( L_a < \frac{b_m - a_m}{2} + a_m = \frac{b_m + a_m}{2} \). Similarly

\[
b_m - L_b < \frac{L_b - L_a}{2} < \frac{b_m - a_m}{2},
\]

Thus there exists a limit \( L \) such that \( S \) is bounded above by \( \varepsilon \) and \( \varepsilon \). We have

\[
L_a - L < \frac{L_b - L_a}{2} < \frac{b_m - a_m}{2} < \varepsilon
\]

and

\[
b_m - L < \frac{L_b - L_a}{2} < \frac{b_m - a_m}{2} < \varepsilon.
\]

So \((a_n)_n\) and \((b_n)_n\) have the same limit.
so \( L_b > b_m - \frac{b_m - a_m}{2} = \frac{b_m + a_m}{2} \). Thus the average of \( a_m \) and \( b_m \) is between \( L_a \) and \( L_b \). But by construction, this average is either \( a_{m+1} \) or \( b_{m+1} \), contradicting the fact that there are no elements of either sequence between \( L_a \) and \( L_b \). Thus \( L_a = L_b \), as desired.

**Claim:** \( L_a = L_b \) is the least upper bound of \( S \).

**Proof of claim.** Every element of \( (b_n)_n \) is an upper bound of \( S \), so \( L_b \) is an upper bound as well. Meanwhile, for all \( \varepsilon > 0 \), there exists \( a_n \) such that \( L_a - a_n < \varepsilon \), which means that \( L_a - \varepsilon < a_n \), where \( a_n \) is an element of \( S \). Thus for all \( \varepsilon > 0 \), \( L_a - \varepsilon \) is not an upper bound of \( S \), so \( L_a = L_b \) is the least upper bound of \( S \).

Thus \( S \) has a least upper bound in \( F \), so BMC implies LUB, as desired.

4. a. Let \( \varepsilon > 0 \), and let \( N \in \mathbb{N} \) be large enough that \( 2^{-(N-1)} < \varepsilon \). Let \( n, m > N \) be arbitrary, and without loss of generality assume that \( n \geq m \). Then

\[
|s_n - s_m| \leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \cdots + |s_{m+1} - s_m|
\]

\[
\leq \sum_{k=m}^{\infty} |s_{k+1} - s_k|
\]

\[
\leq \sum_{k=m}^{\infty} 2^{-k}
\]

\[
\leq 2^{-(m-1)} < 2^{-(N-1)} < \varepsilon,
\]

so \( |s_n - s_m| < \varepsilon \) for all \( n, m > N \). Thus \( (s_n)_n \) is Cauchy.

b. Let \( (t_n)_n \) be given by

\[
t_n = \sum_{k=1}^{n} \frac{1}{k}.
\]

Then

\[
\lim_{n \to \infty} |t_{n+1} - t_n| = \lim_{n \to \infty} \left| \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} \right|
\]

\[
= \lim_{n \to \infty} \frac{1}{n+1} = 0.
\]

On the other hand, \( (t_n) \) consists of partial sums of the harmonic series, which diverge, so \( (t_n)_n \) is not Cauchy.

5. a. Let \( (a_n)_n \) and \( (b_n)_n \) be Cauchy sequences, and let \( \varepsilon > 0 \). Then there exists \( N_a, N_b \in \mathbb{N} \) such that for \( n, m > N_a \), \( |a_n - a_m| < \varepsilon/2 \) and for \( n, m > N_b \), \( |b_n - b_m| < \varepsilon/2 \). Let \( N = \max\{N_a, N_b\} \). Then for all \( n, m > N \),

\[
|(a_n + b_n) - (a_m + b_m)| = |a_n - a_m + b_n - b_m|
\]

\[
\leq |a_n - a_m| + |b_n - b_m|
\]

\[
< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Thus for all \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that if \( n, m > N \), \( |(a_n + b_n) - (a_m + b_m)| < \varepsilon \), so \( (a_n + b_n)_n \) satisfies the Cauchy criterion as well.
b. Let \((a_n)_n\) and \((b_n)_n\) be Cauchy sequences. Then they are both bounded; let \(A, B \in \mathbb{Q}\) be such that \(|a_n| < A\) and \(|b_n| < B\) for all \(n\). Fix \(\varepsilon > 0\). There exists \(N_a \in \mathbb{N}\) such that for all \(n, m > N_a\), \(|a_n - a_m| < \frac{\varepsilon}{2A}\), and similarly there exists \(N_b \in \mathbb{N}\) such that for all \(n, m > N_b\), \(|b_n - b_m| < \frac{\varepsilon}{2B}\). Let \(N = \max\{N_a, N_b\}\). Then for all \(n, m > N\),

\[
|a_n b_n - a_m b_m| = |a_n b_n - a_n b_m + a_n b_m - a_m b_m| \\
= |a_n(b_n - b_m) + (a_n - a_m)b_m| \\
\leq |a_n||b_n - b_m| + |a_n - a_m||b_m| \\
< A|b_n - b_m| + B|a_n - a_m| \\
< A\frac{\varepsilon}{2A} + B\frac{\varepsilon}{2B} = \varepsilon + \varepsilon/2 = \varepsilon,
\]

so \(|a_n b_n - a_m b_m| < \varepsilon\). Thus \((a_n b_n)_n\) is Cauchy as well.

c. Let \((a_n)_n\) be a Cauchy sequence and let \(\mu > 0\) be such that \(a_n \geq \mu\) for all \(n\). Fix \(\varepsilon > 0\). There exists \(N \in \mathbb{N}\) such that for all \(n, m > N\), \(|a_m - a_n| < \varepsilon\mu^2\). Then for all \(n, m > N\),

\[
\left|\frac{1}{a_n} - \frac{1}{a_m}\right| = \left|\frac{a_m - a_n}{a_n a_m}\right| \\
= \frac{|a_m - a_n|}{|a_n||a_m|} \\
\leq \frac{|a_m - a_n|}{\mu^2} \\
< \varepsilon\frac{\mu^2}{\mu^2} = \varepsilon.
\]

Thus for \(n, m > N\), \(\left|\frac{1}{a_n} - \frac{1}{a_m}\right| < \varepsilon\), so \(\left(\frac{1}{a_n}\right)_n\) is also a Cauchy sequence.

Note (as will be relevant for the next part) that this proof works exactly as written if instead we assume only that \(|a_n| \geq \mu\), rather than \(a_n \geq \mu\).

d. Since \((a_n)_n \not\sim (0)_n\), the sequence \((a_n - 0)_n = (a_n)_n\) does not converge to zero. Thus for some \(\varepsilon > 0\), for all \(N \in \mathbb{N}\) there exists \(n \geq N\) with \(|a_n| \geq \varepsilon\). Since \((a_n)_n\) is Cauchy, there exists \(M \in \mathbb{N}\) with \(|a_n - a_m| < \varepsilon/2\) for all \(n, m \geq N\). Let \(k \geq M\) be such that \(|a_k| \geq \varepsilon\). Then for all \(n \geq M\), \(|a_n| \geq |a_k| - |a_k - a_n| > \varepsilon - \varepsilon/2 = \varepsilon/2\).

Define \((b_n)_n\) via

\[
b_n = \begin{cases} 
\frac{1}{a_n} & \text{if } n \geq M \\
1 & \text{otherwise.}
\end{cases}
\]

By part (c), the tail of \(b_n\) when \(n \geq M\) is Cauchy, since for \(n \geq M\), \(a_n \geq \varepsilon/2\). But then \(b_n\) is Cauchy, and

\[
a_n b_n = \begin{cases} 
1 & \text{if } n \geq M \\
a_n & \text{otherwise,}
\end{cases}
\]

so \((a_n b_n)_n\) is eventually equal to \((1)_n\), and thus \((a_n b_n)_n \sim (1)_n\). Thus such a sequence \((b_n)_n\) exists, as desired.
6. a. We define multiplication in \( A/\sim \) via \( f \ast g := f \circ g \), functional composition.

b. Let 

\[
\Phi : \{ \text{almost additive functions } \mathbb{Z} \to \mathbb{Z} \} \to \{ \text{Cauchy sequences in } \mathbb{Q} \}
\]

be given by \( \Phi(f)_n = \frac{f(n)}{n} \).

We’d like to show that for an almost additive function \( f \), \( \Phi(f)_n \) is a Cauchy sequence. By definition of almost additivity, the set \( \{ f(m + n) - f(m) - f(n) : m, n \in \mathbb{Z} \} \) is finite. In particular, it is bounded, so there exists \( M \in \mathbb{Z} \) with \( |f(m + n) - f(m) - f(n)| \leq M \) for all \( m, n \in \mathbb{Z} \).

Claim: For all \( m, n \in \mathbb{Z}, |f(mn) - mf(n)| \leq (m - 1)M \).

Proof of claim. We show this by induction on \( m \). If \( m = 2 \), this is true by how we have defined \( M \). Assume that \( |f((m - 1)n - (m - 1)f(n)| \leq (m - 2)M \). Then

\[
|f(mn) - mf(n)| \leq |f(mn) - f(n) - f((m - 1)n)| + |f((m - 1)n) - (m - 1)f(n)|
\]

\[
\leq M + |f((m - 1)n) - (m - 1)f(n)|
\]

\[
\leq M + (m - 2)M = (m - 1)M,
\]

as desired. \( \square \)

Given this claim, we prove that \( \Phi(f)_n \) is Cauchy. Fix an arbitrary \( \varepsilon > 0 \). Let \( N \in \mathbb{N} \) be large enough that \( \frac{1}{N} < \frac{\varepsilon}{2M} \). Then for any \( n, m \geq N \),

\[
\left| \frac{f(n)}{n} - \frac{f(m)}{m} \right| = \frac{|mf(n) - nf(m)|}{nm}
\]

\[
\leq \left| \frac{mf(n) - f(mn) + f(mn) - nf(m)}{nm} \right|
\]

\[
\leq \left| \frac{mf(n) - f(mn)}{nm} \right| + \left| \frac{f(mn) - nf(m)}{nm} \right|
\]

\[
\leq \left| \frac{(m - 1)M}{nm} \right| + \left| \frac{(n - 1)M}{nm} \right|
\]

\[
\leq \frac{M}{n} + \frac{M}{m}
\]

\[
\leq 2\frac{M}{N} < 2(\varepsilon/2) = \varepsilon,
\]

so \( \Phi(f)_n \) satisfies the Cauchy criterion.

7. Assume by contradiction that \( \lim_{n \to \infty} a_n \neq L \). Then there exists \( \varepsilon > 0 \) such that for all \( N \in \mathbb{N} \), for some \( n \geq N, |a_n - L| \geq \varepsilon \). Thus \( \{ k \in \mathbb{N} : |a_k - L| \geq \varepsilon \} \) is infinite; let \( (a_{n_k})_k \) be the subsequence with all such terms. Since \( (a_n)_n \) is bounded, so is \( (a_{n_k})_k \).

By Bolzano-Weierstrauss, \( (a_{n_k})_k \) has a convergent subsequence. This subsequence is also a convergent subsequence of \( (a_n)_n \), so by assumption it has limit \( L \). But this is a contradiction, because this subsequence has no terms within \( \varepsilon \) of \( L \).
8. a. The sequence \((\alpha_n^2)_n\) converges to \(-1\) with respect to \(|\cdot|_5\). We will prove this by showing that \(\lim_{n \to \infty} |(\alpha_n^2) - (-1)|_5 = 0\). We compute:

\[
\lim_{n \to \infty} |(\alpha_n^2) - (-1)|_5 = \lim_{n \to \infty} |\alpha_n^2 + 1|_5 \\
\leq \lim_{n \to \infty} 5^{-n}, \text{ since } \alpha_n^2 + 1 \text{ is divisible by } 5^n \\
= 0.
\]

b. Let \(\varepsilon > 0\), and let \(N\) be large enough that \(5^{-N} < \varepsilon\). Then for all \(m \geq n \geq N\),

\[
|\alpha_m - \alpha_n|_5 \leq 5^{-n}, \text{ since } \alpha_m - \alpha_n \text{ is divisible by } 5^n \\
\leq 5^{-N} < \varepsilon.
\]

Thus \((\alpha_n)_n\) is Cauchy with respect to \(|\cdot|_5\).

* Assume that \(\alpha_n\) had a limit \(L \in \mathbb{Q}\). Then we’d have \(L^2 = \lim(\alpha_n^2) = -1\), but no rational number \(L\) exists with this property!