Homework 1 - Solutions

1. * If $x \leq y$, then there exists $a \in \mathbb{N}$ with $y = x + a$. If in turn $y \leq z$, then there exists $b \in \mathbb{N}$ with $z = y + b$. But then $z = x + (a + b)$, so $x \leq z$, so $\leq$ is transitive. If $x \leq y$ and $y \leq x$, then there exist $a, b \in \mathbb{N}$ with $x + a = y$ and $y + b = x$. This implies that $x + a + b = x$, so by cancellativity, $a + b = 0$. One can show inductively from the Peano axioms that $a + b = 0$ implies that $a = b = 0$ for $a, b \in \mathbb{N}$, but then $x = y$.

   a. Assume first that $y = 0$. If $x \leq y$, then $x \leq 0$, so $x = 0$. Thus there is no $x$ satisfying the hypotheses, so the conclusion is automatically correct.

   Now assume that $y \neq 0$, and that $x \leq y$ with $x \neq y$. Then there exists $z \in \mathbb{N}$ with $y = x + z$. If $z = 0$, then $y = x + 0 = x$, but $x \neq y$, so $z \neq 0$. Since $z \in \mathbb{N} \setminus \{0\}$, there exists $w \in \mathbb{N}$ with $z = w + 1$. Thus

   $$y = x + w + 1 = (x + 1) + w,$$

   so $y \geq x + 1 = S(x)$.

   b. We proceed by induction on $x$. The base case $x = 0$ is trivial, since $0 \leq y$ for all $y \in \mathbb{N}$. For the induction step, we will assume that $x \leq y$ or $y \leq x$ (this is our induction hypothesis), and use that to prove that $S(x) \leq y$ or $y \leq S(x)$. Suppose that $S(x) \not\leq y$. By the contrapositive of part (a)...

   ...either $x = y$ or $x \not\leq y$. If $x = y$, then $S(x) = S(y) = y + 1$, so $y \leq S(x)$. If $x \not\leq y$, then by our inductive hypothesis $y \leq x$. Thus there exists $z \in \mathbb{N}$ with $x = y + z$, so $x + 1 = y + z + 1 = y + (z + 1)$. Thus $y \leq S(x)$, which completes the proof that $S(x) \leq y$ or $y \leq S(x)$.

2. * Assume that $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$. Then $a + b' = a' + b$ and $c + d' = c' + d$. But then $a + c + b' + d' = (a + b') + (c + d') = a' + b + c' + d = a' + c' + b + d$, so $(a + c, b + d) \sim (a' + b', c' + d')$.

   a. Since the equivalence class $[(a, b)]$ is (implicitly) identified with the integer $a - b$, the product $[(a, b)] \cdot [(c, d)]$ is implicitly identified with the product $(a - b)(c - d)$, which is equal to $ac + bd - ad - bc$. This integer in turn is represented by the equivalence class $[(ac + bd, ad + bc)]$, so we define

   $$[(a, b)] \cdot [(c, d)] = [(ac + bd, ad + bc)].$$

   b. The equivalence class representing $-1$ is $[(0, 1)]$, where $(0, 1)$ is a representative element (chosen arbitrarily). The class representing $n \in \mathbb{N}$ is the class $[(n, 0)]$. 

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Then

\[
[(0, 1)] \cdot [(n, 0)] = [(0 \cdot n + 1 \cdot 0, 0 \cdot 0 + 1 \cdot n)] \\
= [(0, n)],
\]

which is the equivalence class identified with \(-n\). Thus \((-1) \cdot n = -n\).

c. Let \(x\) be given by the equivalence class \([(a, b)]\) and \(y\) by the class \([(c, d)]\). Assume that \(xy = 0\). By problem 1, either \(a \leq b\) or \(b \leq a\); for each of these cases we will show that \([(a, b)] \cdot [(c, d)] = [(0, 0)]\) implies \([(a, b)] = [(0, 0)]\) or \([(c, d)] = [(0, 0)]\).
Assume \(a \leq b\) with \(a + n = b\). Then \((a, b) \sim (0, n)\), so

\[
[(a, b)] \cdot [(c, d)] = [(0, n)] \cdot [(c, d)] = [(0 \cdot c + n \cdot d, 0 \cdot d + n \cdot c)] = [(nd, nc)].
\]

If \([(a, b)] \cdot [(c, d)] = [(nd, nc)] = [(0, 0)], then \(nd = nc\), so by cancellativity of multiplication, either \(n = 0\) or \(d = c\). If \(n = 0\), then \((a, b) \sim (0, 0)\), and if \(c = d\) then \((c, d) \sim (0, 0)\), so \([(a, b)] \cdot [(c, d)] = [(0, 0)]\) implies \([(a, b)] = [(0, 0)]\) or \([(c, d)] = [(0, 0)]\).

Now assume that \(b \leq a\) with \(b + n = a\); this argument is nearly identical to the above. In this case \((a, b) \sim (n, 0)\), so

\[
[(a, b)] \cdot [(c, d)] = [(n, 0)] \cdot [(c, d)] = [(n \cdot c + 0 \cdot d, n \cdot d + 0 \cdot c)] = [(nc, nd)].
\]

Then just as above, \((nc, nd) \sim (0, 0)\) implies that either \(n = 0\), in which case \((a, b) \sim (n, 0) = (0, 0)\), or \(c = d\), in which case \((c, d) \sim (0, 0)\).

Thus \(xy = 0\) implies that \(x = 0\) or \(y = 0\).

3. Assume that \(f\) is an additive function. Then \(f(0) = 0\), since \(f(m) = f(m + 0) = f(m) + f(0)\). Let \(c = f(1)\). For \(m = 0\), certainly \(f(0) = c \cdot 0\). For all \(m \geq 1\),

\[
f(m) = f(1 + \cdots + 1) = f(1) + \cdots + f(1) = f(1)m \cdot cm,
\]

and \(f(m - m) = f(0) = 0\), so \(f(m) + f(-m) = 0\) and thus \(f(-m) = -f(m)\). So for \(m \geq 1\), \(f(m) = cm\), and \(f(-m) = -f(m) = -cm = c(-m)\). Thus \(f(m) = cm\) for all \(m \in \mathbb{Z}\).

a. Transitivity: Let \(f, g, h \in A\) and assume that \(f \sim g\) and \(g \sim h\). Then \(f - g\) and \(g - h\) take on only finitely many values (say they take on \(m\) different values and \(n\) different values, resp.), so \((f - g) + (g - h) = f - h\) also takes on only finitely many values (at most \(m \cdot n\) different values). Hence, \(f \sim h\).

Symmetry: Let \(f, g \in A\), and assume that \(f \sim g\). Then the set \(S_{fg} = \{f(m) - g(m) : m \in \mathbb{Z}\}\) is finite. Let \(S_{gf} = \{g(m) - f(m) : m \in \mathbb{Z}\}\). But then \(S_{gf} = \{-x : x \in S_{fg}\}\), so \(|S_{gf}| = |S_{fg}| < \infty\), and thus \(g \sim f\). Thus \(\sim\) is symmetric.

Reflexivity: For \(f \in A\), \(\{f(m) - f(m) : m \in \mathbb{Z}\} = \{0\}\), which is finite, so \(f \sim f\).
b. For all \( m \in \mathbb{Z} \), the floor function \( f_\lambda(m) = \lfloor \lambda m \rfloor \) has the property that \( \lambda m - 1 < \lfloor \lambda m \rfloor \leq \lambda m \), so \( \lambda m - \lfloor \lambda m \rfloor < 1 \). Thus

\[
|f_\lambda(m + n) - f_\lambda(m) - f_\lambda(n)| = |(\lfloor \lambda(m + n) \rfloor - \lfloor \lambda m \rfloor - \lfloor \lambda n \rfloor|
= |(\lfloor \lambda(m + n) \rfloor - \lfloor \lambda m \rfloor - \lfloor \lambda n \rfloor - (\lambda(m + n) - \lambda(m) - \lambda(n))|
= |((\lfloor \lambda(m + n) \rfloor - \lambda(m + n)) - (\lfloor \lambda m \rfloor - \lambda m)) - (\lfloor \lambda n \rfloor - \lambda(n))|
\leq |\lfloor \lambda(m + n) \rfloor - \lambda(m + n)| + |\lfloor \lambda m \rfloor - \lambda m| + |\lfloor \lambda n \rfloor - \lambda(n)|
< 1 + 1 + 1 = 3.
\]

For all \( x \in D(f_\lambda) \), \( x = f_\lambda(m + n) - f_\lambda(m) - f_\lambda(n) \), so by the above we have \( |x| < 3 \). Thus \( x \in \{-2, -1, 0, 1, 2\} \), so \( D(f_\lambda) \subseteq \{-2, -1, 0, 1, 2\} \). In particular, it is finite, so \( f_\lambda \) is almost additive.

4. a. \( f^{-1}(\cap \mathcal{C}) \subseteq \cap \{f^{-1}(C) : C \in \mathcal{C}\} \): Let \( x \in f^{-1}(\cap \mathcal{C}) \). Then \( f(x) \in \cap \mathcal{C} \), which means that \( f(x) \in C \) for all \( C \in \mathcal{C} \). Thus \( x \in f^{-1}(C) \) for all \( C \in \mathcal{C} \), so \( x \in \cap \{f^{-1}(C) : C \in \mathcal{C}\} \). Thus \( f^{-1}(\cap \mathcal{C}) \subseteq \cap \{f^{-1}(C) : C \in \mathcal{C}\} \).

\( \cap \{f^{-1}(C) : C \in \mathcal{C}\} \subseteq f^{-1}(\cap \mathcal{C}) \): Let \( x \in \cap \{f^{-1}(C) : C \in \mathcal{C}\} \). Then for all \( C \in \mathcal{C} \), \( x \in f^{-1}(C) \), so \( f(x) \in C \) for all \( C \in \mathcal{C} \). Thus \( f(x) \in \cap \mathcal{C} \), so \( x \in f^{-1}(\cap \mathcal{C}) \), which shows that \( \cap \{f^{-1}(C) : C \in \mathcal{C}\} \subseteq f^{-1}(\cap \mathcal{C}) \).

b. First, assume that \( f \) is injective. We want to show that \( f(A \cap B) = f(A) \cap f(B) \).

Let \( y \in f(A \cap B) \). Then there exists \( x \in A \cap B \) with \( f(x) = y \). But then \( x \in A \) and \( x \in B \), so \( y \in f(A) \) and \( y \in f(B) \), so \( y \in f(A) \cap f(B) \). Thus \( f(A \cap B) \subseteq f(A) \cap f(B) \).

Meanwhile, let \( y \in f(A) \cap f(B) \). Then there exist \( x \in A \) and \( z \in B \) with \( f(x) = y = f(z) \). By injectivity, \( x = z \). But then \( x \in A \) and \( x \in B \), so \( x \in A \cap B \). Thus \( y = f(x) \in f(A \cap B) \), so \( f(A) \cap f(B) \subseteq f(A \cap B) \).

This completes the proof that \( f(A \cap B) = f(A) \cap f(B) \).

Now assume that \( f \) is not injective. Then there exist \( x, z \in X \) such that \( f(x) = f(z) \) but \( x \neq z \). Let \( A = \{x\} \) and \( B = \{z\} \); then

\[
f(A \cap B) = f(\{x\} \cap \{z\}) = f(\emptyset) = \emptyset,
\]

but

\[
f(A) \cap f(B) = f(\{x\}) \cap f(\{z\}) = \{f(x)\} \cap \{f(z)\} = \{f(x)\},
\]

since \( f(x) = f(z) \). Whenever \( f \) is not injective, there exist some subsets \( A, B \subseteq X \) with \( f(A \cap B) \neq f(A) \cap f(B) \).

Thus, \( f \) is injective if and only if for all \( A, B \subseteq X \), \( f(A \cap B) = f(A) \cap f(B) \).

5. Let \( f : S \to \mathcal{P}(S) \). We will construct a subset \( T \) of \( S \) that is not contained in the image of \( f \). Define \( T \) by

\[
T = \{s \in S : s \notin f(s)\}.
\]

Assume by contradiction that \( T \) is in the image of \( f \). Then for some \( t \in S \), \( f(t) = T \). Now we ask, is \( t \) contained in \( T \)? If \( t \in T \), then it is not true that \( t \notin f(t) \), so \( t \notin T \).
by definition of \(T\), which is a contradiction. But if \(t \not\in T\), then by definition of \(T\) we have \(t \not\in f(t)\) so \(t \in T\), which is another contradiction. Thus there can be no element \(t \in S\) with \(f(t) = T\). Thus \(T\) is not in the image of \(f\), so \(f\) cannot be surjective/onto.

* Exercise 3.1: Let \(x \in \mathbb{R}\), and let \(y, z \in \mathbb{R}\) be two additive inverses of \(x\). Then
\[
y = 0 + y = (x + z) + y = (x + y) + z = 0 + z = z,
\]
so \(y = z\). Thus the additive inverse of \(x\) is unique.

Exercise 3.5: Assume that \(x \neq 0\) and \(y \neq 0\). Then \(x\) and \(y\) have multiplicative inverses, say \(w\) and \(z\) respectively. But then \((xy)(wz) = xywz = (xz)(yz) = 1 \cdot 1 = 1\), so \(xy\) has a multiplicative inverse, namely \(wz\). However, 0 has no multiplicative inverse, since \(0 \cdot r = 0\) for all \(r \in \mathbb{R}\). Thus \(xy \neq 0\).

Exercise 3.7: We want to show that \(x(-y)\) and \((-x)y\) are each equal to the additive inverse of \(xy\). We compute
\[
xy + x(-y) = x(y + (-y)) = x \cdot 0 = 0,
\]
and
\[
xy + (-x)y = (x + (-x))y = 0 \cdot y = 0.
\]

6. We will prove the contrapositive, that if \(x > y\), then for some \(\varepsilon > 0\), \(x > y + \varepsilon\). If \(x > y\), then \(x - y > 0\). Since \(1 > 0\), \(2 = 1 + 1 > 0\). But then \(2 \cdot \frac{1}{2} = 1 > 0\), so \(\frac{1}{2} > 0\) as well. Thus \(\frac{1}{2}(x - y) > 0\). Let \(\varepsilon = \frac{1}{2}(x - y)\). Then \(x - (y + \varepsilon) = x - y - \frac{1}{2}(x - y) = \frac{1}{2}(x - y) > 0\), so \(x > y + \varepsilon\). Thus if \(x > y\), then for some \(\varepsilon > 0\), \(x > y + \varepsilon\).

7. For all \(x + y \in X + Y\), \(x \leq a\) and \(y \leq b\), so \(x + y \leq a + b\). Thus \(a + b\) is an upper bound of \(X + Y\). Let \(c < a + b\). We will show that \(c\) is not an upper bound of \(X + Y\). Since \(c < a + b\), there exists \(\varepsilon > 0\) with \(\varepsilon < a + b - c\) (or equivalently, \(c < a + b - \varepsilon\)). Since \(a\) is the least upper bound of \(X\), there exists \(x \in X\) with \(x > a - \frac{\varepsilon}{2}\). Likewise, \(b\) is the least upper bound of \(Y\), so there exists \(y \in Y\) with \(y > b - \frac{\varepsilon}{2}\). Thus \(x + y > a + b - 2\frac{\varepsilon}{2} = a + b - \varepsilon > c\), so \(c\) is not an upper bound of \(X + Y\). Thus \(a + b\) is the least upper bound of \(X + Y\).

8. * We want to show that for all \(x, y \in \mathbb{Q}\), \(|xy|_p = |x|_p|y|_p\). If either \(x\) or \(y\) is 0, then \(xy = 0\), and \(|xy|_p = 0 = |x|_p|y|_p\). Assume now that neither \(x\) nor \(y\) is 0.

Let \(x = p^{r_1} s_1\) and let \(y = p^{r_2} s_2\), where \(r_1, r_2, s_1, s_2 \in \mathbb{Z}\) are not divisible by \(p\), and \(v_1, v_2 \in \mathbb{Z}\). Then \(xy = p^{r_1 + v_1} s_1^{r_2} s_2\), where again \(r_1 r_2, s_1 s_2 \in \mathbb{Z}\) are not divisible by \(p\), and \(v_1 + v_2 \in \mathbb{Z}\). But then
\[
|xy|_p = p^{-(v_1 + v_2)} = p^{-v_1} p^{-v_2} = |x|_p|y|_p,
\]
as desired.

a. We want to show that for all \(x, y \in \mathbb{Q}\), \(|x + y|_p \leq |x|_p + |y|_p\). Assume first that \(x = 0\); then \(|x + y|_p = |y|_p = |y|_p = 0 + |y|_p\), so \(|x + y|_p = |x|_p + |y|_p\) and thus the inequality is satisfied. The same is true if \(y = 0\). Meanwhile if \(x + y = 0\), then \(|x + y|_p = 0\), which is automatically a lower bound on \(|x|_p + |y|_p|\).
Now assume that \( x, y, \) and \( x + y \) are all nonzero; let \( x = p^{v_1} \frac{r_1}{s_1} \) and \( y = p^{v_2} \frac{r_2}{s_2} \), as above. Without loss of generality, assume that \( v_1 \leq v_2 \) (if this is not the case, the same proof works where we swap the roles of \( x \) and \( y \)). Note that \( v_1 \leq v_2 \) implies that \(|y|_p = p^{-v_2} \leq p^{-v_1} = |x|_p \). Then

\[
x + y = p^{v_1} \frac{r_1}{s_1} + p^{v_2} \frac{r_2}{s_2} = p^{v_1} \left( \frac{r_1}{s_1} + p^{v_2-v_1} \frac{r_2}{s_2} \right) = p^{v_1} \frac{r_1 s_2 + p^{v_2-v_1} r_2 s_1}{s_1 s_2}.
\]

Note that \( s_1 s_2 \) is not divisible by \( p \), but if \( v_2 = v_1 \) the numerator may be divisible by \( p \). Let \( r_1 s_2 + p^{v_2-v_1} = p^{v_3} r_3 \), where \( r_3 \) is not divisible by \( p \), and \( v_3 \geq 0 \). Then

\[
x + y = p^{v_1+v_3} \frac{r_3}{s_1 s_2},
\]

so

\[
|x + y|_p = p^{-v_1-v_3} \leq p^{-v_1} = |x|_p = \max\{|x|_p, |y|_p\}.
\]

But since \(|x + y|_p \leq \max\{|x|_p, |y|_p\}\), \(|x + y|_p \leq |x|_p + |y|_p\).

b. We will use the fact that

\[
\sum_{n=0}^{N} 2^n = 2^{N+1} - 1.
\]

We then compute:

\[
\lim_{N \to \infty} \left| \sum_{n=0}^{N} 2^n - (-1) \right|_2 = \lim_{N \to \infty} \left| 2^{N+1} - 1 - (-1) \right|_2
\]

\[
= \lim_{N \to \infty} \left| 2^{N+1} \right|_2
\]

\[
= \lim_{N \to \infty} 2^{-(N+1)}, \text{ by definition of } |\cdot|_2
\]

\[
= 0, \text{ as desired.}
\]